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Exact Solutions of (2+1) Dimensional KDV - Burgers Equation with Damping Term

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Abstract:
Homogeneous Balance Method is easy to apply and always yield a special exact solutions of nonlinear Partial Differential Equations. Wang showed the Homogeneous Balance Method is powerful for finding analytic solitary wave solutions of Partial Differential Equations. The idea is the highest nonlinear term partially balanced with the highest derivative term. In this paper, the solitary wave solution of nonlinear Partial Differential Equations has been obtained by this method. The exact solutions of (2+1) dimensional Partial Differential Equation is exponential functions besides the solitary wave solution. Based on the backlund transformations, exact solutions are also obtained.

Keywords: Nonlinear equation, Homogeneous balance method, solitary wave solution.

1. INTRODUCTION

The exact travelling wave solutions of nonlinear evolution equations play an important role in the study of nonlinear physical phenomena. In recent years, the homogeneous balance method has been widely applied to derive the nonlinear transformation and exact solutions.Nonlinear wave phenomena appears in various scientific and engineering fields. The homogeneous balance method (HBM) is introduced by Mingliang Wang [1] to obtain solitary wave solutions of variant Boussinesq equations. Subsequently the same method is applied to derive solitons of the compound KdV equation

$$u_t + puu_x + qu^2u_x - su_{xxx} = 0, \tag{1}$$

the mKdV-Burgers equation

$$u_t + qu^2u_x + ru_{xx} - su_{xxx} = 0, \tag{2}$$

the KdV-Burgers equation

$$u_t + puu_x + ru_{xx} - su_{xxx} = 0, \tag{3}$$

the mKdV equation

$$u_t + qu^2u_x - su_{xxx} = 0, \tag{4}$$

the KdV equation

$$u_t + puu_x - su_{xxx} = 0, \tag{5}$$

and, in general, the compound KdV-Burgers equation

$$u_t + puu_x + qu^2u_x + ru_{xx} - su_{xxx} = 0, \tag{6}$$

by Mingliang Wang [2]; the solitons of the approximate equations for long waves (Whitham [3])

$$u_t - uu_x - v_x + \frac{1}{2}u_{xx} = 0, \tag{7}$$

$$v_t - (uv)_x - \frac{1}{2}v_{xx} = 0, \tag{8}$$

the coupled system of KdV equations

$$u_t + 6\alpha uu_x - 6vv_x + \alpha u_{xxx} = 0, \tag{9}$$

$$v_t + 3\alpha uv_x + \alpha v_{xxx} = 0, \tag{10}$$

and the dispersive long wave equations in $2+1$ dimensions

$$u_{yt} + \eta_{xx} + \frac{1}{2}(u^2)_{xy} = 0, \tag{11}$$

$$\eta_t + (u\eta + u + u_{xy})_x = 0, \tag{12}$$

by Mingliang Wang, Yubin Zhou, Zhibin Li [4]. Lei Yang, Zhengang Zhu and Yinghai Wang [5] proposed a generalized homogeneous balance method and used it to determine the solitary wave solutions of the Boussinesq

$$u_{tt} - u_{xx} - a(u^2)_{xx} + bu_{xxxx} = 0. \tag{13}$$

Recently Engui Fan [6] found the Bäcklund transformation and similarity reductions of the general variable coefficient KdV equation

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0, \tag{14}$$

bases on the idea of the homogeneous balance method.

Homogeneous Balance method yields special exact solutions and solitons for nonlinear partial differential equations (Biao Li, Yong Chen and Hongqing Zhang [7], Feng [8], Lei Yang.et.al [9]. Recently Engui Fan [6] interpreted the transformation used in HBM as an auto-Backlund transformation as it involved first order derivatives. To be precide , any solution u of a pde $L[u] = 0$ is written in term of w and w_x, and a finite set of pdes are derived for w. In the present paper we relate u to w, w_x and w_{xx} and call the relation as a higher order BT.

