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Two Heterogeneous Servers Queueing-Inventory System with Sharing Finite Buffer and a Flexible Server

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Abstract

In this paper, we analyze a continuous review inventory (s, Q) finite buffer queuing model with two heterogeneous servers and mixed priority service. The courses of arrival time and service time of both the customers are all independent exponential processes. One of the two servers is engaged exclusively for high priority customer and the other server serves both the customers along with a mixed priority service scheme. Applying matrix method to obtain the steady state joint distributions of waiting time of mixed priority service customers and the inventory level. Further, the some important measures of system performances in the steady state are derived.

Keywords: (s, Q) model, continuous review inventory, finite buffer, mixed priority service, priority customers, heterogeneous servers.

1. Introduction

Suppose that the cost of customers lost is a significant component of the total expected cost rate. In that situation the customers demand can be fulfilled with increasing the number of servers in the system that will reduce the cost rate further. In these aspects many authors were incorporated in their papers by assuming multiple servers. Multiserver Queueing-Inventory System with infinite orbit size were investigated by Paul Manuel et.al. [1], they studied a continuous review perishable inventory queuing system with multi-server facility in which ordering policy called (s, Q) where s is the reorder level when the order Q is placed. But they assumed that the lead time of the reorders is a phase type distribution. Under the steady state conditions, they obtained the joint probability distribution of the number of busy servers and inventory level. They also calculate the various measures of stationary system performance and the total expected cost rate. Yadavalli et.al. [2] in which Markovian arrival process of customers and exponential distribution of time interval of successive attempts of retrial customers were considered. Yadavalli et al. [3], they analyzed the finite source of retrial customers demanding the inventory with muti homogeneous type of servers. The reader is referred the more details about the finite source inventory system from Sivakumar [4], Shophia Lawrence et al.[5].

All the past papers mentioned, to relate with multi-server facility in which the servers are homogeneous type. That is, the service rate of each server is same and the distribution of service time of each server is identical. But in the case of human servers, the service rate differs from person to person. So therefore we need to develop the multi-server model with heterogeneous type. Yadavalli et al. [6], they studied two heterogeneous servers on a perishable inventory continuous system of a finite population with one unreliable server. Yadavalli and Jeganathan [7], studied the same with one server having multiple vacations instead of unreliable server and also provide the finite waiting capacity hall in addition to the retrial orbit. Recently, jeganathan et.al. [8] studied Two server Markovian inventory systems with server interruptions: Heterogeneous vs. homogeneous servers.

In this paper, we have extended the work of Jeganathan et.al. [9] by two heterogeneous servers(say sever 1 and server 2), in which one of the two servers is dedicatedly served for high priority customers and the other server serves the both types customers (flexible server) according to a mixed priority basis. The major difference between the recent article to this is that they deals with a single server with two classes of customers on a perishable inventory queuing system with fharing finite buffer whereas in this paper we model the system using two heterogeneous servers including one with flexible server with a finite sharing buffer capacity. The main objective of using the two heterogeneous servers with the mixed priority policy is to minimize the total long run expected cost of losing customers of the system.

In the next section we describe the mathematical model in details. Further sections, we study the stability conditions of the mixed priority scheduling of the two types customers in the finite buffer of queuing perishable inventory system, and derive the joint steady state distributions and compute the different measures of the system.

