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Star–In–Coloring Of Half Gear Graphs

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ABSTRACT
A proper coloring of a graph $G = (V, E)$ is a mapping $f: V \rightarrow N$ such that if $e = v_i v_j \epsilon E$, then $f(v_i) \neq f(v_j)$. A graph G is said to admit *star–in–coloring* if it satisfies the following conditions.

1. No path of length three (P_4) is bicolored.
2. If any path of length two (P_3) with end vertices are of the same color, then the edges of P_3 are directed towards the middle vertex.

In this paper, we have proved that the half gear graph, the splitting graph of half gear graph, Cartesian product of path and half gear graph, tensor product of path and half gear graph are star – in – coloring graphs. Further, we have given the general pattern of star – in – coloring, and star – in – chromatic number of each of these graphs.

KEYWORDS: Star–in–coloring; Splitting graph; Cartesian product of two graphs; Tensor product of two graphs.

AMS Subject Classification: 05C15, 05C20.

INTRODUCTION

The concept of acyclic coloring of graphs was introduced by Grunbaum [3], also he introduced the concept of star–coloring of graphs. The star–coloring of graphs have been investigated by Fertin, et al. [2] and Nesetril, et al. [4]. A digraph G is said to be in–coloring if any path of length two with end vertices of same color the edges are always oriented towards the middle vertex. Motivated through the concepts of star–coloring and in–coloring, Sudha and Kanniga [6,7] have introduced the star–in–coloring of graphs. Splitting graph $S(G)$ was defined by Sampathkumar and Walikar [5]. Sugumaran and Kasirajan [8] have found the lower and upper bounds of the star–in–chromatic number of the graphs such as cycle, regular cyclic, gear, fan, double fan, web and complete binary tree.

Let $G = (V, E)$ be a simple, connected digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E, each element of E is a directed edge. An orientation of a graph G is obtained by applying an orientation for each edge $e = v_i v_j \epsilon E$ either from v_i to v_j or v_j to v_i. We provide some of the basic definitions, which are necessary for our present investigations.

Definition 1.1 A *star–coloring* of a graph G is a proper coloring of the graph with the condition that no path of length three (P_4) is 2–colored.
An *n–star–coloring* of a graph G is a star–coloring of G using at most n colors.

Definition 1.2 An *in–coloring* of a graph G is a proper coloring of the graph G if there exist any path P_3 of length two with the end vertices are of the same color, then the edges of P_3 are oriented towards the central vertex.

Definition 1.3 A graph G is said to be *star–in–coloring* graph if the graph G admits both star – coloring and in – coloring.

Definition 1.4 The minimum number of colors required for the star–in–coloring of a graph G is called the *star–in–chromatic number* of G and is denoted by $\chi_{si}(G)$.

First we describe the star–in–coloring of a simple graph as shown in Figure 1. Let v_1, v_2, v_3, v_4 be the verticess, and let the number within the circle indicates, that particular color is assigned to that vertex.

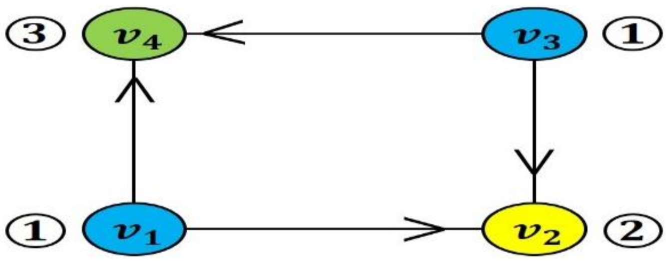

Figure 1: Star – in – coloring of Cycle C_4

In this graph we see that no two adjacent vertices have the same color, no path on four vertices is bicolored, each and every edge in a path of length two in which end vertices have same color are oriented towards the central vertex. Hence it is star–in–colored with orientation. Further the star–in–chromatic number of the above graph is 3.

