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# NON-ARCHIMEDEAN APPROXIMATION OF MIQD AND MIQA FUNCTIONAL EQUATIONS

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**Abstract:**
The objective of this study to examine various classical stability results of the Multiplicative Inverse Quadratic Difference Functional Equation (MIQDF Equation) and Multiplicative Inverse Quadratic Adjoint Functional Equation (MIQAF Equation) in the setting of non-Archimedian fields and non-zero real numbers.

## 1. INTRODUCTION

The crucial point from where the concept of investigating HUS results of functional equations, differential equations, difference equations is the problem of Ulam [24]. Hyers [10] presented a partial solution to the problem of Ulam. Later, Hyers' theorem was extended and generalized in various forms by many mathematicians Aoki [1], T. Rassias [15], J. Rassias [14] and Gavruta [9]. These results instigated many mathematicians to investigate stability of various types of functional equations in different types of spaces. For detailed review of literature on this field, one can refer ([4], [5], [6], [7], [8], [11], [12], [13], [18]).

For the first time, the generalized HUR stability of a new rational type functional equation was dealt in [**?**]. Then, various fundamental stabilities associated with HUS of reciprocal adjoint and difference functional equations were demonstrated in ([16], [17]). In recent times, there are many papers published on the stabilities and applications of some multiplicative inverse functional equations, one can refer [[2], [20], [21], [22]].

The authors in [19] focussed on providing evidence that the stability results are valid for the following multiplicative inverse quadratic functional equation

$$q_r(\alpha+\beta) = \frac{q_r(\alpha)q_r(\beta)}{q_r(\alpha)+q_r(\beta)+2[q_r(\alpha)q_r(\beta)]^{\frac{1}{2}}}. \tag{1.1}$$

Our main results are based on the concepts of non-Archimedean fields. The fundamental notions of non-Archimedean fields are available in [3, 23].

This paper deals with the solution of the Multiplicative Inverse Quadratic Difference Functional Equation (MIQDF Equation)

$$q_d\left(\frac{\alpha+\beta}{2}\right)-q_d(\alpha+\beta) = \frac{3q_d(\alpha)q_d(\beta)}{q_d(\alpha)+q_d(\beta)+2[q_d(\alpha)q_d(\beta)]^{\frac{1}{2}}} \tag{1.2}$$

and the Multiplicative Inverse Quadratic Adjoint Functional Equation (MIQAF Equation)

$$q_a\left(\frac{\alpha+\beta}{2}\right)+q_a(\alpha+\beta) = \frac{5q_a(\alpha)q_a(\beta)}{q_a(\alpha)+q_a(\beta)+2[q_a(\alpha)q_a(\beta)]^{\frac{1}{2}}} \tag{1.3}$$

and then investigation of their generalized HUR stability in the setting of non-Archimedian fields and non-zero real numbers.

## 2. SOLUTION OF MIQDF AND MIQAF EQUATIONS

In the following theorem, we solve equations (1.2) and (1.3).

**Theorem 2.1.** *Suppose $q_r : \mathbb{R}^\star \longrightarrow \mathbb{R}$ is a mapping. Then the ensuing statements are equivalent.*

(a) *$q_r$ satisfies equation* (1.1).
(b) *$q_r$ satisfies equation* (1.2).
(c) *$q_r$ satisfies equation* (1.3).

*Hence, the solution of equations* (1.2) *and* (1.3) *is a multiplicative inverse quadratic function.*

*Proof.* Firstly, assume that $q_r$ satisfies equation (1.1). Plugging $\beta = \alpha$ in (1.1), one obtains that

$$q_r(2\alpha) = \frac{1}{4}q_r(\alpha) \tag{2.1}$$

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for all $\alpha \in \mathbb{R}^\star$. Now, reinstating $\alpha$ by $\frac{1}{2}\alpha$ in (2.1), we get

$$q_r\left(\frac{\alpha}{2}\right) = 4q_r(\alpha) \tag{2.2}$$

for all $\alpha \in \mathbb{R}^\star$. Swapping $(\alpha, \beta)$ with $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ in (1.1) and then in lieu of (2.2), one finds

$$q_r\left(\frac{\alpha+\beta}{2}\right) = \frac{4q_r(\alpha)q_r(\beta)}{q_r(\alpha)+q_r(\beta)+2[q_r(\alpha)q_r(\beta)]^{\frac{1}{2}}} \tag{2.3}$$

for all $\alpha, \beta \in \mathbb{R}^\star$. Subtracting (1.1) from equation (2.3), we arrive at (1.2).