2. Model Description

In this investigation, we consider a finite buffer Markovian inventory system with the following assumptions:
We rely on the finite buffer queueing model consisting of two servers: dedicated server and flexible server and sharing the buffer with two classes of customers: high priority customers and low priority customers. The dedicated server is served absolutely for high priority customer and the flexible server is served both customers. The capacity of the finite buffer is assumed to be N. let N_1 and N_2 are number of high and low priority customers in the buffer whose sum is $N_1 + N_2 \leq N$. The arrival processes of high and low priority customers are assumed to be independent Poisson processes with rates λ_1 and λ_2 , respectively. Let $\lambda = \lambda_1 + \lambda_2$ be the total arrival rate of the system. The servicing time of any class of customer follows a exponential distribution. The commencement of the service of high priority customers is after selecting the purchased item from the inventory and the low priority customers is only arrived for the repaired work. Let μ_1 be the rate of service of dedicated server, μ_2 and μ_3 are rates of service of flexible server respectively for high priority and low priority customers. Assumed that any item is perishable with exponential rate γ but the items in the service not perishable so that the items are perishable with the rate γj where j denotes the number items in the stock and not in the service. The flexible sever takes multiple vacations in the case of either at most one inventory in the system or no high priority customer in the buffer, and no low priority customers in the buffer. The occurrence of the vacation follows the exponential distribution with the rate θ. The ordering policy of the inventory is (s, Q), where Q is the ordering quantity and s is the re order point. The lead time of the order is assumed to be exponential distribution with the rate β.The service policy of both the servers are described as follows:

1. At an arrival instant of a high priority customer, when both servers are busy with high priority customers or the dedicated server busy and the flexible server is on vacation, then the arriving customer joins the queue behind all the high priority customers and in front of all the low priority customers. After finishing the service of any server, he can receive the service by FCFS basis.
2. At an arrival instant of a high priority customer, if both servers are busy but the flexible server busy with low priority customer and the inventory is available for the arriving customer then he can demand the service with the probability p and the rate of demand is $p\lambda_1$. Otherwise, the arriving customer joins the queue behind all the high priority customers and in front of all the low priority customers in the finite buffer with the rate $q\lambda_1$. This is known as mixed priority service policy.
3. At an arrival instant of a low priority customer, if flexible server is either busy or vacation, then the arriving customer joins the queue behind all the high priority customers; After finishing the vacation of the server, he can receive the service immediately by FCFS when either there is no high priority customers in the buffer or no inventory in the stock.
4. A low priority customer is lost when he finds all waiting places are occupied at the arriving instant. Similarly, a high priority customer is lost when he finds all waiting places are occupied and the two servers are busy with high priority customers at the arriving instant.

3. Mathematical Analysis of the Model

Let $L(t) \in \{0, 1, 2, \ldots, S\}$, $Y(t) \in \{0, 1, 2, 3, 4, 5\}$, $X_1(t) \in \{0, 1, 2, \ldots, N\}$ and $X_2(t) \in \{0, 1, 2, \ldots, N\}$ indicate the inventory level, server status, the number of low priority coustomers in the queue at time t and the number of high priority customers in the queue at time t. The server status of the system can be defined as follows,

$$Y(t) = \begin{cases} 0, & \text{if the server 1 is idle and server 2 is on vacation at time t,} \\ 1, & \text{if the server 1 is idle and server 2 is busy with low priority of coustomer at time t,} \\ 2, & \text{if the server 1 is idle and server 2 is busy with high priority of coustomer at time t,} \\ 3, & \text{if the server 1 is busy sever 2 is on vacation at time t,} \\ 4, & \text{if the server 1 is busy and server 2 is busy with low priority of coustomer at time t,} \\ 5, & \text{if the server 1 is busy and server 2 is busy with high priority of coustomer at time t,} \end{cases}$$

Then, the activities of the system can be expressed by a four-dimensional stochastic process
$\{X(t) = (L(t), Y(t), X_1(t)), X_2(t), t \geq 0\}$, with the finite discrete state space E, where

