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A Single-vendor Single-buyer Integrated Model for Deteriorating Items with Partial Backlogging and Price-dependent Market Demand

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Abstract
In this paper, we develop an integrated vendor-buyer model for deteriorating items where the market demand is assumed to be dependent on the selling price of the buyer. The vendor manufactures the product in a lot and delivers it in a number of unequal sized batches to the buyer. Shortages are allowed in the buyer's inventory and are partially backlogged. The objective of this study is to determine the optimal decisions which maximize the joint total profit of the vendor-buyer integrated system. A numerical example is taken to illustrate the developed model and examine the sensitivity of the key parameters involved in the model.

Keywords: Supply chain; vendor-buyer; partial backlogging; shortage; deterioration; price dependent demand.

INTRODUCTION
One of the most successful OR techniques to be applied in businesses or industries or public sector is Inventory Theory. The main resources of any kind having an economic value are inventories and, in the economy of any country, these inventories play an extremely important role. Providing customer service is the primary function of inventory, the factors being availability of goods at the right time, at the right place and at the right cost. Several costs associated with inventories can be broadly classified into fixed cost and variable cost which together form the total cost of the inventory system. The classical Economical Order Quantity (EOQ) model defines isolated inventory problems for vendor and buyer in order to minimize their individual costs or maximize their individual profits. This kind of one sided optimal strategy does not provide suitable solution in today's global market due to rising cost, shrinking resources, short product life cycle, increasing competition etc. Therefore, in recent years, researchers pay more attention on integrated vendor-buyer problems. Collaboration of the vendor and the buyer is one of the key factors for successful management of the supply chain.

Deterioration of inventoried items is a natural phenomenon and it is very common in our daily life. Deteriorating items refers to the items that become decayed, damaged, evaporative, expired, invalid, and devaluation through time. Deteriorating items can be classified broadly into two groups. The first group refers to the items that become decayed, damaged, evaporative, or expired through time; examples include meat, vegetable, fruit, medicine, flower, film and so on. The other group refers to the items that lose part or total value through time because of new technology or the introduction of alternatives. Examples include computer chips, mobile phones, fashion and seasonal goods, and so on. Both the groups have the characteristic of short life cycle. After a specific period (such as durability), the natural attributes of the items change and then lose useable value and economic value. For the second group, the items have a short market life cycle. After a while of popularity in the market, the items lose the original economic value due to the changes in customer's preferences and vendor's product upgradation.

In the classical inventory and supply chain models, the market demand is assumed to be constant. However, in reality, there are many items whose demands are merely not constant but depend on the retail price. If the price is low, then the market demand is more; on the other hand, if the price is high then the market demand of the product is less. So, the selling price plays an important role in determining the inventory or supply chain strategies. In this paper, we will focus on an integrated vendor-buyer model for deteriorating items having price-dependent market demand. Shortages in the buyer's inventory are allowed to occur and are partially backlogged. The objective of this study is to determine the optimal decisions which maximize the joint total profit of the whole supply chain.

LITERATURE REVIEW
The 'lot-for-lot' policy for single-vendor single-buyer supply chain under the assumption of infinite production rate for the vendor was first studied by Goyal(1977). Later, Banerjee (1986) extended this work to consider the case of finite production rate. However, the policy mentioned above is not optimal except when the set up cost of the vendor is significantly smaller than the buyer's ordering cost. Goyal (1988) suggested the vendor to deliver the ordered lot in an integral number of equal size shipments. An alternative shipment policy was introduced by Goyal (1995) where the shipment sizes are all unequal and it should involve successive shipments increased by a fixed geometric growth factor equal to production rate divided by the demand rate. The shipment

policy developed by Lu (1995) was named as 'Identical Delivery Quantity'(IDQ) and the policy given by Goyal (1995) was named as 'Deliver What is Produced' (DWP) strategy by Viswanathan (1998).

Ghare and Schrader (1963) first introduced the concept of deterioration of items in inventory modeling. In his model, he considered the rate of deterioration as constant. This model was further extended by Covert and Philip (1973) considering deterioration rate to be a two-parameter Weibull distribution. After that, numerous research efforts have been made to implement the concept of deterioration in various inventory models. For comprehensive survey of deteriorating inventory literature, the readers may be referred to the review articles contributed by Goyal and Giri (2001) and Janssen et al. (2016).