In the present paper we use the homogenepous balance method to solve the $(2+1)$-dimensional equation

$$u_t + u^pu_x + \alpha u + \beta u^q - u_{xx} + \gamma u_{xxx} + u_{yy} = 0. \tag{15}$$

2. EXPONENTIAL SOLUTIONS OF EQUATION (16)

Specially we consider the case $p = 1, q = 2$, equation (15) reduces to

$$u_t + uu_x + \alpha u + \beta u^2 - u_{xx} + \gamma u_{xxx} + u_{yy} = 0. \tag{16}$$

In order that the highest nonlinear term and the highest derivative term βu^2 ,γu_{xxx} are balanced, we suppose that the solution of (16) is of the form

$$\begin{aligned} u(x,t,y) &= f'''w_x^3 + 3f''w_x w_{xx} + f'w_{xxx} + f''w_x^2 + f'w_{xx} + af'w_x + b, \\ w &= w(x,t,y), \end{aligned} \tag{17}$$

where the functions f and w as well as the constants a and b are to be determined. If follows from (17) that

$$\begin{aligned} u_t &= f^{IV}w_t w_x^3 + 3f'''w_{xt}w_x^2 + 3f'''w_t w_x w_{xx} + f'''w_t w_x^2 + 3f''w_{xt}w_{xx} + 3f''w_x w_{xxt} \\ &\quad + f''w_t w_{xxx} + 2f''w_{xt}w_x + f''w_t w_{xx} + f'w_{xxt} + f'w_{xxxt} + af''w_t w_x + af'w_{xt}, \end{aligned} \tag{18}$$

$$\begin{aligned} uu_x &= f^{IV}f'''w_x^7 + 6f'''^2 w_x^5 w_{xx} + 3f''f'''w_x^3 w_{xx}^2 + 4f''f'''w_x^4 w_{xxx} + f'f'''w_x^3 w_{xxxx} + f'''^2 w_x^6 \\ &\quad + 3f''f'''w_x^4 w_{xx} + f'f'''w_x^3 w_{xxx} + af''f'''w_x^5 + af'f'''w_x^3 w_{xx} + 3\left[f^{IV}f'''w_x^5 w_{xx} + 6f''f'''w_x^3\right. \\ &\quad \left. w_{xx}^2 + 3f''^2 w_x w_{xx}^3 + 4f''^2 w_x^2 w_{xx} w_{xxx} + f'f''w_x w_{xx} w_{xxx} + af''^2 w_x^3 w_{xx} + af'f''w_x w_{xx}^2\right] \\ &\quad + f'f^{IV}w_x^4 w_{xxx} + 6f'f'''w_x^2 w_{xx} w_{xxx} + 4f'f''w_x w_{xxx}^2 + f'^2 w_{xxx} w_{xxxx} + f'f'''w_x^3 w_{xxx} \\ &\quad + 3f'f''w_x w_{xx} w_{xxx} + f'^2 w_{xxx}^2 + af'f''w_x^2 w_{xxx} + af''^2 w_{xx} w_{xxx} + f''f^{IV}w_x^6 + 6f''f'''w_x^4 w_{xx} \\ &\quad + 3f''^2 w_x^2 w_{xx}^2 + 4f''^2 w_x^3 w_{xxx} + f'f''w_x^2 w_{xxxx} + f''f'''w_x^5 + 3f''^2 w_x^3 w_{xx} + f'f''w_x^2 w_{xxx} + af''^2 w_x^4 \\ &\quad + af'f''w_x^2 w_{xx} + f'f^{IV}w_x^4 w_{xx} + 6f'f'''w_x^2 w_{xx} w_{xx}^2 + 3f'f''w_{xx}^3 + 4f'f''w_x w_{xx} w_{xxx} \\ &\quad + f'^2 w_{xx} w_{xxxx} + f'f'''w_x^3 w_{xx} + 3f'f''w_x w_{xx}^2 + f'^2 w_{xx} w_{xxx} + af'f''w_x^2 w_{xx} + af'^2 w_{xx}^2 \\ &\quad + a\left[f'f^{IV}w_x^5 + 6f'f'''w_x^3 w_{xx} + 3f'f''w_x w_{xx}^2 + 4f'f''w_x^2 w_{xxx} + f'^2 w_x w_{xxxx}\right. \\ &\quad \left. + f'f'''w_x^4 + 3f'f''w_x^2 w_{xx} + f'^2 w_x w_{xxx} + af'f''w_x^3 + af'^2 w_x w_{xx}\right] + b\left[f^{IV}w_x^4 + 6f'''w_x^2 w_{xx}\right. \\ &\quad \left. + 3f''w_{xx}^2 + 4f''w_x w_{xxx} + f'w_{xxxx} + f'''w_x^3 + 3f''w_x w_{xx} + f'w_{xxx} + af''w_x^2 + af'w_{xx}\right], \end{aligned} \tag{19}$$

