International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 3 (2019) pp. 600-603

On Upper and Lower Contra-I-Continuous Multifunctions

R. Balaji[(1)], R. Saranya[(2)] and N. Rajesh[(3)]

[(1)]*Department of Mathematics, Agni College of Technology, Chennai-600130, Tamilnadu, India.*

[(2)] *1/82, South Street, Pillaiyarnatham, Thnangudi, Thanjavur 613 601, Tamilnadu, India.*

[(3)]*Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur-613005, Tamilnadu, India.*

Abstract

In this paper, we introduce and study the concept of contra-I-continuous multifunctions on ideal topological spaces.

Keywords and phrases: Topological spaces, I-open set, contra-I-continuous multifunctions.

2000 Mathematics Subject Classification: 54C10, 54C60.

1. INTRODUCTION

One of the important and basic topics in the theory of classical point set topology and several branches of Mathematics, which have been researched by many authors, is continuity of functions. Various types of continuous functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunction. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathaswamy, [11]. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [11] of A with respect to τ and I, is defined as follows: for $A\subset X$, $A^\star(\tau, I) = \{x \in X | U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \,|x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau, I)$ called the *-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, I)$ when there is no chance of confusion, $A^\star(I)$ is denoted by $A^\star$. If I is an ideal on X, then

(X, τ, I) is called an ideal topological space. Akdag [2] introduced and studied the concept of I-continuous multifunctions on ideal topological spaces. In this paper, we define contra-I-continuous multifunctions and obtain some characterizations and some basic properties of such multifunctions.

2. PRELIMINARIES

Let A be a subset of a topological space (X, τ). For a subset A of (X, τ), Cl(A) and Int(A) denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space (X, τ, I) is I-open [1] if $S \subset \mathrm{Int}(S^*)$. The complement of an I-open set is called an I-closed set. The intersection of all I-closed sets containing S is called the I-closure of S and is denoted by ICl(S). The family of all I-open (resp. I-closed) sets of (X, τ, I) is denoted by IO(X) (resp. IC(X)). The family of all I-open (resp. I-closed) sets of (X, τ, I) containing a point $x \in X$ is denoted by IO(X, x) (resp. IC(X, x)). By a multifunction $F : X \to Y$, following [3], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(Y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A\subset X$, $F(A) = \cup_{x\in A} F(x)$. Then F is said to be surjection if $F(x) = y$ and injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Definition 2.1. A multifunction $F : (X, \tau, I) \to (Y,\sigma)$ is said to be[2]:

(1) upper I-continuous if for each point $x \in X$ and each open set V containing F(x), there exists $U \in IO(X, x)$ such that $F(U) \subset V$;

(2) lower I-continuous if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in IO(X, x)$ such that $U\subset F^-(V)$.

3. ON UPPER AND LOWER CONTRA-I-CONTINUOUS MULTIFUNCTIONS

Definition 3.1. A multifunction $F : (X, \tau, I) \to (Y,\sigma)$ is said to be:

(1) upper contra-I-continuous if for each point $x \in X$ and each closed set V containing F(x), there exists $U \in IO(X, x)$ such that $F(U) \subset V$;
(2) lower contra-I-continuous if for each point $x \in X$ and each closed set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in IO(X, x)$ such that $U \subset F^-(V)$.

The following examples show that the concepts of upper I-continuity (resp. lower I-continuity) and upper contra-I -continuity (resp. lower contra-I-continuity) are independent of each other.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$, $\sigma = \{\emptyset, \{b, c\}, X\}$ and $I = \{\emptyset\}$. The multifunction

F : (X, τ, I)→ (X, σ) defined by F(x) ={x} for all x ∈ X is upper I-continuous but is not upper contra-I-continuous.

Example 3.3. Let X ={a, b, c}, τ = {∅, X}, σ ={∅, {a}, X} and I = {∅}. The multifunction

F : (X, τ, I)→ (X, σ) defined by F(x) ={x} for all x ∈ X is upper contra-I-continuous but is not upper I -continuous.

Theorem 3.4. The following statements are equivalent for a multifunction

F : (X, τ, I) → (Y,σ) :

(1) F is upper contra-I-continuous;
(2) F^+(V) ∈ IO(X) for every closed subset V of Y ;
(3) F^-(V) ∈ IC(X) for every open subset V of Y ;
(4) for each x ∈ X and each closed set K containing F(x), there exists U ∈ IO(X, x) such that if y ∈ U, then F(y) ⊂ K.

Proof. (1) ⟺ (2): Let V be a closed subset in Y and x ∈ F^+(V). Since F is upper contra-I-continuous, there exists U ∈ IO(X, x) such that F(U) ⊂ V. Hence, F^+(V) is I-open in X. The converse is similar.

(2) ⟺ (3): It follows from the fact that F^+(Y \V) = X\ F^-(V) for every subset V of Y .

(3) ⟺ (4): This is obvious.