Secondly, presume that $q_r$ satisfies equation (1.2). Letting $\beta = \alpha$ in (1.2), we obtain

$$q_r(2\alpha) = \frac{1}{4}q_r(\alpha) \tag{2.4}$$

for all $\alpha \in \mathbb{R}^\star$. Now, replacing $\alpha$ by $\frac{1}{2}\alpha$ in (2.4), we obtan

$$q_r\left(\frac{\alpha}{2}\right) = 4q_r(\alpha) \tag{2.5}$$

for all $\alpha \in \mathbb{R}^\star$. By the application of equation (2.5) in equation (1.2), we attain equation (1.1). By simple manipulation, it is easy to derive equation (1.3) from the above results.

Finally, suppose $q_r$ satisfies (1.3). Substituting $\beta = \alpha$ (1.3), we obtain

$$q_r(2\alpha) = \frac{1}{4}q_r(\alpha) \tag{2.6}$$

for all $\alpha \in \mathbb{R}^\star$. Now, transforming $\alpha$ to $\frac{1}{2}\alpha$ in 2.7, one gets

$$q_r\left(\frac{\alpha}{2}\right) = 4q_r(\alpha) \tag{2.7}$$

for all $\alpha \in \mathbb{R}^\star$. Using (2.7) in (1.3), we arrive at equation (1.1). This completes the proof. □

## 3. GENERALIZED HUS OF EQUATIONS (1.2) and (1.3) IN NON-ARCHIMEDEAN FIELDS

Throughout this section, let us assume that $\mathbb{S}$ and $\mathbb{T}$ are a non-Archimedean field and a complete non-Archimedean field, respecively. In this section, we prove the validity of stability results of equations (1.2) and (1.3) in non-Archimedean fields. Let $\mathbb{S}^\star$ denote a non-Archimedean field $\mathbb{S} - \{0\}$. For the sake of proving our main results in a concise manner, let $D_{q_d}, D_{q_a} : \mathbb{S}^\star \times \mathbb{S}^\star \longrightarrow \mathbb{T}$ be difference operators defined as follows, respectively:

$$\begin{aligned} D_{q_d}(\alpha,\beta) = {} & q_d\left(\frac{\alpha+\beta}{2}\right) \\ & - q_d(\alpha+\beta) - \frac{3q_d(\alpha)q_d(\beta)}{q_d(\alpha)+q_d(\beta)+2[q_d(\alpha)q_d(\beta)]^{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned} D_{q_a}(\alpha,\beta) = {} & q_a\left(\frac{\alpha+\beta}{2}\right) + q_a(\alpha+\beta) \\ & - \frac{5q_a(\alpha)q_q(\beta)}{q_a(\alpha)+q_a(\beta)+2[q_a(\alpha)q_a(\beta)]^{\frac{1}{2}}} \end{aligned}$$

for all $\alpha, \beta \in \mathbb{S}^\star$.

**Theorem 3.1.** *Suppose $\ell = \pm 1$ be a fixed constant. Let $\nu : \mathbb{S}^\star \times \mathbb{S}^\star \longrightarrow \mathbb{T}^\star$ be a function such that*

$$\lim_{m\to\infty} \left|\frac{1}{4}\right|^{\ell m} \nu\left(\frac{\alpha}{2^{\ell m+\frac{\ell+1}{2}}}, \frac{\beta}{2^{\ell m+\frac{\ell+1}{2}}}\right) = 0 \tag{3.1}$$

*for all $\alpha, \beta \in \mathbb{S}^\star$ and $q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ is a mapping satisfying the inequality*

$$|D_{q_d}(\alpha,\beta)| \le \nu(\alpha,\beta) \tag{3.2}$$

*for all $\alpha, \beta \in \mathbb{S}^\star$. Then a unique multiplicative inverse quadratic functional equation $Q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ exists and it satisfies* (1.2) *with the condition*

(3.3)
$$|q_d(\alpha) - Q_d(\alpha)| \le max\left\{\left|\frac{1}{4}\right|^{k\ell+\frac{\ell-1}{2}} \nu\left(\frac{\alpha}{2^{k\ell+\frac{\ell+1}{2}}}, \frac{\alpha}{2^{k\ell+\frac{\ell+1}{2}}}\right) : k \ge 0 \text{ is an integer}\right\}$$