Definintion 1.5 For any graph G, the *splitting graph* $S(G)$ is obtained by adding to each vertex v_i in G a new vertex v_i' such that v_i' is adjacent to the neighbours of v_i in G.

Definintion 1.6 The half gear graph HG_n is the graph obtained from the fan graph F_n by inserting a vertex between any two adjacent vertices in its path P_n.

Definintion 1.7 Suppose G and H are two graphs with $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(H) = \{v_1, v_2, \dots, v_n\}$. Then the Cartesian product $G \times H$ is the graph with vertex set $V(G \times H) = V(G) \times V(H) = \{(u_i, v_j) : u_i \in V(G), v_j \in V(H)\}$ and e is an edge of $G \times H$ iff $e = (u_i, v_j)(u_k, v_l)$, where either $i = k$ *and* $v_j v_l \in E(H)$ or $j = l$ *and* $u_i u_k \in E(G)$.

Definintion 1.8 The tensor product of two graphs G and H denoted by $G \otimes H$ has the vertex set $V(G \otimes H) = V(G) \times V(H)$ and the edge set $E(G \otimes H) = \{(u_i, v_j)(u_k, v_l) : u_i u_k \in E(G) \text{ and } v_j v_l \in E(H)\}$.
The tensor product of graphs was defined by Alfred North Whitehead, et al. [9] in their Principia Mathematica.

MAIN RESULTS

Theorem 1
The half gear graph HG_n admits star–in–coloring and its star–in–chromatic number is $\chi_{si}\ [HG_n] = 4$.

Proof
Let $V(HG_n) = \{u_i, v_j : 1 \le i \le n-1, 0 \le j \le n\}$, where v_0 is an apex vertex and let $E(HG_n) = \{v_i v_0, v_j u_j, u_k v_{k+1} : 1 \le i \le n, 1 \le j \le n-1, 1 \le k \le n-1\}$. Thus, the graph HG_n consists of $2n$ vertices and $3n-2$ edges.
We define a function $f: V(HG_n) \to \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon\ E(HG_n)$, as follows:

$$f(v_j) = 1, j = 1,2, \dots, n.$$

$$f(u_i) = \begin{cases} 2, if\ i \equiv 1\ (mod\ 2) \\ 3, if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(v_0) = 4.$$

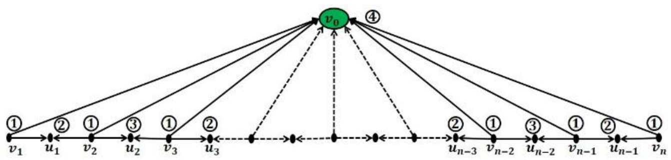

Figure 2: Star-in-coloring of $HG_n, n \equiv 0\ (mod\ 2)$

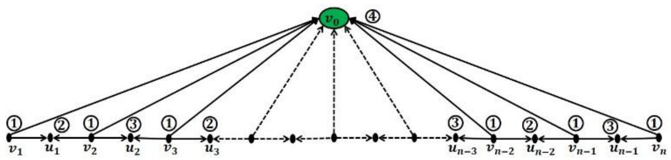

Figure 3: Star-in-coloring of $HG_n, n \equiv 1\ (mod\ 2)$

Hence the star–in–chromatic number of HG_n is $\chi_{si}\ [HG_n] = 4$.

Theorem 2
The splitting graph of half gear graph HG_n admits star–in–coloring and its star–in–chromatic number satisfies the inequality $5 \le \chi_{si}[S(HG_n)] \le 7$.