$$\alpha u = \alpha\left[f'''w_x^3 + 3f''w_x w_{xx} + f'w_{xxx} + f''w_x^2 + f'w_{xx} + af'w_x + b\right], \tag{20}$$

$$\begin{aligned} \beta u^2 &= \beta\Big[f'''^2 w_x^6 + 9f''^2 w_x^2 w_{xx}^2 + f'^2 w_{xxx}^2 f''^2 w_x^4 + f'^2 w_{xx}^2 + a^2 f'^2 w_x^2 + b^2 \\ &\quad + 2\big(3f''f'''w_x^4 w_{xx} + f'f'''w_x^3 w_{xxx} + f''f'''w_x^5 + f'f'''w_x^3 w_{xx} \\ &\quad + af'f'''w_x^4 + bf'''w_x^3 + 3f'f''w_x w_{xx}^2 + 3af'f''w_x^2 w_{xxx} + 3bf''w_x w_{xx} + f'f''w_x^2 w_{xxx} + f''^2 \\ &\quad w_{xx} w_{xxx} + af'^2 w_x w_{xxx} + bf'w_{xxx} + f'f''w_x^2 w_{xx} + af'f''w_x^3 + bf''w_x^2 + af'^2 w_x w_{xx} \\ &\quad + bf'w_{xxx} + f'f''w_x^2 w_{xx} + af'f''w_x^3 + bf''w_x^2 + af'^2 w_x w_{xx} + bf'w_{xx} + abf'w_x\big)\Big] \end{aligned} \tag{21}$$

$$\begin{aligned} -u_{xx} &= -\left[f^V w_x^5 + 10f^{IV}w_x^3 w_{xx} + 15f'''w_x w_{xx}^2 + 10f'''w_x^2 w_{xxx} + 10f''w_{xx} w_{xxx} + 5f''w_x w_{xxxx}\right. \\ &\quad + f'w_{xxxxx} + af^{IV}w_x^4 + 6f'''w_x^2 w_{xx} + 3f''w_{xx}^2 + 4f''w_x w_{xxx} + f'w_{xxxx} + af'''w_x^3 \\ &\quad \left. + 3af''w_x w_{xx} + af'w_{xxx}\right] \end{aligned} \tag{22}$$

$$\begin{aligned} \gamma u_{xxx} &= \gamma\left[f^{VI}w_x^6 + 5f^V w_x^4 w_{xx} + 10f^{IV}w_x^4 w_{xx} + 30f^{IV}w_x^2 w_{xx}^2 + 10f^{IV}w_x^3 w_{xxx} + 15f^{IV}w_x^2 w_{xx}^2\right. \\ &\quad + 15f'''w_{xx}^3 + 30f'''w_x w_{xx} w_{xxx} + 10f^{IV}w_x^3 w_{xxx} w_{xxx} + 30f'''w_x w_{xx} w_{xxx} + 10f'''w_x^2 w_{xxxx} \\ &\quad + 15f''w_{xxx}^2 + 15f''w_{xx} w_{xxxx} + 6f''w_x w_{xxxxx} + f'w_{xxxxxx} + f^V w_x^5 + 4f^{IV}w_x^3 w_{xx} + 6f^{IV}w_x^3 w_{xx} \\ &\quad + 6f'''w_x^2 w_{xx} + 12f'''w_x w_{xx}^2 + 3f''w_x w_{xx}^2 + 6f''w_{xx} w_{xxx} + 4f'''w_x^2 w_{xxx} \\ &\quad + 4f''w_{xx} w_{xxx} + 5f''w_x w_{xxxx} + f'w_{xxxxxx} + f^V w_x^5 + f'w_{xxxxx} + af^{IV}w_x^4 \\ &\quad \left. + 6af'''w_x^2 w_{xx} + 3af''w_{xx}^2 + 4af''w_x w_{xxx} + af'w_{xxxx}\right] \end{aligned} \tag{23}$$