Determination of pricing strategy that influences demand and production inventory decisions is being focused by many practitioners and academics. This strategy also defines the cost of satisfying those demands simultaneously. Whitin (1955) gave the seminal work in this line of research. In his model, he considered EOQ model where the demand function is assumed to be a linear function and price dependent. This work further encouraged so many researchers (Rosenberg, 1991; Giri et al., 1996; Lau and Lau, 2003; Roy, 2008; Khanra et al. 2010) in investigating joint pricing and ordering problems. Chung and Wee (2008) developed joint pricing and ordering problems based on cooperation of the companies with each other in a supply chain. A fuzzy inventory model for deteriorating items with price-dependent demand and allowed shortages was developed by Maragatham and Lakshmidevi (2014). Lin and Lin (2015) defined another model where they considered the demand rate to be a function of unit price charged by the vendor to the buyer. Their objective was to determine the buyer's order quantity and the size of each shipment which minimize the joint total cost per unit time. Giri and Roy (2016) defined a supply chain inventory system with controllable lead time under price-dependent demand. In their model for the lead time demand, they considered two cases - first being followed the normal probability distribution and the second being independent of the distribution. A deterministic inventory model in which the deterioration rate is time proportional, demand rate is a function of selling price and inventory holding cost, ordering cost and deterioration rate are all of functions of time, was introduced by Maragatham and Palani (2017). Several researchers like Giri and Bardhan (2015), Saha (2017), Sheikh and Patel (2017), and Kurdhi, et al. (2017) developed their models considering price-dependent demand. Aliyu and Sani (2018) developed an inventory model for deteriorating items with generalized exponential decreasing demand, constant holding cost and time varying deterioration. An inventory model for non-instantaneous deterioration with price dependent demand, linear holding cost and partial backlogging under inflationary environment was introduced by Agarwal et al. (2018).

This paper presents a vendor-buyer integrated model in which the demand rate depends on the selling price of the buyer, vendor ships the ordered quantity of the buyer in a number of unequal sized batches and items at the buyer inventory deteriorates at a constant rate, shortages are allowed at the buyer's inventory and are partially backlogged. The aim of this study is to determine the buyer's selling price and order quantity and the number of shipments from the vendor to the buyer, which maximize the joint total profit of the vendor-buyer system.

ASSUMPTIONS AND NOTATIONS

The proposed model is developed on the basis of the following assumptions and notations.

Assumptions

(i) There is a single-vendor and a single-buyer for trading a single product.

(ii) The buyer orders Q quantity from the vendor and the vendor delivers it in m shipments. The first shipment size is q and then this first shipment is followed by $(m-1)$ shipments, each is equal to αq in $(m-1)$in size. Therefore, $Q = q+(m-1)\alpha q$.

(iii) The demand rate at the buyer is dependent on the selling price p and is given by $D(p)$.

(iv) Shortages are allowed in the buyer inventory and it is partially backlogged with the backorder rate $R(t)=\dfrac{1}{1+\delta(T-t)}$ where δ (>0) is backlogging parameter.

For δ =0, we have $R(t)$=1 which indicates that shortages are fully backlogged.

We assume that δ <1 and no shortage is allowed in the last shipment cycle.

(v) The vendor's production process is perfect. The production rate(P) is finite and greater than the buyer's demand rate $D(p)$ i.e., $P>D(p)$.

(vi) The buyer's inventory deteriorates at a constant rate θ $(0 \le \theta<1)$.

Notations

s_v = vendor's set up cost per set up
v = vendor's unit production cost
w = wholesale price of the vendor
p = unit selling price of the buyer
h_v = vendor's average inventory holding cost/unit/unit time
m = number of shipments from the vendor to the buyer per production run, a positive integer
α = ratio between the production rate and the demand rate
A_b = buyer's ordering cost per order
Q = buyer's order quantity
Q = size of the first shipment from the vendor to the buyer
h_b = buyer's inventory holding cost/unit/unit time
c_3 = shortage cost/unit/unit time
c_4 = unit lost sale cost
B = maximum shortage quantity at the buyer
θ = deterioration rate
δ = backlogging parameter
d_c = unit deterioration cost
f = vendor's transportation cost per shipment in a cycle

Decision variables

q = size of the first shipment from the vendor to the buyer
p = unit selling price of the buyer
m = number of shipments from the vendor to the buyer

MATHEMATICAL MODEL AND SOLUTION

We now derive the average profit of the buyer and the vendor separately in order to obtain the joint total profit of the whole supply chain.

Buyer's average profit

According to our assumptions, the first shipment size of the vendor is q and then this first shipment is followed by *(m-1)* shipments, each is equal to αq.

First replenishment cycle:

Suppose that the first replenishment is made by the buyer at time $t = 0$.Suppose that during the period $[0,T_{11}]$, the inventory level decreases due to both demand and deterioration. At time $t = T_{11}$, the inventory level reaches to zero and thereafter shortages are allowed to occur during the time interval $[T_{11},T_{12}]$. A fraction of shortages is backlogged at the next replenishment and the other fraction is lost. A graphical representation of the inventory system is reflected in Fig. 1.