Proof
Let $G = S(HG_n)$ and let $V(G), E(G)$ be the vertex and edge set of G respectively. Then $|V(G)| = 4n$ and $|E(G)| = 3(3n-2)$. The vertices of HG_n are denoted by $v_0, v_1, u_1, v_2, \dots, v_{n-1}, u_{n-1}, v_n$ and the new vertices added in HG_n to obtain $S(HG_n)$ are $v'_0, v'_1, u'_1, v'_2, \dots, v'_{n-1}, u'_{n-1}, v'_n$.
We define a function $f: V(G) \to \{1, 2, 3, \dots\}$ such that $f(v_i) \neq f(v_j)$ if $v_i v_j \epsilon\ E(G)$, as follows:

Case 1: Let $n = 2$.

$$f(v_j) = f(v'_j) = 1, j = 1,2$$

$$f(u_1) = 2, f(u'_1) = 3 \text{ and}$$

$$f(v_0) = 4, f(v'_0) = 5$$

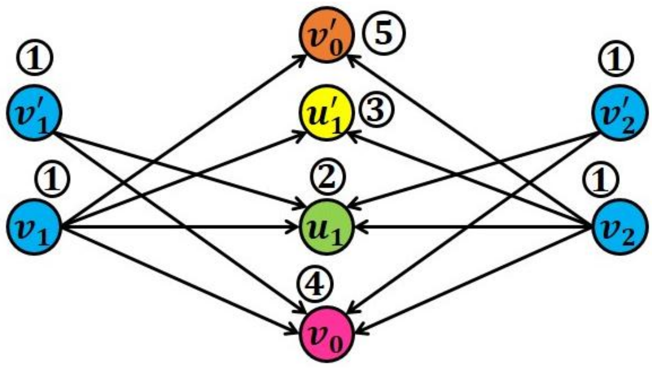

Figure 4: Star-in-coloring of $S(HG_2)$

Case 2: Let $n > 2$.

$$f(v_j) = f(v'_j) = 1, j = 1,2, \dots, n.$$

$$f(u_i) = \begin{cases} 2, if\ i \equiv 1\ (mod\ 2) \\ 3, if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(u'_i) = \begin{cases} 4, if\ i \equiv 1\ (mod\ 2) \\ 5, if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(v_0) = 6 \text{ and } f(v'_0) = 7.$$

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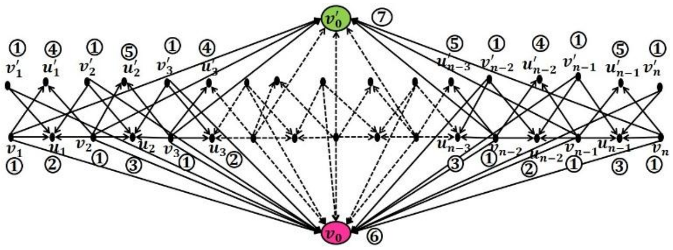

Figure 5: Star-in-coloring of $S(HG_n), n \equiv 1 \ (mod\ 2)$

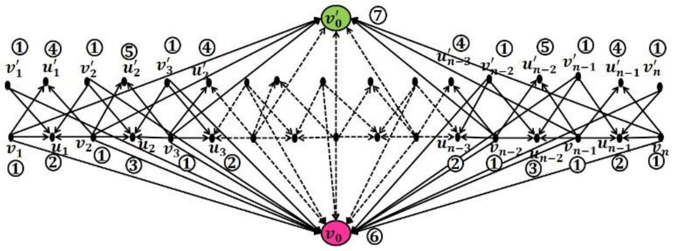

Figure 6: Star-in-coloring of $S(HG_n), n \equiv 0 \ (mod\ 2)$

According to Case 1 and Case 2, the star – in – chromatic number satisfies the inequality

$$5 \leq \chi_{si}[S(\ HG_n\ )] \leq 7.$$

Theorem 3
The Cartesian product of path graph and a half gear graph HG_n admits star–in–coloring and its star–in–chromatic number is given by

$$\begin{cases} 7 \leq \chi_{si}[P_m \times HG_n] \leq 8, m\ is\ odd\ and\ n \geq 4 \\ n+4 \leq \chi_{si}[P_m \times HG_n] \leq n+5, m\ is\ even \end{cases}$$

