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Metric-Like Spaces and Related Fixed Point Theorems with its Various Types

Monika
Department of Mathematics, UIET, MDU, Rohtak, Haryana, India.

Abstract

In this paper, various fixed point results related to metric-like spaces and its various types have been depicted which will be very helpful for further study of metric-like spaces.

Keywords: Metric-like spaces, modified metric-like space, rectangular metric-like space, generalization, fixed point.

1. INTRODUCTION

The generalization of metric space is based on reducing or modifying the metric axioms. Losing or weakening some of metric axioms causes the loss of some properties and so bringing problems in proving some fixed point theorems. These problems force researchers to make efforts to develop new techniques or ways in the development of fixed point theory in order to solve these problems.

To resolve these problems timely, time to new time research work come into existence. This paper is restricted to metric-like spaces and its various types such as rectangular metric-like spaces, b- metric-like spaces, modified rectangular metric-like spaces and so on.

We shall also give some fixed point theorems related to these various types of metric-like spaces.

2. PRELIMINARIES

Definition 2.1[2]. Let U be a non-empty set. A function $\sigma : U \times U \to [0, \infty)$ is said to be a metric-like (dislocated metric) on U if for any u, v, w ∈ U, the following conditions hold:

(σ1) $\sigma(u, v) = 0 \Rightarrow u = v$;

(σ2) $\sigma(u, v) = \sigma(v, u)$;

(σ3) $\sigma(u, v) \leq \sigma(u, w) + \sigma(w, v)$.

Then the pair (U, σ) with σ as metric-like is known as **metric-like (or a dislocated metric) space.**

Following are some metric-like type spaces defined based on modifying one of a metric axiom:

Definition 2.2[3]. Let U be a non-empty set. A function $\sigma_m : U \times U \to [0, \infty)$ is said to be modified metric-like on U if for any u, v, w ∈ U, the following conditions hold:

(σ_m1) $\sigma_m(u, v) = 0 \Rightarrow u = v$;

(σ_m2) $\sigma_m(u, v) = \sigma_m(v, u)$;

(σ_m3) $\sigma_m(u, v) \leq \sigma_m(u, w) + \sigma_m(w, v) - \sigma_m(w, w)$.

Then the pair (U, σ_m) with σ_m as metric-like is known as **modified metric-like space.**

Example 2.3. Let U = {1, 2, 3, 4} and $\sigma_m : U \times U \to [0, \infty)$ be defined as

$$\sigma_m(u, v) = \begin{Bmatrix} 4 & \text{for } u \neq v \\ 0 & \text{for } u = v \neq 1 \\ 5 & \text{for } u = v = 1 \end{Bmatrix}.$$

Then the space (U, σ_m) is modified metric-like space.

Since conditions (σ_m1) and (σ_m2) are already satisfied and as for considering various cases such as u ≠ v, $\sigma_m(u,w) + \sigma_m(w,v) - \sigma_m(w, w) = 4 + 4 - \sigma_m(w, w) \geq \sigma_m(u, v)$. So in all cases, the conditions of definition 2.2 are satisfied. Hence (U, σ_m) is modified metric-like space.

Definition 2.4[3]. Let U be a non-empty set and $\rho_r : U \times U \to [0,\infty)$ be a function. If the following conditions are satisfied for all u, v in U and x, y ∈ X \ {u, v}:

(1) $\rho_r(u, v) = 0 \Rightarrow u = v$;

(2) $\rho_r(u, v) = \rho_r(v, u)$;

(3) $\rho_r(u, v) \leq \rho_r(u, x) + \rho_r(x, y) + \rho_r(y, v)$;
(*rectangular inequality*)

then the pair (U, ρ_r) is known as **rectangular metric-like space.**

Example 2.5. Let U = {1, 2, 3, 4} and define the function $\rho_r : U \times U \to [0,\infty)$ by

$$\rho_r(u, v) = \begin{Bmatrix} 3 & \text{for } u \neq v \\ 5 & \text{for } u = v = 1 \\ 0 & \text{otherwise} \end{Bmatrix}$$

Then, the conditions (1) and (2) of definition (2.4) are clearly satisfied. So after that we need to verify the last condition of definition (2.4).