$$\begin{aligned} u_{yy} &= f^V w_y^2 w_x^3 + f^{IV}w_{yy}w_x^3 + 3f^{IV}w_y w_x^2 w_{xy} + 3f'''w_y w_x^2 w_{xy} + 6f'''w_x w_{xy}^2 \\ &\quad + 3f'''w_x^2 w_{xyy} + 3f^{IV}w_y^2 w_{xx} w_x + 3f'''w_{yy} w_x w_{xx} + 3f'''w_y w_{xy} w_{xx} + 6f'''w_y w_{xxy} w_x \\ &\quad + 3f'''w_y w_{xy} w_{xx} + 3f''w_{xyy} w_{xx} + 6f''w_{xy} w_{xxy} + 3f''w_x w_{xxyy} + f'''w_y^2 w_{xxx} \\ &\quad + f''w_{yy} w_{xxx} + f''w_y w_{xxxy} w_x + f'w_y w_{xxxy} + f'w_{xxxyy} + f'''w_y^2 w_x^2 + f''w_{yy} w_x^2 \\ &\quad + 2f''w_x w_y w_{xy} + 2f'''w_x w_y w_{xy} + 2f''w_{xy}^2 + 2f''w_x w_{xyy} + f'''w_y^2 w_{xx} + f''w_{yy} w_{xx} \\ &\quad + 2f''w_y w_{xyy} + f'w_{xxyy} + af'''w_y^2 w_x + af''w_{yy} w_x + 2af''w_{xy} w_y + af'w_{xyy}. \end{aligned} \tag{24}$$

First collecting the terms with w_x^7 in (19) and setting its coefficient to zero, we obtain $f'''f^{IV} = 0$. Now we take

$$f''' = 0. \tag{25}$$

Then the solution of (25) is

$$f = \frac{c_1}{2}w^2 + c_2 w + c_3. \tag{26}$$

The above transformation yields

$$\begin{aligned} f' &= c_1 w + c_2, \\ f'' &= c_1, \end{aligned} \tag{27}$$

thereby

$$\begin{aligned} f''^2 &= c_1^2, \\ f'^2 &= 2c_1 f + c_4, \quad c_4 = c_2^2 - 2c_1c_3, \\ f'f'' &= c_1 f'. \end{aligned} \tag{28}$$

Using the above results equations (18)-(24) reduces to

$$\begin{aligned} u_t &= 3f''w_x w_{xxt} + f''w_t w_{xxx} + 2f''w_{xt}w_x + f''w_t w_{xx} + f'w_{xxt} + f'w_{xxxt} \\ &\quad +af''w_t w_x + af'w_{xt}, \end{aligned} \tag{29}$$