$$
\begin{aligned}
E \;=\; & \{i_1 = 0,\ i_2 = 0,\ 0 \leq i_3 \leq N,\ 0 \leq i_4 \leq N - i_3\} \cup \\
& \{1 \leq i_1 \leq S,\ i_2 = 0,\ 0 \leq i_3 \leq N, i_4 = 0\} \cup \\
& \{i_1 = 1,\ i_2 = 2,\ 0 \leq i_3 \leq N,\ 0 \leq i_4 \leq N - i_3\} \cup \\
& \{2 \leq i_1 \leq S,\ i_2 = 2,\ 0 \leq i_3 \leq N,\ i_4 = 0\} \cup \\
& \{i_1 = 0,\ i_2 = 1,\ 0 \leq i_3 \leq N,\ 0 \leq i_4 \leq N - i_3\} \cup \\
& \{1 \leq i_1 \leq S,\ i_2 = 1,\ 0 \leq i_3 \leq N,\ i_4 = 0\} \cup \\
& \{1 \leq i_1 \leq S,\ i_2 = 3,\ 0 \leq i_3 \leq N,\ 0 \leq i_4 \leq N - i_3\} \cup \\
& \{2 \leq i_1 \leq S,\ i_2 = 4,\ 0 \leq i_3 \leq N,\ 0 \leq i_4 \leq N - i_3\} \cup \\
& \{1 \leq i_1 \leq S,\ i_2 = 5,\ 0 \leq i_3 \leq N,\ 0 \leq i_4 \leq N - i_3\}
\end{aligned}
$$

The infinitesimal generator of the Markov chain $\{X(t) = (L(t), Y(t), X_1(t)), X_2(t), t \geq 0\}$ is given by:

$$
\Theta = \begin{array}{c} \\ (\mathbf{S}) \\ (\mathbf{S-1}) \\ \vdots \\ (\mathbf{Q+1}) \\ \vdots \\ (\mathbf{s+1}) \\ (\mathbf{s}) \\ (\mathbf{s-1}) \\ \vdots \\ (\mathbf{2}) \\ (\mathbf{1}) \\ (\mathbf{0}) \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} (\mathbf{S}) & (\mathbf{S-1}) & \dots & \dots & (\mathbf{Q}) & \dots & (\mathbf{s+1}) & (\mathbf{s}) & (\mathbf{s-1}) & \dots & (\mathbf{2}) & (\mathbf{1}) & (\mathbf{0}) \end{array} \\
\left(\begin{array}{ccccccccccccc}
A_S & B_S & \mathbf{0} & \cdots & & & & & & & & & \\
\mathbf{0} & A_{S-1} & B_{S-1} & & & & & & & & & & \\
\vdots & & \cdots & \ddots & & & & & & & & & \\
 & & & A_{Q+1} & B_{Q+1} & & & & & & & & \\
 & & & & \cdots & \ddots & & & & & & & \\
\mathbf{0} & \cdots & & & & A_{s+1} & B_{s+1} & & & & & & \\
C & & & & & & & A_s & B_s & & & & \\
\mathbf{0} & C & & & & & & & A_{s-1} & B_{s-1} & & & \\
 & & & & & & & & & \cdots & \ddots & & \\
\vdots & & C & & & & & & & & A_2 & B_2 & \\
 & & & C_1 & & & & & & & & A_1 & B_1 \\
\mathbf{0} & \cdots & & \mathbf{0} & C_0 & \mathbf{0} & \cdots & & & & & & A_0
\end{array}\right)
\end{array},
$$

where

- C_0 denotes the transition from (0) to (Q) and its dimension is $(N+1)(N+2) \times \frac{(N+1)(3N+12)}{2}$
- C_1 denotes the transition from (1) to (Q+1) and its dimension $\frac{(N+1)(3N+10)}{2} \times \frac{(N+1)(3N+12)}{2}$
- C denotes the transition from (i_1) to $(Q + i_1)$, $(i_1 = 2, 3, \dots S)$ and its dimension $\frac{(N+1)(3N+12)}{2} \times \frac{(N+1)(3N+12)}{2}$
- B_1 denotes the transition from (1) to (0) and its dimension $\frac{(N+1)(3N+10)}{2} \times (N+1)(N+2)$
- B_2 denotes the transition from (2) to (1) and its dimension $\frac{(N+1)(3N+12)}{2} \times \frac{(N+1)(3N+10)}{2}$
- B_{i_1} denotes the transition from (i_1) to (i_1), $(i_1 = 3, 4, \dots S)$ and its dimension $\frac{(N+1)(3N+12)}{2} \times \frac{(N+1)(3N+12)}{2}$
- A_0 denotes the transition from (0) to (0) and its dimension $(N+1)(N+2) \times (N+1)(N+2)$
- A_1 denotes the transition from (1) to (1) and its dimension $\frac{(N+1)(3N+10)}{2} \times \frac{(N+1)(3N+10)}{2}$
- A_{i_1} denotes the transition from (i_1) to (i_1), $(i_1 = 2, 3, 4, \dots S$ and its dimension $\frac{(N+1)(3N+12)}{2} \times \frac{(N+1)(3N+12)}{2}$