For this, considering every x, y ∈ U \ {u, v}, we have

$\rho_r(u, x) + \rho_r(x, y) + \rho_r(y, v) = 3 + \rho_r(x, y) + 3 = 6 + \rho_r(x, y) \geq \rho_r(u, v)$, for all u, v ∈ U.

Therefore, (U, ρ_r) is a rectangular metric-like space as all the conditions of definition (2.4) are satisfied.

Definition 2.6[1]. A b-metric-like on a non-empty set U is a function : U × U → [0, ∞) such that for all u, v, w ∈ U and a constant K ≥ 1 the following three conditions hold:

($\mathcal{D}_1$) (u, v) = 0 ⟹ u = v;

($\mathcal{D}_2$) (u, v) = (v, u);

($\mathcal{D}_3$) (u, v) ≤ K($\mathcal{D}$(u, w) + $\mathcal{D}$(w, v)).

The pair (U, $\mathcal{D}$) is known as **b-metric-like space.** Sometimes, we also denote b-metric-like space by (U, $\mathcal{D}$, K).

Definition 2.7[3]. Let U be a non-empty set and ρ_{mr} : U × U → [0,∞) be a function. If the following conditions are satisfied for all u, v in U and x, y ∈ X \ {u, v}:

(1) $\rho_{mr}(u, v) = 0 \Rightarrow u = v$;

(2) $\rho_{mr}(u, v) = \rho_{mr}(v, u)$;

(3) $\rho_{mr}(u, v) \leq \rho_{mr}(u, x) + \rho_{mr}(x, y) + \rho_{mr}(y, v) - \rho_{mr}(x, x) - \rho_{mr}(y, y)$;

then the pair (U, ρ_{mr}) is known as **rectangular modified metric-like space.**

Example 2.8[3]. Let U = (0, 1) and define the mapping ρ_{mr} : U × U → [0,∞) by

$$\rho_{mr}(u, v) = |u - v| = 2.$$

Then (U, ρ_{mr}) is rectangular modified metric-like space.

Definition 2.9[2]. Let (U, σ) be a metric-like space. Then

(i) a sequence $\{u_n\}$ in U **converges** to a point u ∈ U if and only if $\sigma(u, u) = \lim_{n\to\infty} \sigma(u, u_n)$.

(ii) a sequence $\{u_n\}$ in U is said to be **a Cauchy sequence** if $\lim_{n,m\to\infty} \sigma(u_n, u_m)$ exists and is finite.

(iii) the space (U, σ) is said to be **complete** if every Cauchy sequence $\{u_n\}$ in U converges to a point u ∈ U such that $\lim_{n\to\infty} \sigma(u, u_n) = \sigma(u, u) = \lim_{n,m\to\infty} \sigma(u_n, u_m)$.

Here are some main results related to metric-like spaces given as follows:

3. MAIN RESULTS

In 2012, Harandi [2] gave generalizations of well-known fixed point theorems of Rakotch [4], one of which is as follows:

Theorem 3.1[2]. Let (U, σ) be a complete metric-like space, and let T : U → U be a map such that

$\sigma(Tu, Tv) \leq \sigma(u, v) - \varphi(\sigma(u, v))$,

for all u, v ∈ U, where φ : [0, ∞) → [0, ∞) is a non-decreasing continuous function such that

φ(t) = 0 if and only if t = 0. Then T has a unique fixed point."

