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The Maximum Hub Degree Energy of Graphs

Veena Mathad[1] **and Anand**[*2]

[1]*Department of Studies in Mathematics, University of Mysore, Mysuru - 570 006, India.*

[2]*Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru - 570 002, India.*

Abstract
Let G be a graph of order n with vertices labeled as $v_1, v_2, ..., v_n$. Let d_{h_i} be the hub degree of the vertex v_i for $i = 1, 2, ...n$. In this paper, we introduce the Maximum hub degree matrix of G, which is the square matrix of order n whose ij^{th}- entry is equal to $max\{d_{h_i}, d_{h_j}\}$ if v_i is adjacent to v_j and zero otherwise. We further define Maximum hub degree energy and investigate its nature based on the eigenvalues of the Maximum hub degree matrix. In addition, we establish some bounds for the Maximum hub degree energy in terms of minimum hub degree, maximum hub degree and spectral radius of the Maximum hub degree matrix.

Keywords: Hub degree, Maximum hub degree matrix, Maximum hub degree eigenvalue.
2020 Mathematics Subject Classification: 05A05, 05C07, 05C31, 05C50, 15A18.

1 INTRODUCTION

All graphs considered here are simple, finite and undirected. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$ respectively while $|V(G)| = n$ and $|E(G)| = m$ are order and size of G respectively . The open neighborhood $N_G(v)$ of a vertex v consists of the set of vertices adjacent to v, that is, $N_G(v) = \{u \in V : uv \in E\}$. The degree of a vertex v, denoted by $d(v)$, is the cardinality of its neighborhood. By a pendant vertex we mean a vertex of degree one. A double star graph[1] $S_{n,m}$ is a graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their central vertices u and v.
The concept of hub number was introduced by Walsh[2], a subset $H \subseteq V(G)$ is a hub set of G if for any two vertices outside H there exist a path with all internal vertices in H(This includes the degenerate cases where the path consists of single edge uv or a single vertex u if $u = v$; such an H path is trivial). A hubset H of G is a minimal hub set of G if $H - \{v\}$ is not a hub set of G, for any $v \in H$. The minimum cardinality of a minimal hub set is hub number of G and is denoted by $h(G)$. Different variations of hub parameters have been studied in [3, 4, 5, 6].
The adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, ..., v_n$ is a $n \times n$ matrix (a_{ij}) such that $a_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise. The eigen values $\mu_1 .\mu_2, ..., \mu_n$ of a graph G are the eigen values of its adjacency matrix. The set of eigen values of the graph with their multiplicities is known as spectrum of the graph. In 1978, Gutman[7] defined the energy of a graph G as the the sum of absolute values of the eigenvalues of the graph G, denoted by $E(G)$ and is written as $E(G) = \sum_{i=1}^{n} |\mu_i|$. For standard terminology and notations related to Graph theory we follow [8] and for Algebra [9] respectively.

Definition 1.1. *[10] Hub degree of a vertex v in a graph G is the number of minimal hubsets containing v denoted by $d_h(v)$.*

Definition 1.2. *Maximum hub degree of a graph G denoted by $\Delta_h(G)$ is defined as:*
$\Delta_h(G) = max\ \{d_h(v) | v \in V(G)\}$

Definition 1.3. *Minimum hub degree of a graph G denoted by $\delta_h(G)$ is defined as:*
$\delta_h(G) = min\ \{d_h(v) | v \in V(G)\}$

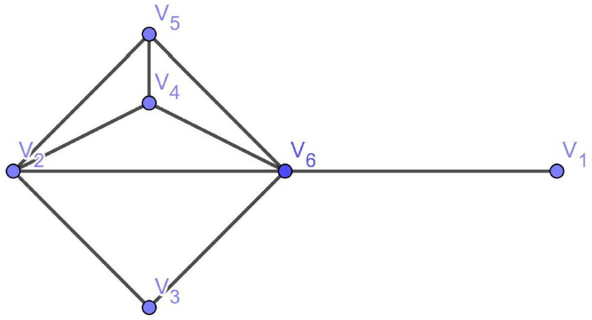
Definition 1.4. *The Maximum hub degree matrix $M^h(G)$ of a graph G of order n is defined as:*

$$M^h(G) = (a_{ij})_{n\times n} = \begin{cases} max\{d_{h_i}, d_{h_j}\}, & v_iv_j \in E(G) \\ 0, & otherwise \end{cases}$$

Definition 1.5. *The Maximum hub degree polynomial of a graph is the characteristic polynomial of the maximum hub degree matrix $M^h(G)$ and is defined as:*
$\phi(G : \mu) = det(\mu I - M^h(G))$
$= \mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + ... + c_n.$
where I denotes the identity matrix of order n.