Proof
Consider a path graph P_m which consists of m vertices denoted by $l_1, l_2, \dots, l_m$ and $m-1$ edges and the half gear graph HG_n which consists of $2n$ vertices denoted by $v_0, v_1, u_1, v_2, \dots, v_{n-1}, u_{n-1}, v_n$ and $(3n-2)$ edges. The Cartesian product $P_m \times HG_n$ consists of $2mn$ vertices and $m(7n-8) - 6(n+2)$ edges.
Let V be the vertex set of $P_m \times HG_n$ and let E be the edge set of $P_m \times HG_n$. We define a function $f: V \rightarrow \{1, 2, 3, \dots\}$ such that $f(u_i v_j) \neq f(u_k v_l)$ if $(u_i v_j)(u_k v_l) \epsilon\ E$, as follows:

Case 1: Let m be an odd and $n \geq 4$.

Subcase 1.1: $For\ k \equiv 1\ (mod\ 4)$

$$f(l_k v_j) = 1, j = 1,2, \dots, n.$$

$$f(l_k u_i) = \begin{cases} 2, if\ i\ \equiv 1\ (mod\ 2) \\ 3, if\ i\ \equiv 0\ (mod\ 2) \end{cases}$$

Subcase 1.2: $For\ k \equiv 2\ (mod\ 4)$

$$f(l_k v_j) = \begin{cases} 4, if\ j\ \equiv 1\ (mod\ 2) \\ 5, if\ j\ \equiv 0\ (mod\ 2)\ and\ j > 0 \end{cases}$$

$$f(l_k u_i) = 1, i = 1,2, \dots, n-1.$$

Subcase 1.3: $For\ k \equiv 3\ (mod\ 4)$

$$f(l_k v_j) = 1, j = 1,2, \dots, n.$$

$$f(l_k u_i) = \begin{cases} 3, if\ i\ \equiv 1\ (mod\ 2) \\ 2, if\ i\ \equiv 0\ (mod\ 2) \end{cases}$$

Subcase 1.4: $For\ k \equiv 0\ (mod\ 4)$

$$f(l_k v_j) = \begin{cases} 5, if\ j\ \equiv 1\ (mod\ 2) \\ 4, if\ j\ \equiv 0\ (mod\ 2)\ and\ j > 0 \end{cases}$$

$$f(l_k u_i) = 1, i = 1,2, \dots, n-1.$$

Also

$$f(l_k v_0) = \begin{cases} 6, if\ k\ \equiv 1\ (mod\ 2) \\ 7, if\ k\ \equiv 2\ (mod\ 4) \\ 8, if\ k\ \equiv 0\ (mod\ 4) \end{cases}$$

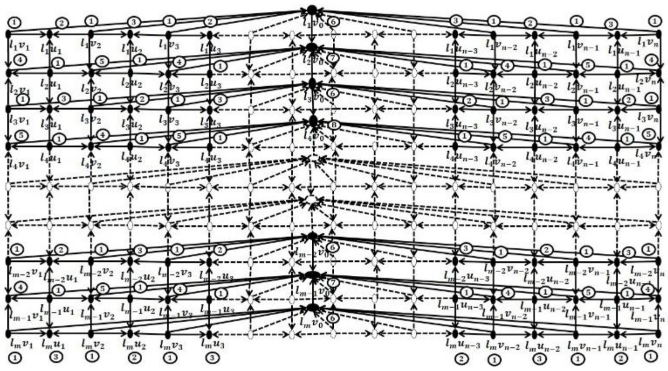

Figure 7: Star-in-coloring of $P_m \times HG_n, m \equiv 3\ (mod\ 4), n \equiv 3\ (mod\ 4)\ and\ n > 3.$

According to Case 1, the Cartesian product $P_m \times HG_n$ is star – in – colored and its star–in–chromatic number satisfies the inequality $7 \leq \chi_{si}[P_m \times HG_n] \leq 8, m\ is\ odd\ and\ n \geq 4.$

Case 2: Let m be an even positive integer.