$$\begin{aligned} uu_x &= w_{xx}^2 + 3f''^2 w_x w_{xx}^3 + 4f''^2 w_x^2 w_{xx} w_{xxx} + f'f''w_x w_{xx} w_{xxx} + af''^2 w_x^3 w_{xx} + af'f''w_x w_{xx}^2 \\ &\quad +4f'f''w_x w_{xxx}^2 + f'^2 w_{xxx} w_{xxxx} + +3f'f''w_x w_{xx} w_{xxx} + f'^2 w_{xxx}^2 + af'f''w_x^2 w_{xxx} + af''^2 w_{xx} w_{xxx} \\ &\quad +3f''^2 w_x^2 w_{xx}^2 + 4f''^2 w_x^3 w_{xxx} + f'f''w_x^2 w_{xxxx} + 3f''^2 w_x^3 w_{xx} + f'f''w_x^2 w_{xxx} + af''^2 w_x^4 \\ &\quad +af'f''w_x^2 w_{xx} + 3f'f''w_{xx}^3 + 4f'f''w_x w_{xx} w_{xxx} + f'^2 w_{xx} w_{xxxx} + 3f'f''w_x w_{xx}^2 \\ &\quad +f'^2 w_{xx} w_{xxx} + af'f''w_x^2 w_{xx} + af'^2 w_{xx}^2 + 3f'f''w_x w_{xx}^2 + 4f'f''w_x^2 w_{xxx} \\ &\quad +f'^2 w_x w_{xxxx} + 3f'f''w_x^2 w_{xx} + f'^2 w_x w_{xxx} + af'f''w_x^3 + af'^2 w_x w_{xx} \\ &\quad +b\left[+3f''w_{xx}^2 + 4f''w_x w_{xxx} + f'w_{xxxx} + 3f''w_x w_{xx} + f'w_{xxx} + af''w_x^2 + af'w_{xx}\right], \end{aligned} \tag{30}$$

$$\alpha u = \alpha\left[3f''w_x w_{xx} + f'w_{xxx} + f''w_x^2 + f'w_{xx} + af'w_x + b\right], \tag{31}$$

$$\begin{aligned} \beta u^2 &= \beta\Big[9f''^2 w_x^2 w_{xx}^2 + f'^2 w_{xxx}^2 f''^2 w_x^4 + f'^2 w_{xx}^2 + a^2 f'^2 w_x^2 + b^2 \\ &\quad +2\Big(3f'f''w_x w_{xx}^2 + 3af'f''w_x^2 w_{xxx} + 3bf''w_x w_{xx} + f'f''w_x^2 w_{xxx} + f''^2 \\ &\quad w_{xx} w_{xxx} + af'^2 w_x w_{xxx} + bf'w_{xxx} + f'f''w_x^2 w_{xx} + af'f''w_x^3 + bf''w_x^2 + af'^2 w_x w_{xx} \\ &\quad +bf'w_{xxx} + f'f''w_x^2 w_{xx} + af'f''w_x^3 + bf''w_x^2 + af'^2 w_x w_{xx} + bf'w_{xx} + abf'w_x\Big)\Big] \end{aligned} \tag{32}$$

$$\begin{aligned} -u_{xx} &= -\left[10f''w_{xx}w_{xxx} + 5f''w_x w_{xxxx} + f'w_{xxxxx} + 3f''w_{xx}^2 \right. \\ &\quad \left. +4f''w_x w_{xxx} + f'w_{xxxx} + +3af''w_x w_{xx} + af'w_{xxx}\right] \end{aligned} \tag{33}$$

$$\begin{aligned} \gamma u_{xxx} &= \gamma\left[15f''w_{xxx}^2 + 15f''w_{xx}w_{xxxx} + 6f''w_x w_{xxxxx} + f'w_{xxxxxx} + 3f''w_x w_{xx}^2 + 6f''w_{xx}w_{xxx} \right. \\ &\quad +4f'''w_x^2 w_{xxx} + 4f''w_{xx}w_{xxx} + 5f''w_x w_{xxxx} + f'w_{xxxxxx} + f'w_{xxxxx} \\ &\quad \left. +3af''w_{xx}^2 + 4af''w_x w_{xxx} + af'w_{xxxx}\right] \end{aligned} \tag{34}$$

$$\begin{aligned} u_{yy} &= 3f''w_{xyy}w_{xx} + 6f''w_{xy}w_{xxy} + 3f''w_x w_{xxyy} + f''w_{yy}w_{xxx} + f''w_y w_{xxxy}w_x + f'w_y w_{xxxy} \\ &\quad +f'w_{xxxyy} + f''w_{yy}w_x^2 + 2f''w_x w_y w_{xy} + 2f''w_{xy}^2 + 2f''w_x w_{xyy} + +f''w_{yy}w_{xx} \\ &\quad +2f''w_y w_{xyy} + f'w_{xxyy} + +af''w_{yy}w_x + 2af''w_{xy}w_y + af'w_{xyy}. \end{aligned} \tag{35}$$