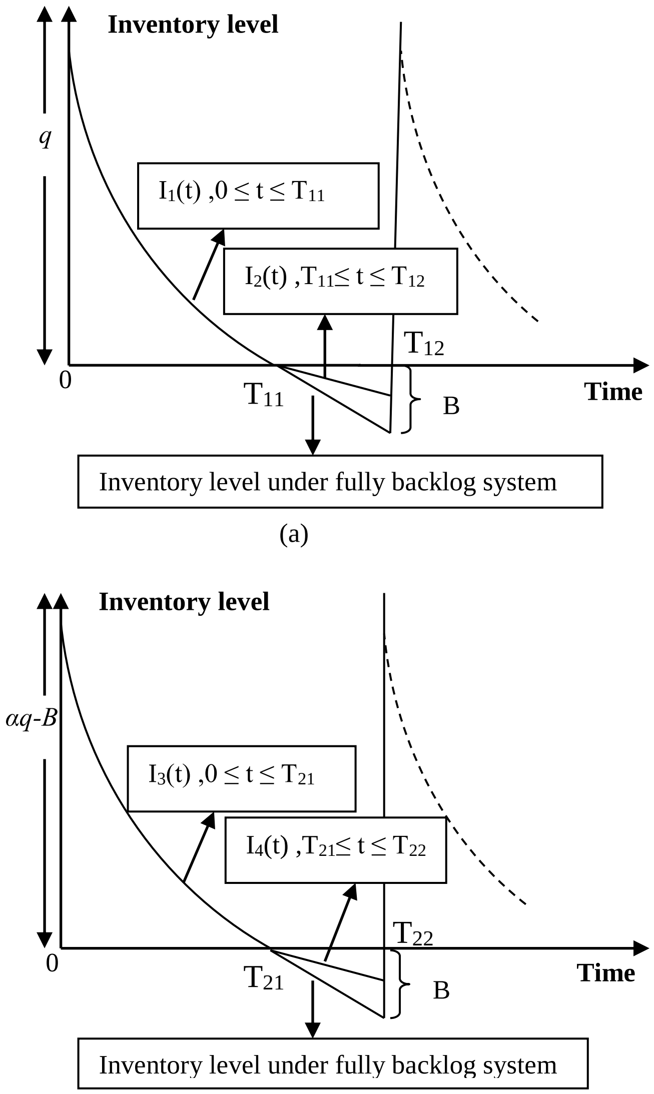

Fig. 1: Graphical representation of the state of buyer's inventory system

Let $I_1(t)$ be the inventory level at any time t during $[0,T_{11}]$. The rate of change of inventory level during the positive stock period $[0,T_{11}]$ is governed by the following differential equation:

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -D(p) \qquad (1)$$

with $I_1(0) = q$.
Solving (1), we get

$$I_1(t) = qe^{-\theta t} - \frac{D(p)}{\theta}(1-e^{-\theta t}) \qquad (2)$$

Now, using the boundary condition $I_1(T_{11})=0$, we get

$$T_{11} = \frac{1}{\theta}\log\left(1+\frac{q\theta}{D(p)}\right) = \frac{q}{D(p)} - \frac{q^2\theta}{2D(p)^2} \qquad (3)$$

Inventory available in the system during time interval $(0, T_{11})$ is given by

$$\int_0^{T_{11}} I_1(t)dt = \frac{q^2}{2D(p)} - \frac{\theta q^3}{3(D(p))^2} \qquad (4)$$

Let $I_2(t)$ be the inventory level at any time t during the period $[T_{11}, T_{12}]$. Then the rate of change of inventory level during the period $[T_{11}, T_{12}]$ is governed by the following differential equation:

$$\frac{dI_2(t)}{dt} = -\frac{D(p)}{1+\delta(T_{12}-t)} \quad where\ \delta<1,\ T_{11} \le t \le T_{12} \qquad (5)$$

with $I_2(T_{11})=0$.
Solving (5) we get,

$$I_2(t) = \frac{D(p)}{\delta}[\log\{1+\delta(T_{12}-t)\} - \log\{1+\delta(T_{12}-T_{11})\}] \qquad (6)$$

Now, using the boundary condition $I_2(T_{12})= -B$, we get

$$T_{12} = \frac{q}{D(p)} - \frac{\theta q^2}{2(D(p))^2} + \frac{1}{\delta}\left[e^{\frac{\delta B}{D(p)}} - 1\right] \qquad (7)$$