Theorem 3.5. The following statements are equivalent for a multifunction

F : (X, τ, I) → (Y,σ)

(1) F is lower contra-I-continuous;
(2) F^-(V) ∈ IO(X) for every closed subset V of Y ;
(3) F^+(K) ∈ IC(X) for every open subset K of Y ;
(4) for each x ∈ X and each closed set K such that F(x) ∩K ≠ ∅, there exists U ∈ IO(X, x) such that if y ∈ U, then F(y) ⊂ K ≠ ∅.

Proof. The proof is similar to that of Theorem 3.4.

Definition 3.6. A topological space (X,τ) is said to be semi-regular [10] if for each open set U of X and for each point x ∈ U, there exists a regular open set V such that x ∈ V ⊂ U.

Definition 3.7. [12] Let (X,τ) be a topological space and A a subset of X and x a point of X. Then

(1) x is called δ-cluster point of A if A ∩ Int(Cl(U)) ≠ ∅, for each open set U containing x.
(2) the family of all δ-cluster point of A is called the δ-closure of A and is denoted by Cl δ (A).
(3) A is said to be δ-closed if Cl δ (A) = A. The complement of a δ-closed set is said to be a δ-open set.

Theorem 3.8. For a multifunction F : (X, τ, I) → (Y,σ), where Y is semi-regular, the following are equivalent:

(1) F is upper contra-I-continuous;
(2) F^+(Cl δ (B)) ∈ IO(X) for every subset B of Y ;
(3) F^+(K) ∈ IO(X) for every δ-closed subset K of Y ;
(4) F^-(V) ∈ IC(X) for every δ-open subset V of Y .

Proof. (1) ⟹ (2): Let B be any subset of Y . Then Cl δ (B) is closed and by Theorem 3.4, F^+(Cl δ (B)) ∈ IO(X).

(2) ⟹ (3): Let K be a δ-closed set of Y . Then Cl δ (K) = K. By (2), F^+(K) is I-open.

(3) ⟹ (4): Let V be a δ-open set of Y . Then Y \V is δ-closed. By (3),

F^+(Y \V) = X\ F^-(V) is I-open. Hence, F^- (V) is I-closed.

(4) ⟹ (1): Let V be any open set of Y . Since Y is semi-regular, V is δ-open. By (4), F^-(V) is

I-closed and by Theorem 3.4, F is upper contra-I-continuous.

Theorem 3.9. For a multifunction F : (X, τ, I) → (Y,σ) , where Y is semi-regular, the following are equivalent:

(1) F is lower contra-I-continuous;
(2) F^- (Cl δ (B)) ∈ IO(X) for every subset B of Y ;
(3) F^- (K) ∈ IO(X) for every δ-closed subset K of Y ;
(4) F^+(V) ∈ IC(X) for every δ-open subset V of Y .

Proof. The proof is similar to that of Theorem 3.8.

Definition 3.10. [8] A subset A of an ideal topological space (X, τ, I) is said to be I-compact relative to X if for every cover {Uα : α ∈ △} of A by open subsets of X, there exists a finite subset △0 of △ such that A\ ∪ {Uα : α ∈ △0} ∈ I.

Definition 3.11. A subset A of an ideal topological space (X, τ, I) is said to be I-closed relative to X if for every cover {Uα: α ∈ △} of A by closed subsets of X, there exists a finite subset △0 of △ such that A\ ∪ {Uα : α ∈ △0} ∈ I.

Theorem 3.12. Let F : (X, τ, I) → (Y,σ) be an upper contra-I-continuous surjective multifunction and F(x) is strongly S-closed relative to Y for each x ∈ X. If A is I-compact relative to X, then F(A) is F(I)-closed relative to Y .

Proof. Let {V_i : i ∈ △} be any cover of F(A) by closed sets of Y. For each x ∈ A, there exists a finite subset △(x) of △ such that F(x) ⊂∪ {V_i : i ∈ △ (x)}. Put V (x) = ∪ {V_i : i ∈ △ (x)}. Then F(x) ⊂ V (x) and there exists U(x) ∈ IO(X, x) such that F(U(x)) ⊂ V (x). Since

{U(x) : x ∈ A} is a cover of A by I-open sets in X, there exists a finite number of points of A, say, x_1, x_2,....x_n such that A\∪ {U(x_i) : 1 = 1, 2,….n} ∈ I. Therefore, we obtain F(A)\ $\bigcup_{i=1}^{n} \bigcup_{i=\Delta(xi)}$ Vi ∈ F(I). This shows that F(A) is I-closed relative to Y .

Corollary 3.13. Let F : (X, τ, I) → (Y,σ) be an upper contra-I-continuous surjective multifunction and F(x) is I-compact relative to Y for each x ∈ X. If X is I-compact, then Y is F(I)-closed.

Definition 3.14. [4] Let A and B be subsets of an ideal topological space (X, τ, I) such that A ⊂ B ⊂ X. Then (B, τ|B, I |B) is an ideal topological space with an ideal I |B = {I ∈ I |I ⊂ B} = {I ∩ B|I ∈ I }.

Lemma 3.15. Let A and B be subsets of an ideal topological space (X, τ, I).