Subcase 2.1: $For\ k \equiv 1\ (mod\ 4)$

$$f(l_k v_j) = 1, j = 1,2, \dots, n.$$

$$f(l_k u_i) = \begin{cases} 2, if\ i\ \equiv 1\ (mod\ 2) \\ 3, if\ i\ \equiv 0\ (mod\ 2) \end{cases}$$

Subcase 2.2: $For\ k \equiv 2\ (mod\ 4)$

$$f(l_k v_j) = j + 3, j = 1,2, \dots, n.$$

$$f(l_k u_i) = 1, i = 1,2, \dots, n-1.$$

Subcase 2.3: $For\ k \equiv 3\ (mod\ 4)$

$$f(l_k v_j) = 1, j = 1,2, \dots, n.$$

$$f(l_k u_i) = \begin{cases} 3, if\ i \equiv 1\ (mod\ 2) \\ 2, if\ i \equiv 0\ (mod\ 2) \end{cases}$$

Subcase 2.4: $For\ k \equiv 0\ (mod\ 4)$

$$f(l_k v_j) = \begin{cases} j+4, if\ j < n \\ 4, if\ j = n \end{cases}$$

$$f(l_k u_i) = 1, i = 1,2, \dots, n-1.$$

Also

$$f(l_k v_0) = \begin{cases} 1, if\ k \equiv 0\ (mod\ 2) \\ n+4, if\ k \equiv 1\ (mod\ 4) \\ n+5, if\ k \equiv 3\ (mod\ 4) \end{cases}$$

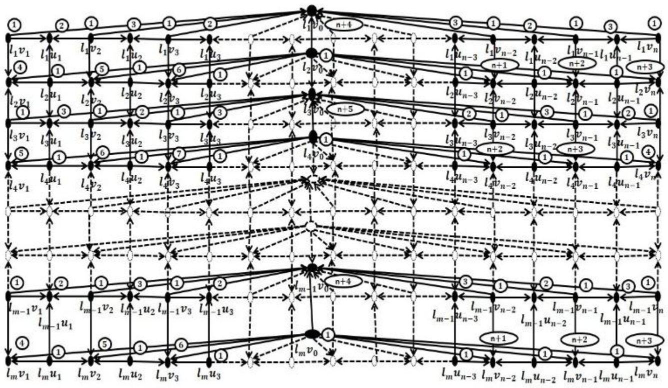

Figure 8: Star-in-coloring of $P_m \times HG_n, m \equiv 2\ (mod\ 4)\ and\ n \equiv 3\ (mod\ 4)$

According to Case 2, the Cartesian product $P_m \times HG_n$ is star – in – colored and its star–in–chromatic number satisfies the inequality $n + 4 \le \chi_{si}[P_m \times HG_n] \le n + 5, m$ is even.

Theorem 4

The tensor product of path graph and a half gear graph HG_n admits star–in–coloring and its star–in–chromatic number is given by

$$\chi_{si}\ [P_m \otimes HG_n] = \begin{cases} n+1, if\ m = 2,3 \\ 2n+1, if\ m \ge 4 \end{cases}$$

Proof

Consider a path graph P_m which consists of m vertices denoted by $l_1, l_2, \dots, l_m$ and $m-1$ edges and the half gear graph HG_n which consists of $2n$ vertices denoted by $v_0, v_1, u_1, v_2, \dots, v_{n-1}, u_{n-1}, v_n$ and $(3n-2)$ edges. The tensor product $P_m \otimes HG_n$ consists of $2mn$ vertices and $2(m-1)(3n-3)$ edges.