*for all $\alpha, \beta \in \mathbb{S}^\star$.*

*Proof.* Plugging $(\alpha, \beta)$ into $\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)$ in (3.2), we get

$$\left|\frac{1}{4^\ell}q_d\left(\frac{\alpha}{2^\ell}\right) - q_d(\alpha)\right| \le |4|^{\frac{|\ell-1|}{2}} \nu\left(\frac{\alpha}{2^{\frac{\ell+1}{2}}}, \frac{\alpha}{2^{\frac{\ell+1}{2}}}\right) \tag{3.4}$$

for all $\alpha \in \mathbb{A}^*$. Now, transforming $\alpha$ to $\dfrac{\alpha}{2^{m\ell}}$ in (3.4) and then multiplying by $\left|\frac{1}{4}\right|^{m\ell}$, we obtain

$$\begin{aligned} & \left|\frac{1}{4^{m\ell}}q_d\left(\frac{\alpha}{2^{m\ell}}\right) - \frac{1}{4^{(m+1)\ell}}q_d\left(\frac{\alpha}{2^{(m+1)\ell}}\right)\right| \\ & \le \left|\frac{1}{4}\right|^{\ell m+\frac{\ell-1}{2}} \nu\left(\frac{\alpha}{2^{\ell m+\frac{\ell+1}{2}}}, \frac{\alpha}{2^{\ell m+\frac{\ell+1}{2}}}\right) \end{aligned} \tag{3.5}$$

for all $\alpha \in \mathbb{S}^*$. In lieu of (3.1), we find that the the right hand side of (3.5) tends to zero as $m \longrightarrow \infty$, which implies that the sequence $\left\{4^{-m\ell}q_d(2^{-m\ell}\alpha)\right\}$ is Cauchy. Also, due to completness of $\mathbb{T}$, a mapping $Q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ exists such that

$$Q_d(\alpha) = \lim_{m\to\infty} \frac{1}{4^{m\ell}}q_d\left(\frac{\alpha}{2^{m\ell}}\right). \tag{3.6}$$

Also, for every $\alpha \in \mathbb{S}^\star$ and an integer $m \geq 0$, we have

$$\left|\frac{1}{4^{m\ell}} q_d\left(\frac{\alpha}{2^{m\ell}}\right) - q_d(\alpha)\right|$$
$$= \left|\sum_{k=0}^{m-1} \frac{1}{4^{(k+1)\ell}} q_d\left(\frac{\alpha}{2^{(k+1)\ell}}\right) - \frac{1}{4^{k\ell}} q_d\left(\frac{\alpha}{2^{k\ell}}\right)\right| \leq max$$
$$\left\{\left|\frac{1}{4^{(k+1)\ell}} q_d\left(\frac{\alpha}{2^{(k+1)\ell}}\right) - \frac{1}{4^{k\ell}} q_d\left(\frac{\alpha}{2^{k\ell}}\right)\right| : 0 \leq k \leq m-1\right\}$$
$$\leq max\left\{\left|\frac{1}{4}\right|^{k\ell} \nu\left(\frac{\alpha}{2^{(k+1)\ell}}, \frac{\alpha}{2^{(k+1)\ell}}\right) : 0 \leq k \leq m-1\right\}.$$

By the application of (3.6) and taking $m \to \infty$, we find that the inequality (3.3) holds good. From (3.1), (3.2), (3.6), for all $\alpha, \beta \in \mathbb{S}^\star$

$$|D_{Q_d}(\alpha,\beta)| = \lim_{m\to\infty} \left|\frac{1}{4}\right|^{m\ell} \left|D_{q_d}\left(\frac{\alpha}{2^{m\ell}}, \frac{\beta}{2^{m\ell}}\right)\right|$$
$$\leq \lim_{m\to\infty} \left|\frac{1}{4}\right|^{m\ell} \nu\left(\frac{\alpha}{2^{(m+1)\ell}}, \frac{\beta}{2^{(m+1)\ell}}\right) = 0.$$