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3.1 Steady-state Analysis

It can be noticed from the formation of Θ that the homogeneous Markov process $\{X(t) = (L(t), Y(t), X_1(t)), X_2(t), t \geq 0\}$ on the finite discrete state space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\pi^{(i_1,i_2,i_3,i_4)} = \lim_{t\to\infty} Pr[L(t) = i_1, Y(t) = i_2, X_1(t) = i_3, X_2(t) = i_4 | L(0), Y(0), X_1(0), X_2(0)]$$

exists and is independent of the initial state.

$$\mathbf{\Pi} = (\pi^{(\mathbf{0})}, \pi^{(\mathbf{1})}, \ldots, \pi^{(\mathbf{S})})$$

satisfies

$$\mathbf{\Pi}\Theta = \mathbf{0} \text{ and} \tag{3.1}$$

$$\sum\sum\sum\sum_{(i_1,i_2,i_3,i_4)} \pi^{(i_1,i_2,i_3,i_4)} = 1 \tag{3.2}$$

The equation $\mathbf{\Pi}\Theta = \mathbf{0}$ yields the following set of equations:

$$\boldsymbol{\pi}^{(\boldsymbol{i_1})} A_{i_1} + \boldsymbol{\pi}^{(\boldsymbol{i_1+1})} B_{i_1+1} = \mathbf{0}, \qquad i_1 = 0, 1, 2, \ldots, Q-1, \tag{3.3}$$

$$\boldsymbol{\pi}^{(\boldsymbol{i_1})} A_{i_1} + \boldsymbol{\pi}^{(\boldsymbol{i_1+1})} B_{i_1+1} + \boldsymbol{\pi}^{(\boldsymbol{i_1-Q})} C_{(i_1-(Q))} = \mathbf{0}, \qquad i_1 = Q, Q+1, \tag{3.4}$$

$$\boldsymbol{\pi}^{(\boldsymbol{i_1})} A_{i_1} + \boldsymbol{\pi}^{(\boldsymbol{i_1+1})} B_{i_1+1} + \boldsymbol{\pi}^{(\boldsymbol{i_1-Q})} C = \mathbf{0}, \qquad i_1 = Q+2, \ldots, S-1, \tag{3.5}$$

$$\boldsymbol{\pi}^{(\boldsymbol{i_1})} A_{i_1} + \boldsymbol{\pi}^{(\boldsymbol{i_1-Q})} C = \mathbf{0} \qquad i_1 = S \tag{3.6}$$

After lengthy simplifications, the above equations, except $\pi^{(Q)} A_Q + \pi^{(Q+1)} B_{Q+1} + \pi^{(0)} C_1 = 0$, yields

$$\begin{aligned}
\Omega_{i_1} &= (-1)^{Q-i_1} \mathop{\Omega}_{j=Q}^{i_1+1} B_j A_{j-1}^{-1}, \qquad i_1 = Q-1, Q-2, \ldots, 0 \\
&= (-1)^Q I, \ i_1 = Q, \\
&= (-1)^Q \left[\left(\mathop{\Omega}_{k=Q}^{2} B_k A_{k-1}^{-1} \right) C_1 A_{i_1}^{-1} + \right. \\
&\quad \left. \sum_{j=0}^{s-2} \left[\left\{ \left(\mathop{\Omega}_{k=Q}^{(s+1)-j} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \right\} \left\{ \left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1} \right) \right\} \right] \right], i_1 = Q+1 \\
&= (-1)^{2Q-i_1+1} \sum_{j=0}^{S-i_1} \left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1} \right) \right], \\
&\qquad i_1 = S, S-1, \ldots, Q+2
\end{aligned}$$