Proof: Let $u_0 \in U$ and define $u_{n+1} = Tu_n$ for n ≥ 0. Then by our supposition,

$$\sigma(u_{n+1}, u_{n+2}) = \sigma(Tu_n, Tu_{n+1}) \leq \sigma(u_n, u_{n+1}) - \varphi(\sigma(u_n, u_{n+1})), \qquad (3.1.1)$$

for each n ∈ ℕ. Then $\{\sigma(u_n, u_{n+1})\}$ is a non-negative non-increasing sequence and hence have a limit $r_0 \geq 0$. Since φ is non-decreasing, then from (3.1.1), we get

$\sigma(u_{n+1}, u_{n+2}) \leq \sigma(u_n, u_{n+1}) - \varphi(r_0)$

for each n ∈ ℕ. Then $r_0 \leq r_0 - \varphi(r_0)$ and so $r_0 = 0$. Therefore,

$\lim_{n\to\infty} \sigma(u_n, u_{n+1}) = 0.$

Now, we show that $\{u_n\}$ is a Cauchy sequence. Taking ε > 0 and choose N such as

$$\sigma(u_n, u_{n+1}) < \min\left\{\frac{\varepsilon}{2}, \varphi\left(\frac{\varepsilon}{2}\right)\right\} \text{ for every } n \geq N.$$

We shall now depict that if $\sigma(u, u_N) \leq \varepsilon$, then $\sigma(Tu, u_N) \leq \varepsilon$. To depict the claim, let us first assume that $\sigma(u, u_N) \leq \frac{\varepsilon}{2}$. Then

$$\sigma(Tu, u_N) \leq \sigma(Tu, Tu_N) + \sigma(Tu_N, u_N)$$

$$\leq \sigma(u, u_N) - \varphi(\sigma(u, u_N)) + \sigma(u_{N+1}, u_N) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Now we assume that $\frac{\varepsilon}{2} < \sigma(u, u_N) \leq \varepsilon$. Then $\varphi(\sigma(u, u_N)) \geq \varphi\left(\frac{\varepsilon}{2}\right)$. So, from the above, we have

$$\sigma(Tu, u_N) \leq \sigma(u, u_N) - \varphi(\sigma(u, u_N)) + \sigma(u_{N+1}, u_N)$$

$$\leq \sigma(u, u_N) - \varphi\left(\frac{\varepsilon}{2}\right) + \varphi\left(\frac{\varepsilon}{2}\right)$$

$$= \sigma(u, u_N) \leq \varepsilon .$$

Since $\sigma(u_{N+1}, u_N) \leq \varepsilon$, then from above, we determine that $\sigma(u_n, u_N) \leq \varepsilon$ for every n ≥ N. As

ε > 0 is arbitrary, we have $\lim_{m,n\to\infty} \sigma(u_m, u_n) = 0$ and so $\{u_n\}$ is a Cauchy sequence u. As U is complete, so by completeness of U, there is some

w ∈ U such that $\lim_{n\to\infty} u_n = w$, i.e. ,

$$\lim_{n\to\infty} \sigma(u_n, w) = \sigma(w, w) = \lim_{m,n\to\infty} \sigma(Tu_n, Tu_m) = 0. \qquad (3.1.2)$$

As

$$\sigma(u_{n+1}, Tw) = \sigma(Tu_n, Tw) \leq \sigma(u_n, w) - \varphi(\sigma(u_n, w)) \qquad (3.1.3)$$

and φ is continuous, then using (3.1.2) and (3.1.3), we get

$$\lim_{n\to\infty} \sigma(u_n, Tw) = 0. \qquad (3.1.4)$$

Since

$\sigma(w, Tw) \leq \sigma(u_n, w) + \sigma(u_n, Tw)$

then by (3.1.2) and (3.1.4), we conclude that $\sigma(w, Tw) = 0$ and so $Tw = w$.

To show the uniqueness, let z be another fixed point of T, i.e., $Tz = z$. Then

$\sigma(w, z) = \sigma(Tw, Tz) \le \sigma(w, z) - \varphi(\sigma(w, z))$,

which gives us $\varphi(\sigma(w, z)) = 0$ and so $w = z$.