Hence the mapping $Q_d$ satisfies (1.2) and therefore the mapping $Q_d$ is multiplicative inverse quadratic. Now, let $Q_d' : \mathbb{S}^\star \longrightarrow \mathbb{T}$ be another multiplicative inverse quadratic mapping satisfying (3.3). Then

$$|Q_d(\alpha) - Q_d(\alpha)'| = \lim_{m\to\infty} \left|\frac{1}{4}\right|^{m\ell} \left|Q_d\left(\frac{\alpha}{2^{m\ell}}\right) - Q_d'\left(\frac{\alpha}{2^{m\ell}}\right)\right|$$
$$\leq \lim_{m\to\infty} \left|\frac{1}{4}\right|^{m\ell} max\left\{\left|Q_d\left(\frac{\alpha}{2^{m\ell}}\right) - q_d\left(\frac{\alpha}{2^{m\ell}}\right)\right|,\right.$$
$$\left.\left|q_d\left(\frac{\alpha}{2^{m\ell}}\right) - Q_d\left(\frac{\alpha}{2^{m\ell}}\right)\right|\right\} \leq \lim_{m\to\infty} \lim_{p\to\infty}$$
$$max\left\{max\left\{\left|\frac{1}{4}\right|^{(k+m)\ell} \nu\left(\frac{\alpha}{2^{(k+m+1)\ell}}, \frac{\alpha}{2^{(k+m+1)\ell}}\right)\right.\right.$$
$$\left.\left. : m \leq k \leq p+m\right\}\right\}$$
$$= 0$$

for all $\alpha \in \mathbb{S}^\star$, which proves that $q_d$ is unique. This completes the proof of theorem. ☐

**Corollary 3.2.** *Let $\delta_1$ be a fixed non-negative real number and $\rho \neq -2$. Suppose $q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ is a mapping such that*

$$|D_{q_d}(\alpha,\beta)| \leq k_1\{|\alpha|^\rho + |\beta|^\rho\}$$

*for all $\alpha, \beta \in \mathbb{S}^\star$. Then a unique multiplicative inverse quadratic mapping $Q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ existis and satisfies* (1.2) *and*

$$|q_d(\alpha) - Q_d(\alpha)| \leq \begin{cases} \frac{2\delta_1|\alpha|^\rho}{|2|^\rho} & \text{if } \rho < -2 \\ 8\delta_1\,|\alpha|^\rho & \text{if } \rho > -2 \end{cases}$$

*for every $\alpha \in \mathbb{S}^\star$.*

*Proof.* Replacing $\nu(\alpha,\beta)$ by $\delta_1\left(|\alpha|^\rho + |\beta|^\rho\right)$ and continuing with similar arguments as in Theorem 3.1, we arrive at the desired result. ☐

**Corollary 3.3.** *Let $q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ be a mapping and $\mu$, $\tau$ be two real numbers such that $\rho = \mu + \tau \neq 2$. Let there exists a non-negative real number $\delta_2$ such that*

$$|D_{q_d}(\alpha,\beta)| \leq \delta_2\,|\alpha|^\mu\,|\beta|^\tau$$

*for all $\alpha, \beta \in \mathbb{S}^\star$. Then a unique multiplicative inverse quadratic mapping $Q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ exists and satisfies* (1.2) *and,*

$$|q_d(\alpha) - Q_d(\alpha)| \leq \begin{cases} \frac{\delta_2|\alpha|^\rho}{|2|^\rho} & \text{if } \rho < -2 \\ 4\delta_2\,|\alpha|^\rho & \text{if } \rho > -2 \end{cases}$$

*for every $\alpha \in \mathbb{S}^\star$.*

*Proof.* Assuming $\nu(\alpha,\beta) = \delta_2|\alpha|^\mu\,|\beta|^\tau$ in Theorem 3.1, we achieve the required result. ☐

**Corollary 3.4.** *Let $\delta_3$ be a fixed non-negative real number and $\rho \neq -2$. Suppose $q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ is a mapping such that*

$$|D_{q_d}(\alpha,\beta)| \leq \delta_3\left(|\alpha|^{\frac{\rho}{2}}\,|\beta|^{\frac{\rho}{2}} + (|\alpha|^\rho + |\beta|^\rho)\right)$$

*for all $x, y \in \mathbb{A}^*$. Then a unique multiplicative inverse quadratic mapping $Q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ exists and satisfies* (1.2) *and,*

$$|q_d(\alpha) - Q_d(\alpha)| \leq \begin{cases} \frac{3\delta_3|\alpha|^\rho}{|2|^\rho} & \text{if } \rho < -2 \\ 12\delta_3\,|\alpha|^\rho & \text{if } \rho > -2 \end{cases}$$

*for every $\alpha \in \mathbb{S}^\star$.*

*Proof.* The proof follows directly by taking $\nu(\alpha,\beta) = \delta_3\left(|\alpha|^{\frac{\rho}{2}}\,|\beta|^{\frac{\rho}{2}} + (|\alpha|^\rho + |\beta|^\rho)\right)$ in Theorem 3.1. ☐

In the following theorem and corollaries, we present the stability results of equation (1.3). The arguments of proving stability results of equation (1.3) are akin to the proof of Theorem 3.1. For the sake of completness, we furnish below the statement of theorems and corollaries only pertinent to various fundamental stabilities.