where $\boldsymbol{\pi}^{(\boldsymbol{Q})}$ can be obtained by solving the following equations:

$$\boldsymbol{\pi}^{(\boldsymbol{Q})} \left[(-1)^Q \left(\mathop{\Omega}_{k=Q}^{2} B_k A_{k-1}^{-1} \right) C_1 A_{Q_1+1}^{-1} + \sum_{j=0}^{s-2} \left[\left\{ \left(\mathop{\Omega}_{k=Q}^{(s+1)-j} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \right\} \left\{ \left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1} \right) \right\} \right] B_{Q+1} \right.$$
$$\left. + A_Q + \left\{ (-1)^Q \mathop{\Omega}_{j=Q}^{1} B_j A_{j-1}^{-1} \right\} C_0 \right] = \mathbf{0}, \tag{3.7}$$

and

$$\boldsymbol{\pi}^{(Q)}\left[\sum_{i_1=0}^{Q-1}\left((-1)^{Q-i_1}\underset{j=Q}{\overset{i_1+1}{\Omega}}B_jA_{j-1}^{-1}\right)+I+\right.$$
$$(-1)^{Q}\left[\left(\underset{k=Q}{\overset{2}{\Omega}}B_kA_{k-1}^{-1}\right)C_1A_{i_1}^{-1}+\sum_{j=0}^{s-2}\left[\left\{\left(\underset{k=Q}{\overset{(s+1)-j}{\Omega}}B_kA_{k-1}^{-1}\right)CA_{S-j}^{-1}\right\}\left\{\left(\underset{l=S-j}{\overset{i_1+1}{\Omega}}B_lA_{l-1}^{-1}\right)\right\}\right]\right] \tag{3.8}$$
$$\left.+\sum_{i_1=Q+1}^{S}\left((-1)^{2Q-i_1+1}\sum_{j=0}^{S-i_1}\left[\left(\underset{k=Q}{\overset{s+1-j}{\Omega}}B_kA_{k-1}^{-1}\right)CA_{S-j}^{-1}\left(\underset{l=S-j}{\overset{i_1+1}{\Omega}}B_lA_{l-1}^{-1}\right)\right]\right)\right]\mathbf{e} \;=\; 1.$$

Using $\boldsymbol{\pi}^{(Q)}$ and $\Omega_{i_1}, i_1 = 0, 1, \ldots, S$ can calculate the value of $\boldsymbol{\pi}^{(i_1)}$. That is,

$$\boldsymbol{\pi}^{(i_1)} \;=\; \pi^{(Q)}\Omega_{i_1}, \quad i_1 = 0, 1, \ldots, S. \tag{3.9}$$

4. System performance measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

4.1 Expected Inventory Level η_I:

$$\eta_I = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_2=0}^{1} i_1\pi^{(i_1,i_2,i_3,0)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=0}^{N-i_3}\sum_{i_2=3}^{4} i_1\pi^{(i_1,i_2,i_3,i_4)} + \sum_{i_3=0}^{N}\sum_{i_4=0}^{N-i_3} i_1\pi^{(1,2,i_3,i_4)}$$
$$+\sum_{i_1=2}^{S}\sum_{i_3=0}^{N}[i_1\pi^{(i_1,2,i_3,0)} + \sum_{i_4=0}^{N-i_3} i_1\pi^{(i_1,5,i_3,i_4)}]$$

4.2 Expected Reorder Rate η_R:

$$\eta_R = \sum_{i_3=0}^{N}\sum_{i_2=0}^{1}(s+1)\gamma\pi^{(s+1,i_2,i_3,0)} + \sum_{i_3=0}^{N}\sum_{i_4=0}^{N-i_3}\sum_{i_2=3}^{4}(s\gamma+\mu_1)\pi^{(s+1,i_2,i_3,i_4)}$$
$$+\sum_{i_3=0}^{N}(s\gamma+\mu_2)\pi^{(s+1,i_2,i_3,0)} + \sum_{i_3=0}^{N}\sum_{i_4=0}^{N-i3}[(s-1)\gamma+\mu_1+\mu_2]\pi^{(s+1,i_2,i_3,0)}$$