The total shortage quantity is given by

$$\int_{T_{11}}^{T_{12}} -I_2(t)dt = -\left[\frac{D(p)}{\delta^2}\log\{1+\delta(T_{12}-T_{11})\} - \frac{D(p)}{\delta}(T_{12}-T_{11})\right] \qquad (8)$$

During the period $[T_{11}, T_{12}]$, the shortages are accumulated but not all customers are willing to wait for the next batch to arrive. Hence, there is a loss of sale and the lost sale quantity is given by

$$\int_{T_{11}}^{T_{12}} \{1-\frac{1}{1+\delta(T_{12}-t)}\}D(p)dt = D(p)(T_{12}-T_{11}) - B \qquad (9)$$

Second replenishment cycle:

We now calculate the inventory and shortage quantities for the second replenishment cycle, which will be repeated (m-2) times (as in the final replenishment cycle, there will be no shortage). Without any loss of generality, we assume that the second replenishment cycle starts at time $t = 0$ with inventory level αq-$B>0$, see Fig. 1(b).Let $I_3(t)$ be the inventory level at any time t, during the period $[0, T_{21}]$. The rate of change of inventory during this stock period is governed by the following differential equation:

$$\frac{dI_3(t)}{dt} + \theta I_3(t) = -D(p) \qquad (10)$$

with $I_3(0)= \alpha q-B$. Solving eqn. (10), we get with $I_3(0)= \alpha q-B$. Solving eqn. (10), we get

$$I_3(t) = -\frac{D(p)}{\theta}(1-e^{-\theta t}) + (\alpha q - B)e^{-\theta t}$$

$$= -D(p)[t - \frac{\theta t^2}{2}] + (\alpha q - B)(1-\theta t) \quad (11)$$

(ignoring second and higher powers of θ which are small)
Now, using the boundary condition $I_3(T_{21})=0$, we get

$$T_{21} = \frac{1}{\theta}\log\{1+\frac{\theta(\alpha q - B)}{D(p)}\} \quad = \frac{\alpha q - B}{D(p)} - \frac{\theta(\alpha q - B)^2}{2(D(p))^2} \quad (12)$$

Therefore, inventory available in the system during the time period [0, T_{21}] is given by

$$\int_0^{T_{21}} I_3(t)dt = \frac{(\alpha q - B)^2}{2D(p)} - \frac{\theta(\alpha q - B)^3}{3(D(p))^2} \quad (13)$$

Let $I_4(t)$ be the inventory level at any time t, $T_{21} \le t \le T_{22}$. Then the rate of change of inventory level during the period $[T_{21}, T_{22}]$ is governed by the differential equation

$$\frac{dI_4(t)}{dt} = -\frac{D(p)}{1+\delta(T_{22}-t)} \quad , \quad \delta < 1, \ T_{21} \le t \le T_{22} \quad (14)$$

with $I_4(T_{21})=0$. Solving eqn.(14), we get

$$I_4(t) = \frac{D(p)}{\delta}\left[\log\{1+\delta(T_{22}-t)\} - \log\{1+\delta(T_{22}-T_{21})\}\right] \quad (15)$$

Now, using the boundary condition $I_4(T_{22})= -B$ in (15), we get

$$T_{22} = T_{21} + \frac{1}{\delta}\left(e^{\frac{\delta B}{D(p)}} - 1\right) = \frac{1}{\theta}\log\{1+\frac{\theta(\alpha q - B)}{D(p)}\} + \frac{1}{\delta}\left(e^{\frac{\delta B}{D(p)}} - 1\right) \quad (16)$$

Due to stock out situation during the period [T_{21}, T_{22}], shortages are accumulated but not all customers are willing to wait for the next batch to arrive. This results in some loss of sale.
Total shortage quantity

$$= \int_{T_{21}}^{T_{22}} -I_4(t)dt$$

$$= -\left[\frac{D(p)}{\delta^2}\log\{1+\delta(T_{22}-T_{21})\} - \frac{D(p)}{\delta}(T_{22}-T_{21})\right] \quad (17)$$

and the lost sale quantity

$$= \int_{T_{21}}^{T_{22}} \{1 - \frac{1}{1+\delta(T_{22}-t)}\}D(p)dt$$

$$= D(p)(T_{22}-T_{21}) - B \quad (18)$$

From the above calculations, the buyer's total inventory in a cycle is given by

$$\int_0^{T_{11}} I_1(t)dt + (m-1)\int_0^{T_{21}} I_3(t)dt$$

$$= \frac{q^2}{2D(p)} - \frac{\theta q^3}{3(D(p))^2} + (m-1)\{\frac{(\alpha q - B)^2}{2D(p)} - \frac{\theta(\alpha q - B)^3}{3(D(p))^2}\} \quad (19)$$