**Theorem 3.5.** *Suppose $\ell = \pm 1$ be a fixed constant. Let $\nu : \mathbb{S}^\star \times \mathbb{S}^\star \longrightarrow \mathbb{T}^\star$ be a function such that*

$$\lim_{m\to\infty} \left|\frac{1}{4}\right|^{\ell m} \nu\left(\frac{\alpha}{2^{\ell m + \frac{\ell+1}{2}}}, \frac{\beta}{2^{\ell m + \frac{\ell+1}{2}}}\right) = 0 \tag{3.7}$$

*for all $\alpha, \beta \in \mathbb{S}^\star$ and $q_a : \mathbb{S}^\star \longrightarrow \mathbb{T}$ is a mapping satisfying the inequality*

$$|D_{q_a}(\alpha,\beta)| \leq \nu(\alpha,\beta)$$

*for all $\alpha, \beta \in \mathbb{S}^\star$. Then a unique multiplicative inverse quadratic functional equation $Q_a : \mathbb{S}^\star \longrightarrow \mathbb{T}$ exists and it satisfies*

(1.3) *with the condition*

$$|q_a(\alpha) - Q_a(\alpha)| \leq max\left\{\left|\frac{1}{4}\right|^{k\ell+\frac{\ell-1}{2}} \nu\left(\frac{\alpha}{2^{k\ell+\frac{\ell+1}{2}}}, \frac{\alpha}{2^{k\ell+\frac{\ell+1}{2}}}\right) : k \geq 0 \text{ is an integer}\right\}$$

*for all* $\alpha, \beta \in \mathbb{S}^\star$.

**Corollary 3.6.** *Let* $\delta_1$ *be a fixed non-negative real number and* $\rho \neq -2$. *Suppose* $q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ *is a mapping such that*

$$|D_{q_a}(\alpha, \beta)| \leq k_1\{|\alpha|^\rho + |\beta|^\rho\}$$

*for all* $\alpha, \beta \in \mathbb{S}^\star$. *Then a unique multiplicative inverse quadratic mapping* $Q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ *existis and satisfies* (1.3) *and*

$$|q_a(\alpha) - Q_a(\alpha)| \leq \begin{cases} \frac{2\delta_1|\alpha|^\rho}{|2|^\rho} & \text{if } \rho < -2 \\ 8\delta_1 |\alpha|^\rho & \text{if } \rho > -2 \end{cases}$$

*for every* $\alpha \in \mathbb{S}^\star$.

**Corollary 3.7.** *Let* $q_a : \mathbb{S}^\star \longrightarrow \mathbb{T}$ *be a mapping and* $\mu$, $\tau$ *be two real numbers such that* $\rho = \mu + \tau \neq 2$. *Let there exists a non-negative real number* $\delta_2$ *such that*

$$|D_{q_a}(\alpha, \beta)| \leq \delta_2 |\alpha|^\mu |\beta|^\tau$$

*for all* $\alpha, \beta \in \mathbb{S}^\star$. *Then a unique multiplicative inverse quadratic mapping* $Q_d : \mathbb{S}^\star \longrightarrow \mathbb{T}$ *exists and satisfies* (1.3) *and,*

$$|q_a(\alpha) - Q_a(\alpha)| \leq \begin{cases} \frac{\delta_2|\alpha|^\rho}{|2|^\rho} & \text{if } \rho < -2 \\ 4\delta_2 |\alpha|^\rho & \text{if } \rho > -2 \end{cases}$$

*for every* $\alpha \in \mathbb{S}^\star$.