Next substituting(29)-(35) into the left hand side of (16) and using (28), we obtain

$$\begin{aligned}
u_t + uu_x + \alpha u + \beta u^2 - u_{xx} & \\
+\gamma u_{xxx} + u_{yy} \quad =\quad & [(3w_xw_{xxt} + w_tw_{xxx} + 2w_{xt}w_x + w_tw_{xx} + aw_tw_x) \\
& +b(w_{xx}^2 + 4w_xw_{xxx} + 3w_xw_{xx} + aw_x^2) + \alpha(3w_xw_{xx} + w_x^2) \\
& +\beta(6bw_xw_{xx} + 4bw_x^2) - 10w_{xx}w_{xxx} - 5w_xw_{xxxx} + \gamma(15w_{xxx}^2 \\
& +15w_{xx}w_{xxxx} + 6w_xw_{xxxxx} + 3w_xw_{xx}^2 + 10w_{xx}w_{xxx} + 4w_xw_{xxx} \\
& +5w_xw_{xxxx} + 4aw_xw_{xxx}) + 3(w_{xyy}w_{xx} + 6w_{xy}w_{xxy} + 3w_xw_{xxyy} \\
& +w_{yy}w_{xxx} + w_yw_{xxxy}w_x + w_{yy}w_x^2 + 2w_xw_yw_{xy} + 2w_{xy}^2 + 2w_xw_{xyy} \\
& +w_{yy}w_{xx} + 2w_yw_{xyy} + aw_{yy}w_x + 2aw_{xy}w_y)]\, f'' \\
& + \big[(w_{xxt} + w_{xxx} + aw_{xt}) + c_1(w_xw_{xx}w_{xxx} + aw_xw_{xx}^2 + 4w_xw_{xxx}^2 \\
& +7w_xw_{xx}w_{xxx} + aw_x^2w_{xxx} + w_x^2w_{xxxx} + 5w_x^2w_{xxx} + 2aw_x^2w_{xx} + 3w_{xx}^3 \\
& +6w_xw_{xx}^2 + aw_x^3) + b(w_{xxxx} + w_{xxx} + aw_{xx}) \\
& +\alpha(w_{xxx} + w_{xx} + aw_x) + \beta(4bw_{xxx} + 2bw_{xx} + abw_x) - (w_{xxxxx} + w_{xxxx} \\
& +aw_{xxx}) + \gamma(2w_{xxxxxx} + w_{xxxxx} + aw_{xxxx}) + (w_yw_{xxxy} + w_{xxxyy} + w_{xxyy} \\
& +aw_{xyy})\big]\, f' + 2c_1\left[w_{xxx}w_{xxxx} + w_{xxx}^2 + w_{xx}w_{xxxx} + w_{xx}w_{xxx} + aw_{xx}^2\right. \\
& +w_xw_{xxxx} + w_xw_{xxx} + aw_xw_{xx} + \beta(w_{xxx}^2 + w_{xx}^2 + a^2w_x^2 + aw_xw_{xxx} \\
& \left.+2aw_xw_{xx})\right] f + c_4\left[w_{xxx}w_{xxxx} + w_{xxx}^2 + w_{xx}w_{xxxx} + w_{xx}w_{xxx}\right. \\
& +aw_{xx}^2 + w_xw_{xxxx} + w_xw_{xxx} + aw_xw_{xx} + \beta(w_{xxx}^2 + w_{xx}^2 + 2aw_xw_{xxx} \\
& \left.+4aw_xw_{xx})\right] + \alpha b + b^2\beta + c_1^2(3w_xw_{xx}^3 + 4w_x^2w_{xx}w_{xxx} + aw_x^3w_{xx} \\
& +aw_{xx}w_{xxx} + 3w_x^2w_{xx}^2 + 4w_x^3w_{xxx} + 3w_x^3w_{xx} + aw_x^4 + 9\beta w_x^2w_{xx}^2 \\
& +2w_{xx}w_{xxx}). \qquad (36)
\end{aligned}$$