Therefore, the buyer's holding cost and deterioration cost is given by

$$(h_b + \theta d_c)\left[\frac{q^2}{2D(p)} - \frac{\theta q^3}{3(D(p))^2} + (m-1)\{\frac{(\alpha q - B)^2}{2D(p)} - \frac{\theta(\alpha q - B)^3}{3(D(p))^2}\}\right]$$

The buyer's shortage cost is

$$\frac{c_3 D(p)}{\delta^2}(m-1)\left(e^{\frac{\delta B}{D(p)}} - \frac{\delta B}{D(p)} - 1\right)$$

and lost sale cost is

$$c_4(m-1)(e^{\frac{\delta B}{D(p)}} - 1) + \frac{c_4 D(p)}{\delta^2}\left(e^{\frac{\delta B}{D(p)}} - \frac{\delta B}{D(p)} - 1\right)$$

The sum of the ordering and purchase costs is

$A_b+w(q+(m-1)\alpha q)$.

Since shortage is not allowed in the last replenishment cycle, the cycle length is given by
$T_{12}+(m-2)T_{22}+T_{22}$

$$= \frac{q}{D(p)} - \frac{\theta q^2}{2(D(p))^2} + (m-1)\frac{1}{\delta}\left(e^{\frac{\delta B}{D(p)}} - 1\right) + (m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q - B)}{D(p)}\}$$

The buyer's total cost includes holding cost, deteriorating cost, shortage cost, lost sale cost, ordering cost and purchase cost. Hence the average profit of the buyer is given by

$TPB(q,p,m) =$

$$\Big[p\{Q - \frac{\theta q^2}{2D(p)} - (m-1)\theta\frac{(\alpha q - B)^2}{2D(p)}\} - (h_b + \theta d_c)[\frac{q^2}{2D(p)} - \frac{\theta q^3}{3D(p)^2}$$
$$+ (m-1)\{\frac{(\alpha q - B)^2}{2D(p)} - \frac{\theta(\alpha q - B)^3}{3D(p)^2}\}] - \frac{c_3 D(p)}{\delta^2}(m-1)(e^{\frac{\delta B}{D(p)}} - \frac{\delta B}{D(p)} - 1) - c_4(m-1)(e^{\frac{\delta B}{D(p)}} - 1)$$
$$- \frac{c_4 D(p)}{\delta^2}(e^{\frac{\delta B}{D(p)}} - \frac{\delta B}{D(p)} - 1)] - A_b - wQ\Big]/$$
$$\Big[\frac{q}{D(p)} - \frac{\theta q^2}{2D(p)^2} + (m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q - B)}{D(p)}\} + (m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}} - 1)\Big] \quad (20)$$

Vendor's average profit

Suppose that the vendor's initial stock is $qD(p)/P$. Once the production run is started, the total stock increases at a rate of $(P-D(p))$ until the complete batch quantity of Q has been manufactured.

Therefore, the maximum inventory level in the supply chain system

$$= \quad q\frac{D(p)}{P} + Q\frac{P-D(p)}{P}$$

Hence, the vendor's average inventory is given by

$$q\frac{D(p)}{P} + Q\frac{P-D(p)}{P} - \Big[\Big\{\frac{q^2}{2D(p)} - \frac{\theta q^3}{3D(p)^2} + (m-1)\Big(\frac{(\alpha q - B)^2}{2D(p)} - \frac{\theta(\alpha q - B)^3}{3D(p)^2}\Big)\Big\}/$$
$$\Big\{\frac{q}{D(p)} - \frac{\theta q^2}{2D(p)^2} + (m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q - B)}{D(p)}\} + (m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}} - 1)\Big\}\Big]$$

The vendor's total cost which includes holding cost, production cost, set up cost and transportation cost is given by

$$\Big[q\frac{D(p)}{P} + Q\frac{P-D(p)}{P} -$$
$$[\{\frac{q^2}{2D(p)} - \frac{\theta q^3}{3(D(p))^2} + (m-1)\{\frac{(\alpha q - B)^2}{2D(p)} - \frac{\theta(\alpha q - B)^3}{3(D(p))^2}\}\}/$$

$$\{\frac{q}{D(p)}-\frac{\theta q^2}{2(D(p))^2}+\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)+(m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q-B)}{D(p)}\}$$
$$+(m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)\}\big]h_v+\big[(Qv+S_v)/\{\frac{q}{D(p)}-\frac{\theta q^2}{2(D(p))^2}$$
$$+\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)+(m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q-B)}{D(p)}\}$$
$$+(m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)\}\big]+mf$$