In 2010, the theorems which Mlaiki et al. [3] introduced, associated with rectangular metric-like spaces, one of them is as follows:

Theorem 3.2[3]. Let (U, ρ_r) be a ρ_r – complete rectangular metric-like space, and T a self mapping on U. If there exists $0 < k < 1$ such that

$$\rho_r(Tu, Tv) \le k\,\rho_r(u, v) \quad \text{for all } u, v \in U. \tag{3.2.1}$$

Then T has a unique fixed point w in U, where $\rho_r(w, w) = 0$.

Proof: To prove the uniqueness of the fixed point, we show two things: First the existence and second the uniqueness of the fixed point.

Existence of the fixed point:

For this, let $u_0 \in U$ and define the sequence $\{u_n\}$ by

$$u_1 = Tu_0,\ u_2 = Tu_1 = T^2u_0,\ \dots,\ u_n = Tu_{n-1} = T^nu_0,\ \dots \tag{3.2.2}$$

Consider, if ∃ a natural number n such that $\rho_r(u_n, u_{n+1}) = 0$, then $u_n = u_{n+1} = Tu_n$ (using 3.2.2), which give that u_n is a fixed point of T and we get the result.

Also, if $u_n = u_{n+1}$, for some n, then u_n is the fixed point of T and then also we get the desired result.

So, we can assume that $\rho_r(u_n, u_{n+1}) > 0$, and $u_n \ne u_{n+1}$ for all n.

Now to show the existence of the fixed point, consider the following notations:

$\rho_n = \rho_r(u_n, u_{n+1})$, $\rho_n^* = \rho_r(u_n, u_{n+2})$ and $\rho_n' = \rho_r(u_n, u_n)$

Hence,

$$\begin{aligned}\rho_n' = \rho_r(u_n, u_n) &= \rho_r(Tu_{n-1}, Tu_{n-1})\\ &\le k\,\rho_r(u_{n-1}, u_{n-1}) \quad \text{(using 3.2.1)}\\ &= k\,\rho_{n-1}'\\ &\le \dots\\ &\le k^n\,\rho_0'\end{aligned}$$

i.e. $$\rho_n' \le k^n\,\rho_0' \tag{3.2.3}$$

Taking $n \to \infty$ in R.H.S. of (3.2.3), we get $\rho_n' \to 0$.

Thereby, $\lim_{n\to\infty} \rho_n' = 0$.

Also, by using (3.2.1), we obtain

$$\begin{aligned}\rho_n = \rho_r(u_n, u_{n+1}) = \rho_r(Tu_{n-1}, Tu_n) \le k\,\rho_r(u_{n-1}, u_n) &= k.\,\rho_{n-1}\\ &\le k^2\,\rho_{n-2}\\ &\le \dots\\ &\le k^n\,\rho_0\end{aligned}$$

Therefore,

$$\rho_n \le k^n\,\rho_0 \tag{3.2.4}$$

Similarly, we can have that

$$\rho_n^* \le k^n\,\rho_0^* \tag{3.2.5}$$

Now, if we have $u_0 = u_n$ for some $n > 0$, then

$$\begin{aligned}\rho_0 &= \rho_r(u_0, Tu_0)\\ &= \rho_r(u_n, Tu_n)\\ &= \rho_n\\ &\le k^n\,\rho_0\end{aligned}$$

and from this we arrive at a contradiction as $0 < k < 1$.

Thus, in this case we have $\rho_0 = 0$ and that is $u_0 = u_1$, therefore x_0 is a fixed point of T. Thus, we may assume now that $u_n \ne u_m$ for all natural numbers $n \ne m$.