Setting the coefficients of f'', f', f andf^0 to zero respectively, we obtain the following equations for the determinationof $w(x, t, y)$

$$\begin{aligned}
[(3w_xw_{xxt} + w_tw_{xxx} + 2w_{xt}w_x + w_tw_{xx} + aw_tw_x) & \\
+b(w_{xx}^2 + 4w_xw_{xxx} + 3w_xw_{xx} + aw_x^2) + \alpha(3w_xw_{xx} + w_x^2) & \\
+\beta(6bw_xw_{xx} + 4bw_x^2) - 10w_{xx}w_{xxx} - 5w_xw_{xxxx} + \gamma(15w_{xxx}^2 & \\
+15w_{xx}w_{xxxx} + 6w_xw_{xxxxx} + 3w_xw_{xx}^2 + 10w_{xx}w_{xxx} + 4w_xw_{xxx} & \\
+5w_xw_{xxxx} + 4aw_xw_{xxx}) + 3(w_{xyy}w_{xx} + 6w_{xy}w_{xxy} + 3w_xw_{xxyy} & \\
+w_{yy}w_{xxx} + w_yw_{xxxy}w_x + w_{yy}w_x^2 + 2w_xw_yw_{xy} + 2w_{xy}^2 + 2w_xw_{xyy} & \\
+w_{yy}w_{xx} + 2w_yw_{xyy} + aw_{yy}w_x + 2aw_{xy}w_y)] &= 0, \qquad (37)
\end{aligned}$$

$$\begin{aligned}
\big[(w_{xxt} + w_{xxx} + aw_{xt}) + c_1(w_xw_{xx}w_{xxx} + aw_xw_{xx}^2 + 4w_xw_{xxx}^2 & \\
+7w_xw_{xx}w_{xxx} + aw_x^2w_{xxx} + w_x^2w_{xxxx} + 5w_x^2w_{xxx} + 2aw_x^2w_{xx} + 3w_{xx}^3 & \\
+6w_xw_{xx}^2 + aw_x^3) + b(w_{xxxx} + w_{xxx} + aw_{xx}) + \alpha(w_{xxx} + w_{xx} + aw_x) & \\
+\beta(4bw_{xxx} + 2bw_{xx} + abw_x) - (w_{xxxxx} + w_{xxxx} + aw_{xxx}) & \\
+\gamma(2w_{xxxxxx} + w_{xxxxx} + aw_{xxxx}) + (w_yw_{xxxy} + w_{xxxyy} + w_{xxyy} + aw_{xyy})\big] &= 0, \qquad (38) \\
\big[w_{xxx}w_{xxxx} + w_{xxx}^2 + w_{xx}w_{xxxx} + w_{xx}w_{xxx} + aw_{xx}^2 + w_xw_{xxxx} + w_xw_{xxx} & \\
+aw_xw_{xx} + \beta(w_{xxx}^2 + w_{xx}^2 + a^2w_x^2 + aw_xw_{xxx} + 2aw_xw_{xx})\big] &= 0, \qquad (39) \\
\big[w_{xxx}w_{xxxx} + w_{xxx}^2 + w_{xx}w_{xxxx} + w_{xx}w_{xxx} + aw_{xx}^2 + w_xw_{xxxx} & \\
+w_xw_{xxx} + aw_xw_{xx} + \beta(w_{xxx}^2 + w_{xx}^2 + 2aw_xw_{xxx} + 4aw_xw_{xx})\big] &= 0, \qquad (40) \\
(3w_xw_{xx}^3 + 4w_x^2w_{xx}w_{xxx} + aw_x^3w_{xx} + aw_{xx}w_{xxx} + 3w_x^2w_{xx}^2 & \\
+4w_x^3w_{xxx} + 3w_x^3w_{xx} + aw_x^4 + 9\beta w_x^2w_{xx}^2 + 2w_{xx}w_{xxx}) &= 0, \qquad (41) \\
\alpha b + b^2\beta &= 0. \qquad (42)
\end{aligned}$$