Let V be the vertex set of $P_m \otimes HG_n$ and let E be the edge set of $P_m \otimes HG_n$. We define a function $f: V \to \{1, 2, 3, \dots\}$ such that $f(u_i v_j) \ne f(u_k v_l)$ if $(u_i v_j)(u_k v_l) \epsilon E$, as follows:

Case 1: Let $m = 2,3$.

$$f(l_k v_j) = \begin{cases} 1, if\ k = 1,3 \\ 2, if\ k = 2\ and\ j = 0 \\ j+1, if\ k = 2\ and\ j > 0 \end{cases}$$

$$f(l_k u_i) = \begin{cases} 3, if\ k = 2\ and\ i \equiv 1\ (mod\ 2) \\ 4, if\ k = 2\ and\ i \equiv 0\ (mod\ 2) \end{cases}$$

Case 2: Let $m \ge 4$.

Subcase 2.1: $For\ k \equiv 1\ (mod\ 2)$

$$f(l_k v_j) = f(l_k u_i) = 1$$

Subcase 2.2: $For\ k \equiv 2\ (mod\ 4)$

$$f(l_k v_j) = \begin{cases} 2, if\ j = 0 \\ j+1, if\ j > 0 \end{cases}$$

$$f(l_k u_i) = \begin{cases} 3, if\ i \equiv 1\ (mod\ 2) \\ 4, if\ i \equiv 0\ (mod\ 2) \end{cases}$$

Subcase 2.3: $For\ k \equiv 0\ (mod\ 4)$

$$f(l_k v_j) = \begin{cases} n+2, if\ j = 0 \\ j+n+1, if\ j > 0 \end{cases}$$

$$f(l_k u_i) = \begin{cases} n+3, if\ i \equiv 1\ (mod\ 2) \\ n+4, if\ i \equiv 0\ (mod\ 2) \end{cases}$$

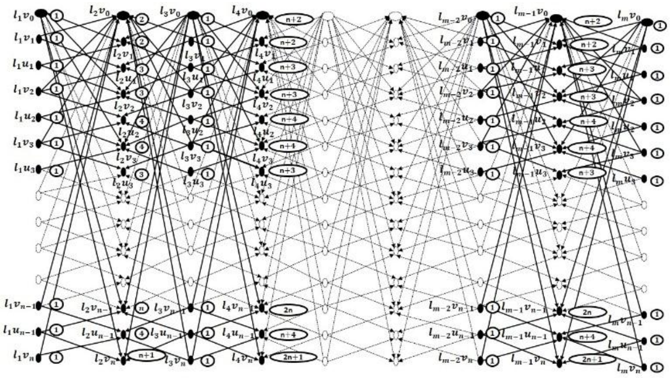

Figure 9: Star-in-coloring of $P_m \otimes HG_n$

According to Case 1 and Case 2, the tensor product $P_m \otimes HG_n$ is star – in – colored and its star–in–chromatic number is $\chi_{si}\ [P_m \otimes HG_n] = n+1$, when $m = 2,3$ and $\chi_{si}\ [P_m \otimes HG_n] = 2n+1$ when $m \ge 4$.

CONCLUSION

A graph which satisfies both star – coloring and in – coloring concepts is called a star – in – coloring graph. The minimum number of colors required for the star – in – coloring of a graph is called its star – in – chromatic number. In this paper, we have found the star – in – chromatic number of the graphs are as follows:

1. The star–in–chromatic number of HG_n is 4
2. $5 \le \chi_{si}[S(\ HG_n\ )] \le 7$.
3. $\begin{cases} 7 \le \chi_{si}[P_m \times HG_n] \le 8, m\ is\ odd\ and\ n \ge 4 \\ n+4 \le \chi_{si}[P_m \times HG_n] \le n+5, m\ is\ even \end{cases}$

4. $$\chi_{si}[P_m \otimes HG_n] = \begin{cases} n+1, if\ m = 2,3 \\ 2n+1, if\ m \geq 4 \end{cases}$$

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