4.3 Expected Perishable Rate η_P:

$$\eta_P = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_2=0}^{1} i_1\gamma\pi^{(i_1,i_2,i_3,0)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=0}^{N-i_3}\sum_{i_2=3}^{4}[i_1-1]\gamma\pi^{(i_1,i_2,i_3,i_4)}$$
$$+\sum_{i_1=2}^{S}\sum_{i_3=0}^{N}[(i_1-1)\gamma\pi^{(i_1,2,i_3,0)} + \sum_{i_4=0}^{N-i_3}(i_1-2)\gamma\pi^{(i_1,5,i_3,i_4)}]$$

4.4 Expected Number of low priority Customers in the Queue η_L:

$$\eta_L = \sum_{i_3=1}^{N}\sum_{i_4=0}^{N-i_3}\sum_{i_2=0}^{1} i_3\pi^{(0,i_2,i_3,i_4)} + \sum_{i_1=2}^{S}\sum_{i_3=1}^{N} i_3\pi^{(i_1,2,i_3,0)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{N-i_3}\sum_{i_2=3}^{4} i_3\pi^{(i_1,i_2,i_3,i_4)}$$
$$+ \sum_{i_3=1}^{N}\sum_{i_4=0}^{N-i_3}[i_3\pi^{(1,2,i_3,i_4)} + \sum_{i_1=2}^{S} i_3\pi^{(i_1,5,i_3,i_4)}] + \sum_{i_1=1}^{S}\sum_{i_2=0}^{1}\sum_{i_3=1}^{N} i_3\pi^{(i_1,i_2,i_3,0)}$$

4.5 Expected Number of High Priority Customers in the Queue η_H:

$$\eta_H = \sum_{i_3=0}^{N-1}\sum_{i_4=1}^{N-i_3}\sum_{i_2=0}^{1} i_4\pi^{(0,i_2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\sum_{i_4=1}^{N-i_3}\sum_{i_2=3}^{4} i_4\pi^{(i_1,i_2,i_3,i_4)}$$
$$+ \sum_{i_1=2}^{S}\sum_{i_3=0}^{N-1}\sum_{i_4=1}^{N-i_3} i_4\pi^{(i_1,5,i_3,i_4)} + \sum_{i_3=0}^{N-1}\sum_{i_4=1}^{N-i_3} i_4\pi^{(1,2,i_3,i_4)}$$

4.6 Expected Loss Rate of Low Priority customers η_{LRL}:

$$\eta_{LRL} = \sum_{i_3=0}^{N}\sum_{i_2=0}^{1} \lambda_2\pi^{(0,i_2,i_3,N-i_3))} + \sum_{i_3=1}^{S} \lambda_2\pi^{(i_1,i_2,N,0)}$$
$$+ \sum_{i_1=1}^{1}\sum_{i_3=0}^{N}\sum_{i_2=3}^{4} \lambda_2\pi^{(i_1,i_2,i_3,N-i_3)} + \sum_{i_1=2}^{S}\sum_{I_3=0}^{N} \lambda_2\pi^{(i_1,5,i_3,N-i_3)}$$
$$+ \sum_{i_3=0}^{N} \lambda_2\pi^{(1,2,i_3,N-i_3)} + \sum_{i_1=2}^{S} \lambda_2\pi^{(i_1,i_2,N,0)}$$