The solution of equations (37)-(42) is of the form

$$w(x, t) = w_0 + e^{k_1x + k_2t + k_3y + k_4}, \qquad (43)$$

where w_0, k_4 are arbitrary constants and k_1, k_2, k_3 are constants to be determined by the following equations

$$
\begin{aligned}
4k_1^3k_2 + 2k_1^2k_2 + ak_1k_2 + (5k_1^4 + 3k_1^3 + ak_1^2) + \alpha(3k_1^3 + k_1^2) + \beta(6bk_1^3 + 4bk_1^2) & \\
-15k_1^5 + \gamma(30k_1^6 + 21k_1^5 + 4k_1^4 + 4ak_1^4) + 3(11k_1^3k_3^2 + 7k_1^2k_3^2 + 2K_1k_3^3 + 3ak_1k_3^2 &= 0 \quad (44)\\
11k_1^3k_3^2 + 7k_1^2k_3^2 + 2k_1k_3^3 + 3ak_1k_3^2 &= 0, \quad (45)\\
k_1^2k_2 + k_1^3 + ak_1k_3 + b(k_1^4 + k_1^3 + ak_1^3) + \alpha(k_1^3 + k_1^2 + ak_1) & \\
+\beta(4bk_1^3 + 2bk_1^2 + abk_1) - (k_1^5 + k_1^4 + ak_1^3) + \gamma(2k_1^6 + k_1^5 + ak_1^4) &= 0, \quad (46)\\
2k_1^2 + k_1 + a &= 0, \quad (47)\\
12k_1^3 + ak_1^3 + 4k_1^4 + ak_1^2 + 11k_1^2 + 2ak_1 + a &= 0, \quad (48)\\
k_1^5 + 2k_1^4 + 2k_1^3 + ak_1^2 + k_1^2 + ak_1 + \beta(k_1^4 + k_1^2 + a^2 + ak_1^2 + 2ak_1) &= 0, \quad (49)\\
k_1^4 + 2k_1^3 + 2k_1^2 + ak_1^2 + ak_1 + k_1 + a + \beta(k_1^3 + k_1 + 2ak_1 + 4a) &= 0, \quad (50)\\
a &= -2, \quad (51)\\
\alpha b + b^2\beta &= 0. \quad (52)
\end{aligned}
$$

Substituting (43) and (27) in (17), we obtain a exact solution of equation (16)

$$
\begin{aligned}
u(x,y,t) = \; & 3c_1k_1^3e^{2(k_1x+k_2t+k_3y+k_4)} \\
& +k_1^3\left[c_1(w_0 + e^{k_1x+k_2t+k_3y+k_4}) + c_2\right]e^{(k_1x+k_2t+k_3y+k_4)} \\
& +c_1^2k_1^2e^{2(k_1x+k_2t+k_3y+k_4)} \\
& +k_1^2\left[c_1(w_0 + e^{k_1x+k_2t+k_3y+k_4}) + c_2\right]e^{(k_1x+k_2t+k_3y+k_4)} \\
& -2\left[c_1(w_0 + e^{k_1x+k_2t+k_3y+k_4}) + c_2\right]e^{(k_1x+k_2t+k_3y+k_4)} + b. \quad (53)
\end{aligned}
$$

3. RESULTS AND CONCLUTIONS

In this paper, we obtain the exact solution of (16) are

$$
\begin{aligned}
u(x,y,t) = \; & 3c_1k_1^3e^{2(k_1x+k_2t+k_3y+k_4)} \\
& +k_1^3\left[c_1(w_0 + e^{k_1x+k_2t+k_3y+k_4}) + c_2\right]e^{(k_1x+k_2t+k_3y+k_4)} \\
& +c_1^2k_1^2e^{2(k_1x+k_2t+k_3y+k_4)} \\
& +k_1^2\left[c_1(w_0 + e^{k_1x+k_2t+k_3y+k_4}) + c_2\right]e^{(k_1x+k_2t+k_3y+k_4)} \\
& -2\left[c_1(w_0 + e^{k_1x+k_2t+k_3y+k_4}) + c_2\right]e^{(k_1x+k_2t+k_3y+k_4)} + b. \quad (54)
\end{aligned}
$$

The exact solution of (16) are in terms of exponential functions.

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