The roots $\mu_1, \mu_2, ..., \mu_n$ assumed in non-increasing order of $\phi(G : \mu) = 0$ are the maximum hub degree eigen values of $M^h(G)$. Then the maximum hub degree energy of G is defined as $E^h_M(G) = \sum_{i=1}^{n} |\mu_i|$.
To illustrate this concept, we study the following examples.

Example 1.6. *Let G_1 be a graph in Figure 1.*



[*]Corresponding Author.

Minimal hub sets are:
$\{v_1, v_3\}, \{v_6\}, \{v_1, v_2\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}$.

$d_h(v_1) = 3$, $d_h(v_2) = 2$, $d_h(v_3) = 2$, $d_h(v_4) = 2$, $d_h(v_5) = 2$, $d_h(v_6) = 1$

Its Maximum Hub Degree Matrix is as follow:

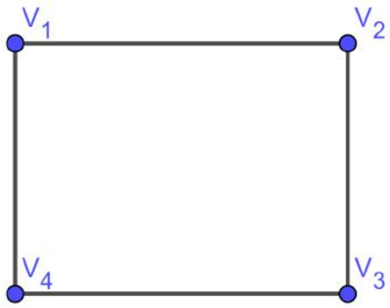
$$\begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 \\ 3 & 2 & 2 & 2 & 2 & 0 \end{pmatrix} \end{array}$$

Its Characteristic polynomial is : $\mu^6 - 41\mu^4 - 80\mu^3 + 128\mu^2 + 208\mu - 144 = 0$.

And its Eigen values are: $\mu_1 = 7.0184$, $\mu_2 = 1.4769$, $\mu_3 = 0.5821$,

$\mu_4 = -2$, $\mu_5 = -2.7713$, $\mu_6 = -4.3061$.

Example 1.7. *Let G_2 be a graph in Figure 2.*



Minimal hub sets are: $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$

$d_h(v_1) = 1, d_h(v_2) = 1, d_h(v_3) = 1, d_h(v_4) = 1$

Its Maximum Hub Degree Matrix is as follow:

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Its Characteristic polynomial is : $\mu^4 - 4\mu^2 = 0$.

And its Eigen values are: $\mu_1 = 2$, $\mu_2 = 0$, $\mu_3 = 0$, $\mu_4 = -2$

2 CHARACTERIZATION OF COEFFICIENTS OF MAXIMUM HUB DEGREE POLYNOMIAL

Theorem 2.1. *Consider the coefficient C_i of μ^{n-i}, $i = 0, 1, 2, ...$ in the characteristic polynomial of the maximum hub degree matrix $M^h(G)$.Then:*
1. $C_0 = 1$
2. C_1=Trace of $M^h(G)$= 0

3. $C_2 = \sum_{1 \le j < k \le n} \begin{vmatrix} 0 & d_{jk} \\ d_{kj} & 0 \end{vmatrix}$

4. $C_3 = -2 \sum_{\substack{\Delta v_i v_j v_k \\ d_h(v_i) \le d_h(v_j) \le d_h(v_k)}} [d_h(v_k)]^2 d_h(v_j)$

5. $C_n = det(M^h(G))$

Proof. We know that $\phi(G : \mu) = det(\mu I - M^h(G))$
$= c_0\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + ... + c_n$.
$= (-1)^n\mu^n + (-1)^{n-1}Tr(M^h(G))\mu^{n-1} + ... + det(M^h(G))$
and hence (1), (2) and (5) follows.
Now,

$$\begin{vmatrix} 0 & d_{jk} \\ d_{kj} & 0 \end{vmatrix} = \begin{cases} -[max\{d_{h_j}, d_{h_k}\}]^2, & v_j v_k \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

So, $C_2 = -\sum_{i=1}^{n} (a_{h_i} + b_{h_i}) d_{h_i}^2$

$= -\sum_{i=1}^{n} (a_{h_i} + b_{h_i}) [d_h(v_i)]^2$

where a_{h_i}: $\left|\{v_j \in N(v_i) | \, d_h(v_j) < d_h(v_i)\}\right|$

b_{h_i}: $\left|\{v_j \in N(v_i) | \, j > i \text{ and } d_h(v_j) = d_h(v_i)\}\right|$

Remark 2.2. *Note that $\sum_{i=1}^{n} (a_{h_i} + b_{h_i})$ denotes the number of edges in G.*

$$C_3 = (-1)^3 \sum_{1 \le i < j < k \le n} \begin{vmatrix} d_{ii} & d_{ij} & d_{ik} \\ d_{ji} & d_{jj} & d_{jk} \\ d_{ki} & d_{kj} & d_{kk} \end{vmatrix}$$

$$= -2 \sum_{1 \le i < j < k \le n} d_{ij} d_{jk} d_{ki}$$

$$= -2 \sum_{\substack{\Delta v_i v_j v_k \\ d_h(v_i) \le d_h(v_j) \le d_h(v_k)}} [d_h(v_k)]^2 d_h(v_j)$$

Remark 2.3. *(a) Number of terms in the above sum is equal to the number of triangles in the graph.*
(b) If there is no triangle in G, then $C_3 = 0$.