(1) If A ∈ IO(X) and B ∈ τ , then A ∩ B ∈ IO(B) [5];
(2) If A ∈ IO(B) and B ∈ I O(X), then A ∈ I O(X) [7].

Theorem 3.16. Let F : (X, τ, I) → (Y,σ) be a multifunction and U an open subset of X. If F is a upper contra-I-continuous (resp. lower contra-I-continuous), then $F_{|U}$: U → Y is an upper contra-I-continuous (resp. lower contra-I-continuous) multifunction.

Proof. Let V be any closed set of Y . Let x ∈ U and x ∈ $F^{-}_{|U}$ (V). Since F is lower contra-I-continuous multifunction, there exists a I-open set G containing x such that G ⊂ F^{-} (V). Then x ∈ G∩ U ∈ I O(A) and G ∩ U ⊂ $F^{-}_{|U}$ (V) . This shows that $F_{|U}$ is a lower contra-I-continuous. The proof of the upper contra-I-continuous of $F_{|U}$ is similar.

Theorem 3.17. Let {U_i : i ∈ △} be an open cover of a topological space X. A multifunction

F : (X, τ, I) → (Y,σ) is upper contra-I-continuous if and only if the restriction $F_{|U}$: U_i → Y is upper contra-I-continuous for each i ∈ △.

Proof. Suppose that F is upper contra-I-continuous. Let i ∈ △ and x ∈ U_i and V be a closed set of Y containing $F_{|Ui}(x)$. Since F is upper contra-I-continuous and F(x) = $F_{|Ui}(x)$, there exists G ∈ I O(X, x) such that F(G) ⊂ V. Set U = G ∩ U_i, then x ∈ U ∈ IO(U_i, x) and $F_{|Ui}$ (U) = F(U) ⊂ V. Therefore, $F_{|Ui}$ is upper contra-I-continuous.

Conversely, let x ∈ X and V be a closed subset of Y containing F(x).

There exists i ∈ △ such that x ∈ U_i. Since $F_{|Ui}$ is upper contra-I- continuous and F(x) = $F_{|Ui}$ (x), there exists U ∈ I O(U_i, x) such that $F_{|Ui}$ (U) ⊂ V. Then we have U ∈ IO(X, x) and F(U) ⊂ V. Therefore, F is upper contra-I-continuous.

Theorem 3.18. Let X and X_j be topological spaces for i ∈ I. If a multifunction F : X → $\coprod_{i\in I} X_i$ is an upper (lower) contra-I-continuous multifunction, then $P_i o F$ is an upper (lower) contra-I-continuous multifunction for each i ∈ I, where P_i : $\coprod_{i\in I} X_i \to X_i$ is the projection for each i ∈ I.

Proof. Let H_i be a closed subset of X_i. We have $(P_i\ o\ F)^+(H_i)$ $=F^+(P_i{}^+(H_i)) = F^+(H_i \times \coprod_{i\neq j} X_j)$. Since F an upper contra-I-continuous multifunction, $F^+(H_i \times \coprod_{i\neq j} X_j)$ is I-open in X. Hence,

P_i *o* F is an upper (lower) contra-I-continuous.

Definition 3.19. [9] A topological space (X, τ) is said to be ultranormal if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets.

Definition 3.20. [7] An ideal topological space (X, τ, I) is said to be I-normal if each pair of nonempty disjoint closed sets can be separated by disjoint I-open sets.

Theorem 3.21. If F : (X, τ, I) → (Y,σ) is an upper contra-I-continuous punctually closed multifunction and (Y, σ) is ultranormal, then (X, τ, I) is I-normal.

Proof. The proof follows from the respective definitions.

Definition 3.22. Let A be a subset of an ideal topological space (X, τ, I). The I-frontier of A denoted by IFr(A), is defined as follows: IFr(A) =I Cl(A) ∩ I Cl(X\A).

Theorem 3.23. The set of points x of X at which a multifunction F : (X, τ, I) → (Y,σ) is not upper contra-I-continuous (resp. upper contra-I-continuous) is identical with the union of the I-frontiers of the upper (resp. lower) inverse images of closed sets containing (resp.meeting) F(x).

Proof. Let x be a point of X at which F is not upper contra-I-continuous. Then there exists a closed set V of Y containing F(x) such that U∩ (X\$F^+$(V)) ≠ ∅ for each U ∈ IO(X, x). Then x ∈ I Cl(X\$F^+$(V)). Since x ∈ F^+(V), we have x ∈ I Cl(F^+(Y)) and hence x ∈ IFr(F^+(A)). Conversely, let V be any closed set of Y containing F(x) and x ∈ IFr(F^+(V)). Now, assume that F is upper contra-I-continuous at x, then there exists U ∈ IO(X, x) such that F(U) ⊂ V . Therefore, we obtain x ∈ U ⊂ I Int(F^+(V)). This contradicts that x ∈ IFr(F^+(V)).

Thus, F is not upper contra-I-continuous. The proof of the second case is similar.

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