**Corollary 3.8.** *Let* $\delta_3$ *be a fixed non-negative real number and* $\rho \neq -2$. *Suppose* $q_a : \mathbb{S}^\star \longrightarrow \mathbb{T}$ *is a mapping such that*

$$|D_{q_a}(\alpha, \beta)| \leq \delta_3 \left(|\alpha|^{\frac{\rho}{2}} |\beta|^{\frac{\rho}{2}} + (|\alpha|^\rho + |\beta|^\rho)\right)$$

*for all* $x, y \in \mathbb{A}^*$. *Then a unique multiplicative inverse quadratic mapping* $Q_a : \mathbb{S}^\star \longrightarrow \mathbb{T}$ *exists and satisfies* (1.3) *and,*

$$|q_a(\alpha) - Q_a(\alpha)| \leq \begin{cases} \frac{3\delta_3|\alpha|^\rho}{|2|^\rho} & \text{if } \rho < -2 \\ 12\delta_3 |\alpha|^\rho & \text{if } \rho > -2 \end{cases}$$

*for every* $\alpha \in \mathbb{S}^\star$.

## 4. GENERALIZED HUS OF EQUATIONS (1.2) and (1.3) IN NON ZERO REAL NUMBERS

In this section, we prove the validity of stability results of equations (1.2) and (1.3) with non-zero real numbers as domain. For the purpose of easy computation, in the sequel, let us define the operators $\Delta_{q_d}, \Delta_{q_a} : \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow \mathbb{R}$ as, respectively:

$$\Delta_{q_d}(\alpha, \beta) = q_d\left(\frac{\alpha+\beta}{2}\right) - q_d(\alpha+\beta) - \frac{3q_d(\alpha)q_d(\beta)}{q_d(\alpha) + q_d(\beta) + 2[q_d(\alpha)q_d(\beta)]^{\frac{1}{2}}}$$

$$\Delta_{q_a}(\alpha, \beta) = q_a\left(\frac{\alpha+\beta}{2}\right) + q_a(\alpha+\beta) - \frac{5q_a(\alpha)q_q(\beta)}{q_a(\alpha) + q_a(\beta) + 2[q_a(\alpha)q_a(\beta)]^{\frac{1}{2}}}$$

for all $\alpha, \beta \in \mathbb{R}^\star$.

**Theorem 4.1.** *Let* $\ell = \pm 1$ *be a fixed constant and* $q_d : \mathbb{R}^\star \longrightarrow \mathbb{R}$ *be a function. Suppose* $\upsilon : \mathbb{R}^\star \times \mathbb{R}^\star \longrightarrow [0, \infty)$ *be a mapping satisfying*

$$|\Delta_{q_d}(\alpha, \beta)| \leq \upsilon(\alpha, \beta) \tag{4.1}$$

*for all* $\alpha, \beta \in \mathbb{R}^\star$, *where* $\upsilon$ *satisfies*

$$\sum_{j=0}^{\infty} \frac{1}{4^{\ell j}} \upsilon\left(\frac{\alpha}{2^{\ell(j+1)}}, \frac{\beta}{2^{\ell(j+1)}}\right) < +\infty \tag{4.2}$$

*for all* $\alpha, \beta \in \mathbb{R}^\star$. *Then a unique multiplicative inverse quadratic mapping* $Q_d : \mathbb{R}^\star \longrightarrow \mathbb{R}$ *which satisfies* (1.2) *and the inequality*

$$|q_d(\alpha) - Q_d(\alpha)| \leq \sum_{j=0}^{\infty} \frac{1}{4^{\ell j}} \upsilon\left(\frac{\alpha}{2^{\ell(j+1)}}, \frac{\beta}{2^{\ell(j+1)}}\right) \tag{4.3}$$

*for all* $\alpha \in \mathbb{R}^\star$.

*Proof.* Replacing $(\alpha, \beta)$ by $(\frac{\alpha}{2}, \frac{\beta}{2})$ in (4.1), we obtain

$$\left|\frac{1}{4^\ell} q_d\left(\frac{\alpha}{2^\ell}\right) - q_d(\alpha)\right| \leq |4|^{\frac{|\ell-1|}{2}} \upsilon\left(\frac{\alpha}{2^{\frac{\ell+1}{2}}}, \frac{\alpha}{2^{\frac{\ell+1}{2}}}\right) \tag{4.4}$$

for all $\alpha \in \mathbb{R}^\star$. Now, changing $\alpha$ with by $\frac{\alpha}{2^{m\ell}}$ in (4.4) and then multiplying by $\left|\frac{1}{4}\right|^{m\ell}$, we obtain

$$\left|\frac{1}{4^{m\ell}} q_d\left(\frac{\alpha}{2^{m\ell}}\right) - \frac{1}{4^{(m+1)\ell}} q_d\left(\frac{\alpha}{2^{(m+1)\ell}}\right)\right| \leq \frac{1}{4^{m\ell}} \upsilon\left(\frac{\alpha}{2^{(m+1)\ell}}, \frac{\alpha}{2^{(m+1)\ell}}\right). \tag{4.5}$$