Hence the average profit of the vendor is given by

$$\mathrm{TPV}(q,m,p)$$
$$=\frac{(w-v)Q-S_v-mf}{\frac{q}{D(p)}-\frac{\theta q^2}{2(D(p))^2}+(m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q-B)}{D(p)}\}+(m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)}$$
$$-h_v q\frac{D(p)}{P}+Q\frac{P-D(p)}{P}h_v-[\{\frac{q^2}{2D(p)}-\frac{\theta q^3}{3(D(p))^2}+(m-1)\{\frac{(\alpha q-B)^2}{2D(p)}$$
$$-\frac{\theta(\alpha q-B)^3}{3(D(p))^2}\}\}h_v/\{\frac{q}{D(p)}-\frac{\theta q^2}{2(D(p))^2}+(m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q-B)}{D(p)}\}$$
$$+(m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)\}] \quad (21)$$

Since m is discrete, and p and q are continuous, it is not possible to optimize TPV with respect to m, p and q simultaneously. Therefore, firstly we consider m as a real number, not just an integer. Then, for a fixed positive values of p and q, taking the first and second order partial derivatives of $TPV(q,m,p)$ with respect to m, we get

$$\frac{\partial TPV(q,m,p)}{\partial m}=\frac{mG(q)+E(q)}{mH(q)+F(q)}-mh_v\alpha q\frac{P-D}{P}+I(q)$$

and

$$\frac{\partial^2 TPV(q,m,p)}{\partial m^2}=-\frac{2H(q,p)(G(q,p)F(q,p)-H(q,p)E(q,p))}{(mH(q,p)+F(q,p))^3}$$

where

$$E(q,p)=(w-v)q(1-\alpha)-s_v-h_v\Big[\frac{q^2}{2D(p)}-\frac{\theta q^3}{3(D(p))^2}-\frac{(\alpha q-B)^2}{2D(p)}+\frac{\theta(\alpha q-B)^3}{3(D(p))^2}\Big]$$

$$F(q,p)=\frac{q}{D(p)}-\frac{\theta q^2}{2D(p)^2}-\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)-\frac{(\alpha q-B)}{D(p)}+\frac{\theta(\alpha q-B)^2}{2(D(p))^2}$$

$$G(q,p)=(w-v)\alpha q-f-\{\frac{(\alpha q-B)^2}{2D(p)}-\frac{\theta(\alpha q-B)^3}{3(D(p))^2}\}h_v$$

$$H(q,p)=\frac{1}{\delta}(e^{\frac{\delta B}{D(p)}}-1)+\frac{(\alpha q-B)}{D(p)}-\frac{\theta(\alpha q-B)^2}{2(D(p))^2}$$

Theorem. For given values of p and q, the average profit function $TPV(q, m, p)$ attains the maximum value at

$$m=\frac{1}{H(q,p)}\left(\sqrt{\frac{P}{h_v\alpha q(P-D)}}-F(q,p)\right)$$

if $(G(q,p)F(q,p)-H(q,p)E(q,p))$ is positive.

Proof: The maximum profit per unit time for the vendor will occur at the point m which satisfies $\frac{\partial TPV(q,m,p)}{\partial m}=0.$

Hence, setting $\frac{\partial TPV(q,m,p)}{\partial m}$ equal to zero, we obtain

$$m=\frac{1}{H(q,p)}\left(\sqrt{\frac{P}{h_v\alpha q(P-D)}}-F(q,p)\right)$$

Now

$\frac{\partial^2 TPV(q,m,p)}{\partial m^2}<0$ if $G(q,p)F(q,p)-H(q,p)E(q,p)$ is positive.

Hence the theorem follows.

Since m is an integer, we choose $m*$, the optimum integer value of m such that it yields $TPV(q,m^*,p)=Max\{TPV(m-),TPV(m+)\}$ with regard to the fact that $TPV(q,m^*,p)$ is a concave function, where $m+$ and $m-$ represent the nearest integers larger and smaller than the optimal $m*$.

Supply chain's average profit

In this subsection, we consider the integrated vendor-buyer cooperative model to achieve the overall maximum total profit. Once the vendor and the buyer establish a long term strategic partnership, they can coordinate their production and inventory strategies and jointly determine the optimal policy for the integrated supply chain system.