□

Theorem 2.4. *If $\mu_1, \mu_2, ..., \mu_n$ are maximum hub degree eigen values of a graph G then*
$\sum_{i=1}^{n} \mu_i^2 = -2C_2.$

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Proof. $\sum_{i=1}^{n} \mu_i^2$ = Trace of $(M^h(G))^2$

$$= \sum_{i=1}^{n} \left[\sum_{k=1}^{n} d_{ik} d_{ki} \right]$$

$$= 2 \sum_{i=1}^{n} (a_{h_i} + b_{h_i}) d_h^2(v_i)$$

$$= -2C_2$$ □

Example 2.5. *For the Graph G_1 in Fig 1, the coeffecient C_2 of μ^2 in the characteristic polynomial of the maximum degree matrix $M^h(G_1)$ is equal to*

$$C_2 = -\sum_{i=1}^{6} (a_{h_i} + b_{h_i}) d_h^2(v_i)$$

$= -[(1 + 0)9 + (1 + 3)4 + (1 + 0)4 + (1 + 1)4 + (1 + 0)4 + (0 + 0)1]$

$= -[9 + 16 + 4 + 8 + 4 + 0]$

$= -41$

Example 2.6. *For the Graph G_2 in Fig 2, the coeffecient C_2 of μ^2 in the characteristic polynomial of the maximum degree matrix $M^h(G_2)$ is equal to*

$$C_2 = -\sum_{i=1}^{4} (a_{h_i} + b_{h_i}) d_h^2(v_i)$$

$= -[(0 + 2)1 + (0 + 1)1 + (0 + 1)1 + (0 + 1)0]$

$= -[2 + 1 + 1 + 0]$

$= -4$

Example 2.7. *In figure 1,*

$$C_3 = -2 \sum_{\substack{\Delta v_i v_j v_k \\ d_h(v_i) \leq d_h(v_j) \leq d_h(v_k)}} [d_h(v_k)]^2 d_h(v_j)$$

$$= -2[2^2.2 + 2^2.2 + 2^2.2 + 2^2.2 + 2^2.2]$$

$$= -2[8 + 8 + 8 + 8 + 8]$$

$$= -80$$

Theorem 2.8. *If the maximum hub degree energy of a graph is rational, then it must be an even integer.*

Proof. Let $\mu_1, \mu_2, ..., \mu_n$ be the maximum hub degree eigenvalues of a graph G with n vertices. Then we have $\sum_{i=1}^{n} \mu_i = 0$. Let $\mu_1, \mu_2, ..., \mu_r$ be positive and $\mu_{r+1}, \mu_{r+2}, ..., \mu_n$ be non-positive. Then $E_M^h(G) = 2(\mu_1 + \mu_2 + ... + \mu_r)$.

Since $\mu_1, \mu_2, ..., \mu_r$ are algebraic numbers, so is their sum, and hence must be integer if $E_M^h(G)$ is rational. Thus $E_M^h(G)$ is an even positive integer if it is rational. □

2.1 Spectral moments of the Maximum hub degree matrix

In this section, We calculate $tr(M^h(G))^2$, $tr(M^h(G))^3$, where tr denotes the trace of the respective matrix. Denote by S_k the k^{th} spectral moment of the hub Zagreb Matrix $M^h(G)$, i.e $S_k = \sum_{i=1}^{n} (\mu_i)^k$ and note that $S_k = tr(M^h(G))^k$.