Now, letting $m \to \infty$ in (4.5) and using (4.2), we have $\{\frac{1}{4^{m\ell}} q_d(\frac{\alpha}{2^{(m+1)\ell}})\}$ is a cauchy sequaence. Since $\mathbb{R}$ is complete, the sequence $\{\frac{1}{4^{m\ell}} q_d(\frac{\alpha}{2^{(m+1)\ell}})\}$ converges to a function $Q_d$ defined by

$$Q_d(\alpha) = \lim_{n\to\infty} \frac{1}{4^{m\ell}} q_d(2^{(m+1)\ell}\alpha). \tag{4.6}$$

Now, we claim that $Q_d$ satisfies (1.2). Plugging $(\alpha, \beta)$ into $\left(\frac{\alpha}{2^{m\ell}}, \frac{\beta}{2^{m\ell}}\right)$ in (4.1) and then multiplying by $\frac{1}{4^{m\ell}}$, we obtain

$$\left|\frac{1}{4^{m\ell}} \Delta_{q_d}\left(\frac{\alpha}{2^{m\ell}}, \frac{\beta}{2^{m\ell}}\right)\right| \leq \frac{1}{4^{m\ell}} \upsilon\left(\frac{\alpha}{2^{m\ell}}, \frac{\beta}{2^{m\ell}}\right) \tag{4.7}$$

for all $\alpha, \beta \in \mathbb{R}^\star$ and for all positive integer $m$. Now, using (4.2), (4.6) in (4.7), we find that $Q_d$ satisfies (1.2) for all $\alpha, \beta \in \mathbb{R}^\star$. For each $\alpha \in \mathbb{R}^\star$ and each integer $m$, we have

$$\begin{aligned}&\left|\frac{1}{4^{m\ell}} q_d\left(\frac{\alpha}{2^{m\ell}}\right) - q_d(\alpha)\right| \\ &\quad \leq \sum_{i=0}^{m-1}\left|\frac{1}{4^{i\ell}} q_d\left(\frac{\alpha}{2^{(i+1)\ell}}\right) - \frac{1}{4^{(i-1)\ell}} q_d\left(\frac{\alpha}{2^{i\ell}}\right)\right| \\ &\quad \leq \sum_{i=0}^{m-1} \frac{1}{4^{i\ell}} \upsilon\left(\frac{\alpha}{2^{(i+1)\ell}}, \frac{\alpha}{2^{(i+1)\ell}}\right).\end{aligned}$$

Applying (4.6) and letting $m \to \infty$, we obtain (4.3). Now, we prove that $Q_d$ is unique. Let $D'_q : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be another multiplicative inverse quadratic mapping satisfying (1.2) and (4.3). Therefore

$$Q'_d\left(\frac{\alpha}{2^{m\ell}}\right) = 4^{m\ell} Q'_d(\alpha) \quad \text{and} \quad Q_d\left(\frac{\alpha}{2^{m\ell}}\right) = 4^{m\ell} Q_d(\alpha)$$

for all $\alpha \in \mathbb{R}^\star$. By the application of (4.3), we obtain

$$\begin{aligned}&|Q'_d(\alpha) - Q_d(\alpha)| \\ &= 4^{-m\ell}\left|Q'_d(2^{-m\ell}\alpha) - q_d(2^{-m\ell}\alpha)\right| \\ &\leq 4^{-m\ell}\left|q'_d(2^{-m\ell}\alpha) - q_d(2^{-m\ell}\alpha)\right| \\ &+ 4^{-m\ell}\left|q_d(2^{-m\ell}\alpha) - q_d(2^{-m\ell}\alpha)\right| \\ &\leq 2\sum_{j=0}^{\infty} \frac{1}{4^{(m+j)\ell}} \upsilon\left(\frac{\alpha}{2^{(m+j+1)\ell}}, \frac{\alpha}{2^{(m+j+1)\ell}}\right) \\ &\leq 2\sum_{j=m}^{\infty} \frac{1}{4^{j\ell}} \upsilon\left(\frac{\alpha}{2^{(j+1)\ell}}, \frac{\alpha}{2^{(j+1)\ell}}\right)\end{aligned} \tag{4.8}$$

for all $\alpha \in \mathbb{R}^\star$. Allowing $m \to \infty$in (4.8), we find $Q_d$ is unique and this competes the proof. □

The following corollary is pertinent to HUS of equation (1.2). The proof directly follows from Theorem 4.1 for $\ell = -1$. Hence, we omit the proof.