The joint total profit per unit time for the vendor and the buyer is given by

$$JTP(q,m,p)=TPV(q,m,p)+TPB(q,m,p)$$
$$=\frac{(w-v)Q-S_v-mf}{\frac{q}{D}-\frac{\theta q^2}{2D^2}+(m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q-B)}{D(p)}\}+(m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D}}-1)}-$$
$$h_v q\frac{D(p)}{P}+Q\frac{P-D}{P}h_v-[\{\frac{q^2}{2D}-\frac{\theta q^3}{3D^2}+(m-1)\{\frac{(\alpha q-B)^2}{2D}-\frac{\theta(\alpha q-B)^3}{3D^2}\}\}h_v/$$
$$\{\frac{q}{D}-\frac{\theta q^2}{2D^2}+(m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q-B)}{D(p)}\}+(m-1)\frac{1}{\delta}(e^{\frac{\delta B}{D}}-1)\}]$$
$$+[p\{Q-\frac{\theta q^2}{2D}-(m-1)\theta\frac{(\alpha q-B)^2}{2D}\}-(h_b+\theta d_c)[\frac{q^2}{2D}-\frac{\theta q^3}{3D^2}+(m-1)\{\frac{(\alpha q-B)^2}{2D}-\frac{\theta(\alpha q-B)^3}{3D^2}]$$
$$-\frac{c_3D(p)}{\delta^2}(m-1)(e^{\frac{\delta B}{D(p)}}-\frac{\delta B}{D(p)}-1)-c_4(m-1)(e^{\frac{\delta B}{D(p)}}-1)-\frac{c_4D(p)}{\delta^2}(e^{\frac{\delta B}{D(p)}}-\frac{\delta B}{D(p)}-1)]-A_b-wQ]/$$
$$\{\frac{q}{D}-\frac{\theta q^2}{2D^2}+(m-1)\frac{1}{\theta}\log\{1+\frac{\theta(\alpha q-B)}{D(p)}\}+\frac{1}{\delta}(m-1)(e^{\frac{\delta B}{D}}-1)\} \quad (22)$$

Our objective is to determine the optimal number of shipments m^*, optimal shipment size q^* and optimal selling price p^* so that the average profit $JTP(m,q,p)$ is maximized. Due to complex form of $JTP(m,q,p)$, it is not possible to prove the concavity of the average profit function. We will therefore examine the concavity behavior of the profit function $JTP(m,q,p)$ numerically.

NUMERICAL EXAMPLE

To illustrate the proposed model numerically, an example is considered in this section. We determine the solution of the vendor-buyer problem, calculating the shipment size, selling price and the number of shipments from the vendor to the buyer. We consider the demand function as $D(p)=ap^{-b}$ where $a>0$ is a scaling factor and $b>1$ is the price elasticity coefficient, and the parameter-values for the supply chain system :
a = 10000, A_b = 100, b = 1.5, w = 8, A_b = 500, α = 1.5, f = 20,

P = 150000, v = 4, S_v = 150, h_v = 0.5, h_b = 1.2, d_c = 1.5, B = 20,θ = 0.02, δ = 0.95, c_3 = 2, c_4 = 3 in appropriate units. Figure 2 shows the concavity of the profit function *JTP(q,m,p)* for some given value of m.

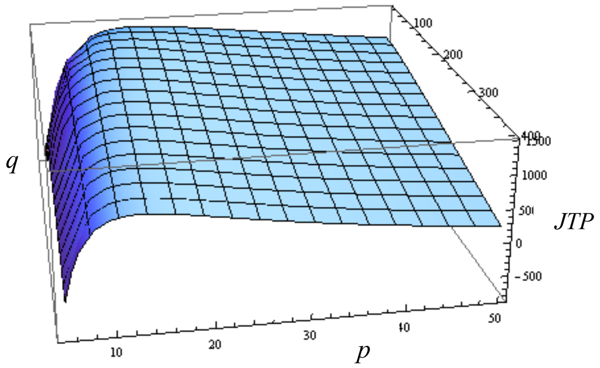

Fig. 2: Concavity of JTP with respect to p and q

The results obtained for different values of m are shown in Table 1. From Table 1, we see that the optimal number of shipments m is 6, the optimal value of q is 78.68 units, the optimal selling price p is 15.18 and the corresponding maximum joint total profit is 1475.97.