Now, next we claim that $\rho_r(u_n, u_{n+p}) \to 0$ as $n, p \to \infty$. To prove the claim, we need to consider the following two cases:

Case 1: p is odd i.e. $p = 2m+1$

Hence, by (3.2.1) and (3.2.4), we have

$$\begin{aligned}\rho_r(u_n, u_{n+2m+1}) &\le \rho_r(u_n, u_{n+1}) + \rho_r(u_{n+1}, u_{n+2}) + \rho_r(u_{n+2}, u_{n+2m+1})\\ &\le \dots\\ &\le \rho_r(u_n, u_{n+1}) + \rho_r(u_{n+1}, u_{n+2}) + \dots + \rho_r(u_{n+2m}, u_{n+2m+1})\\ &\le k^n\,\rho_0 + k^{n+1}\,\rho_0 + \dots + k^{n+2m}\,\rho_0\\ &= k^n\,\rho_0\,[1 + k + k^2 + \dots + k^{2m}]\\ &= k^n\,\rho_0\left(\frac{1-k^{2m+1}}{1-k}\right).\end{aligned}$$

Taking the limit in above inequality, we obtain

$$\rho_r(u_n, u_{n+2m+1}) \to 0 \text{ as } n, m \to \infty$$

Case 2: p is even, i.e., $p = 2m$. Hence, from (3.2.1), (3.2.4) and (3.2.5), we have

$$\begin{aligned}\rho_r(u_n, u_{n+2m}) &\le \rho_r(u_n, u_{n+1}) + \rho_r(u_{n+1}, u_{n+2}) + \rho_r(u_{n+2}, u_{n+2m})\\ &\le \dots\\ &\le \rho_r(u_n, u_{n+1}) + \rho_r(u_{n+1}, u_{n+2}) + \dots + \rho_r(u_{n+2m-3}, u_{n+2m-2})\\ &\quad + \rho_r(u_{n+2m-2}, u_{n+2m})\\ &\le k^n\,\rho_0 + k^{n+1}\,\rho_0 + \dots + k^{n+2m-3}\,\rho_0 + k^{n+2m-2}\,\rho_0^*\end{aligned}$$

Using the fact that $0 < k < 1$ and taking the limit in above inequality we obtain:

$$\rho_r(u_n, u_{n+2m}) \to 0 \text{ as } n, m \to \infty$$

Thus, $\{u_n\}$ is a ρ_r – Cauchy sequence. Since (X, ρ) is a ρ_r – complete rectangular metric-like space, we deduce that $\{u_n\}$ converges to some $w \in U$, such that

$$\lim_{n\to\infty} \rho_r(u_n, u_n) = \rho_r(w, w)$$

On the other hand, we have

$$0 = \lim_{n\to\infty} \rho_n' = \lim_{n\to\infty} \rho_r(u_n, u_n) = \rho_r(w, w)$$

Hence, $\rho_r(w, w) = 0$.

Now,

$$\rho_r(w, Tw) \leq \rho_r(w, u_n) + \rho_r(u_n, u_{n+1}) + \rho_r(u_{n+1}, Tw)$$
$$\leq \rho_r(w, u_n) + \rho_r(u_n, u_{n+1}) + k\, \rho_r(u_n, w)$$

Here taking limit as $n \to \infty$ and using above cases, we get that $\rho_r(w, Tw) = 0$ and using definition we have, $Tw = w$ i.e. w is the fixed point of T.

Hence, T has a fixed point in U.

Uniqueness of the fixed point:

Assume that T has two fixed points, say x and y, hence

$$\rho_r(x, y) = \rho_r(Tx, Ty) \leq k\, \rho_r(x, y) < \rho_r(x, y)$$

Thus, $\rho_r(x, y) = 0 \Rightarrow x = y$, which shows the uniqueness of the fixed point.

Hence the theorem.

4. CONCLUSION

As we can conclude from this paper that the main difference between one type of metric-like space to another is only of basic properties as shown in their definitions. Based on the difference in properties we name metric-like spaces as of different types. And then based on these metric-like spaces, we derive various fixed point results related to kind of map involved, such as either of single valued map or multivalued map.

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