**Corollary 4.2.** *Let $q_d : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a function, for which there exists a constant $\lambda \geq 0$ (independent of $\alpha$ and $\beta$) such that the functional inequality*

$$|D_{q_d}(\alpha, \beta)| \leq \frac{\lambda}{4}$$

*holds for all $(\alpha, \beta) \in \mathbb{R}^\star \times \mathbb{R}^\star$. Then the limit*

$$Q_d(\alpha) = \lim_{m\to\infty} \frac{1}{4^{m\ell}} q_d\left(\frac{\alpha}{2^{m\ell}}\right)$$

*exists for all $\alpha \in \mathbb{R}^\star$, $m \in \mathbb{N}$ and $Q_d : \mathbb{R}^\star \to \mathbb{R}$ is the unique multiplicative inverse quadratic mapping satisfying the equation (1.2) such that*

$$|q_d(\alpha) - Q_d(\alpha)| \leq \lambda.$$

*Also, functional identity $q_d(\alpha) = \frac{1}{4^m} q_d\left(\frac{\alpha}{2^m}\right)$ holds for all $\alpha \in \mathbb{R}^\star$ and $m \in \mathbb{N}$.*

**Corollary 4.3.** *For any fixed $\lambda_1 \geq 0$ and $\rho \neq -2$, if $q_d : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfies*

$$|D_{q_d}(\alpha, \beta)| \leq \lambda_1(|\alpha|^\rho + |\beta|^\rho)$$

*for all $\alpha, \beta \in \mathbb{R}^\star$. Then there exist a unique multiplicative inverse quadratic mapping $Q_d : \mathbb{R}^\star \longrightarrow \mathbb{R}$ such that*

$$|q_d(\alpha) - Q_d(\alpha)| \leq \frac{8k_1}{2^{\alpha+2} - 1}|\alpha|^\rho \quad \text{for } \rho \neq -2$$

*for all $\alpha \in \mathbb{R}^\star$.*

*Proof.* The desired results are obtained by choosing $\upsilon(\alpha, \beta) = \lambda_1(|\alpha|^\rho + |\beta|)^\rho$, for all $\alpha, \beta \in \mathbb{R}^\star$ in Theorem 4.1 and proceeding by similar arguments. □

**Corollary 4.4.** *Let $q_d : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a function. If there exists $\mu$, $\tau$ such that $\rho = \mu + \tau \neq -2$ and $\lambda_2 \geq 0$ such that for all $\alpha, \beta \in \mathbb{R}^\star$,*

$$|D_{q_d}(\alpha, \beta)| \leq \lambda_2 |\alpha|^\mu |\beta|^\tau$$

*then, there exists a unique multiplicative inverse quadratic mapping $Q_d : \mathbb{R}^\star \longrightarrow \mathbb{R}$ such that*

$$|q_d(\alpha) - Q_d(\alpha)| \leq \frac{4\lambda_2}{|2^{\rho+2} - 1|}|\alpha|^\rho \quad \text{for } \rho \neq -2$$

*holds and $Q_d$ satisfies (1.2) for all $\alpha, \beta \in \mathbb{R}^\star$.*

*Proof.* Assuming $\upsilon(\alpha, \beta) = \lambda_2|\alpha|^\mu |\beta|^\tau$ in Theorem 4.1, we achieve the required result. □

We omit the the proof of investigating various stabilities of equation (1.3) since they are obtained by similar arguments as in the proof of stability results concerning equation (1.2).

## 5. CONCLUSION

We wind up this paper with a conclusion that we have proved various stability results of MIQDF and MIQAF equations involving a general control function, sum of powers of norms, product of different powers of norms, and mixed product-sum of powers of norms pertinent to the results established by Gavruta and Rassias. We observe from our main results, that the stability results still hold good for MIQDF and MIQAF equations in non-Archimedean fields and non zero real numbers.

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