4.7 Expected Loss Rate of High Priority customers η_{LRH}:

$$\eta_{LRH} = \sum_{i_3=0}^{N}\sum_{i_2=0}^{1} \lambda_1\pi^{(0,i_2,i_3,N-i_3))} + \sum_{i_3=1}^{S} \lambda_1\pi^{(i_1,i_2,N,0)}$$
$$+ \sum_{i_1=1}^{1}\sum_{i_3=0}^{N}\sum_{i_2=3}^{4} \lambda_1\pi^{(i_1,i_2,i_3,N-i_3)} + \sum_{i_1=2}^{S}\sum_{I_3=0}^{N} \lambda_1\pi^{(i_1,5,i_3,N-i_3)}$$
$$+ \sum_{i_3=0}^{N} \lambda_1\pi^{(1,2,i_3,N-i_3)} + \sum_{i_1=2}^{S} \lambda_1\pi^{(i_1,i_2,N,0)}$$

4.8 Expected number of High Priority customers Enter into the Queue η_{HE}:

$$\eta_{HE} = \sum_{i_3=0}^{N-1}\sum_{i_4=0}^{N-(i_3-1)} \lambda_1\pi^{(0,i_2,i_3,i_4)} + \sum_{i_2=2}^{4}\sum_{i_3=0}^{N-1}\sum_{i_4=0}^{N-(i_3+1)} \lambda_1\pi^{(1,i_2,i_3,i_4)}$$
$$+ \sum_{i_1=2}^{S}\sum_{i_3=0}^{N-1}\sum_{i_4=0}^{N-(i_3+1)} [\lambda_1\pi^{(i_1,3,i_3,i_4)} + q\lambda_1\pi^{(i_1,4,i_3,i_4)} + \lambda_1\pi^{(i_1,5,i_3,i_4)}]$$

4.9 Expected number of Low Priority customers Enter into the Queue η_{LE}:

$$\eta_{LE} = \sum_{i_3=0}^{N-1} \sum_{i_4=0}^{N-(i_3-1)} \lambda_2 \pi^{(0,i_2,i_3,i_4)} + \sum_{i_1=1}^{S} \sum_{i_2=0}^{1} \sum_{i_3=0}^{N-1} \lambda_2 \pi^{(i_1,i_2,i_3,0)} + \sum_{i_2=2}^{4} \sum_{i_3=0}^{N-1} \sum_{i_4=0}^{N-(i_3+1)} \lambda_2 \pi^{(1,i_2,i_3,i_4)} + \sum_{i_1=2}^{S} \sum_{i_2=3}^{5} \sum_{i_3=0}^{N-1} \sum_{i_4=0}^{N-(i_3+1)} \lambda_2 \pi^{(i_1,i_2,i_3,i_4)} + \sum_{i_1=2}^{S} \sum_{i_3=0}^{N-1} \lambda_2 \pi^{(i_1,2,i_3,0)}$$

4.10 Expected Waiting Time of Low Priority customers η_{WL}:

$$\eta_{WL} = \frac{\eta_{LE}}{\eta_L}$$

4.11 Expected Waiting Time of High Priority customers η_{WH}:

$$\eta_{WH} = \frac{\eta_{HE}}{\eta_H}$$

4.12 Fraction of time Server is on Vacation η_{FV}:

$$\eta_{FV} = \sum_{i_3=0}^{N} \sum_{i_4=0}^{N-i_3} \pi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S} \sum_{i_3=0}^{N} \pi^{(i_1,0,i_3,0)} + \sum_{i_1=1}^{S} \sum_{i_3=0}^{N} \sum_{i_4=0}^{N-i_3} \pi^{(i_1,3,i_3,i_4)}$$

4.13 Expected total cost rate

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined as c_h:The inventory carrying cost per unit item per unit time, c_s:Setup cost per order, c_p:Perishable cost per unit item per unit time, c_{wh}:Waiting time cost of a high priority customer per unit time, c_{wl}:Waiting time cost of a low priority customer per unit time, c_{hl}:Shortage cost of a high priority customer per unit time, c_{ll}:Shortage cost of a low priority customer per unit time. Then the expected total cost three variable function is defined as

$$TC(S,s,N) = c_h \times \eta_I + c_s \times \eta_R + c_p \times \eta_P + c_{wh} \times \eta_{WH} + c_{wl} \times \eta_{WL} + c_{hl} \times \eta_{LRH} + c_{ll} \times \eta_{LRL}.$$

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