Table 1. Optimal results

m	*q**	*p**	*JTP(m,q*,p*)*
1	379.19	17.32	1346.61
2	195.04	16.16	1429.49
3	138.63	15.70	1458.84
4	109.37	15.45	1470.77
5	91.97	15.29	1475.27
6	78.68	15.18	1475.97
7	69.53	15.10	1474.79
8	62.53	15.04	1471.68
9	56.98	14.98	1468.01
10	52.48	14.94	1463.77

Let us now consider the situation for m = 1, i.e., lot for lot situation. If we consider the model from buyer's perspective, then the buyer's optimum order quantity is $Q_b^* = Q^*$ = 206.43 units, optimum selling price p^* = 31.28 units and the average profit is TPB^* = 1041.19 units. Substituting Q^* in average profit function of the vendor, we get TPV^* = 162.21 units. Then the average profit for the whole supply chain becomes TPB^*+TPV^* = 1203.40. This shows an increase in the average total profit of (1475.9-1203.40) = 272.5 units in the proposed supply chain. Similarly, if we think of the model from vendor's perspective then taking m = 1 and p = 31.28units (taken from buyer's decision) we get TPV^* = 257.197 and Q^* = 1662.41. Substituting Q^* in the average profit function of the buyer, we get TPB^* = 235.74. Then the average profit for the supply chain becomes 492.90. This shows an increase of profit of1475.9-492.90 = 983 units in the proposed supply chain. Hence the integrated approach appears superior to either the modeling approach from buyer's perspective or that from vendor's perspective. It is also observed that the average profit of the integrated supply chain in any cycle is closer to the average profit from the buyer's perspective than that of the vendor's perspective.

Sensitivity analysis

In this sub-section, we will analyze the effects of P, v, h_v, α, A_b, and δ on the joint total profit of the vendor and the buyer. We change the value of one parameter at a time and keep other parameter-values unchanged. The results are shown in Table 2.

Table 2 indicates that the optimal result of the integrated model is highly sensitive with respect to the parameter v and moderately sensitive with respect to the parameter h_v. As v increases, unit selling price of the buyer increases but *JTP* and q decrease, and as v decreases, p decreases but *JTP* and q increase. As the holding cost of the vendor increases, the joint total profit decreases and as the holding cost is lower, the profit inflates itself. We check that the optimal results are insensitive with respect to the remaining parameters P, A_b, δ, and α.

In this paper, we have considered an integrated vendor-buyer model with price-dependent demand where the vendor delivers the buyer's order quantity in a number of unequal shipments. . We have considered shortage in the buyer's inventory, which is partially backlogged. The proposed model can assist the manufacturer and the retailer in accurately determining the optimal order quantity, selling price charged by the buyer and the joint total profit of the supply chain. Most firms have no pricing power in today's business competition. As a result, they are not able to change price. In order to reflect this fact, the selling price is very important in this model in which both the vendor and the buyer can determine the selling price for best profit.

Table 2: Sensitivity analysis

Parameter	**% change in parameter-value**	***p****	***q****	**% change in *JTP***
P	50	14.84	85.47	- 0.005
	20	14.84	85.47	0.002
	10	14.80	85.48	0.001
	-10	14.84	85.49	- 0.001
	-20	14.84	85.50	- 0.0035
	-50	14.84	85.54	- 0.01
v	50	20.98	70.15	- 18.14
	20	17.37	77.87	- 8.44
	10	16.12	81.35	- 4.48
	-10	13.54	90.48	5.11
	-20	12.21	96.64	11.05
	-50	8.04	127.75	37.29

h_v	50 20 10 -10 -20 -50	15.50 15.12 14.98 14.70 14.54 14.03	68.65 77.57 81.28 90.32 95.95 120.8	- 6.67 - 2.85 - 1.46 1.54 3.16 8.78
A_b	50 20 10 -10 -20 -50	14.96 14.89 14.87 14.82 14.79 14.72	86.89 86.05 85.77 85.20 84.90 84.02	- 0.68 - 0.28 - 0.14 0.14 0.28 0.71
α	50 20 10 -10 -20 -50	14.87 14.82 14.79 14.72 14.65 14.76	85.77 85.20 84.90 84.02 87.32 86.23	-0.03 -0.009 -0.005 0.004 0.005 -0.04
δ	50 20 10 -10 -20 -50	14.65 14.76 14.80 14.88 14.92 15.05	87.24 86.23 85.86 85.11 84.73 83.57	- 0.55 - 0.22 - 0.1 0.11 0.22 0.56

CONCLUSION

In this paper, we have developed a vendor-buyer integrated supply chain model with price dependent demand for deteriorating items allowing shortages in the buyer's inventory, which are partially backordered. The vendor supplies the buyer's order quantity in a number of unequal sized shipments. The proposed model is solved by maximizing the average total profit of the supply chain. Finally, the model has been demonstrated by a numerical example and performance of the integrated approach has been established. In our study, due to complexity we are unable to derive exact formulae for optimal values of the decision variables. The proposed model can further be enriched by taking more realistic assumptions such as stochastic demand, trade credit offer by the vendor and the buyer, a combination of equal and unequal shipments from the vendor, etc.

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