Theorem 2.9. *Let $\mu_1, \mu_2, ..., \mu_n$ be the maximum hub degree eigenvalues of $M^h(G)$. Then*

1. $\sum_{i=1}^{n} \mu_i = 0$

2. $\sum_{i=1}^{n} \mu_i^2 = 2 \sum_{i=1}^{n} (a_{h_i} + b_{h_i}) d_{h_i}^2$.

where, a_{h_i}: $\left| \{v_j \in N(v_i) | \; d_h(v_j) < d_h(v_i)\} \right|$

b_{h_i}: $\left| \{v_j \in N(v_i) | \; j > i \text{ and } d_h(v_j) = d_h(v_i)\} \right|$

3. $\sum_{i=1}^{n} \mu_i^3 = 6 \sum_{\substack{\Delta v_i v_j v_k \\ d_h(v_i) \leq d_h(v_j) \leq d_h(v_k)}} [d_h(v_k)]^2 d_h(v_j)$

Proof. The proof is the consequences of Newton's identity [11] and theorem 2.1 □

3 BOUNDS FOR THE MAXIMUM HUB DEGREE ENERGY

Theorem 3.1. *We have*

$$\sqrt{2 \sum_{i=1}^{n} (a_{h_i} + b_{h_i}) d_h^2(v_i) + n(n-1) p^{\frac{2}{n}}} \leq E_M^h(G) \leq \sqrt{2n \sum_{i=1}^{n} (a_{h_i} + b_{h_i}) d_h^2(v_i)}$$

Proof. We have, $E_M^{h^2}(G) = \left(\sum_{i=1}^{n} |\mu_i| \right)^2 = \sum_{i=1}^{n} |\mu_i|^2 + \sum_{i \neq j} |\mu_i||\mu_j|$

$$E_M^{h^2}(G) \geq 2 \sum_{i=1}^{n} (a_{h_i} + b_{h_i}) d_h^2(v_i) + n(n-1) p^{\frac{2}{n}},$$

Where $p = \prod_{i=1}^{n} |\mu_i|$, the last inequality is due to Theorem 2.4 and Arithmetic mean, Geometric mean inequality [12]. On employing Holder's inequality [12], we obtain

$$E_M^h(G) = \sum_{i=1}^{n} |\mu_i| \leq \sqrt{\sum_{i=1}^{n} |\mu_i|^2} . \sqrt{n}$$

$$= \sqrt{\sum_{i=1}^{n} 2n (a_{h_i} + b_{h_i}) d_h^2(v_i)}$$ □

Theorem 3.2. *Let G be a connected graph of order n and size m and let $\delta_h(G)$ and $\Delta_h(G)$ be the minimum hub degree and maximum hub degree of G respectively. Then*

$$\delta_h(G)\sqrt{2m} \leq E_M^h(G) \leq \Delta_h(G)\sqrt{2nm}$$

Proof. Consider the Cauchy - Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choosing $a_i = 1$ and $b_i = |\mu_i|$ and by theorem 2.9, we get

$$\begin{aligned}\left(E_M^h(G)\right)^2 &= \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \\ &\leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \mu_i^2\right) \\ &\leq n\left(2\sum_{i=1}^{n}(a_{h_i}+b_{h_i})d_h^2(v_i)\right).\end{aligned}$$

Since $d_h(v) \leq \Delta_h(G)$, for every $v \in V(G)$ and by Remark 2.2, $\sum_{i=1}^{n}(a_{h_i}+b_{h_i}) = m$,

it follows that $E_M^h(G) \leq \Delta_h(G)\sqrt{2nm}$

Now, since $\left(\sum_{i=1}^{n} |\mu_i|\right)^2 \geq \sum_{i=1}^{n} \mu_i^2$,

it follows that $\left(E_M^h(G)\right)^2 \geq 2\sum_{i=1}^{n}(a_{h_i}+b_{h_i})d_h^2(v_i)$.

Since $d_h(v) \geq \delta_h(G)$, for every $v \in V(G)$, then

$E_M^h(G) \geq \delta_h(G)\sqrt{2m}$. □

Theorem 3.3. *Let G be a connected graph of order n and size m and let $\delta_h(G)$ be the minimum hub degree of G. Then*

$$E_M^h(G) \geq \sqrt{2m\left[\delta^2(G) + (det(M^h(G)))^{\frac{2}{n}}\right]}.$$

Proof. We have, $E_M^{h^2}(G) = \left(\sum_{i=1}^{n} |\mu_i|\right)^2 = \sum_{i=1}^{n} |\mu_i|^2 + \sum_{i\neq j} |\mu_i||\mu_j|$

Using the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\mu_i||\mu_j| \geq \left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{[n(n-1)]}}$$

Hence, by the above inequality and theorem 2.9 we get

$$\begin{aligned}\left(E_M^h(G)\right)^2 &\geq \sum_{i=1}^{n}|\mu_i|^2 + n(n-1)\left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{[n(n-1)]}} \\ &= \sum_{i=1}^{n}|\mu_i|^2 + n(n-1)\left(\prod_{i=1}^{n}|\mu_i|^{2(n-1)}\right)^{\frac{1}{[n(n-1)]}} \\ &= \sum_{i=1}^{n}|\mu_i|^2 + n(n-1)\left|\prod_{i=1}^{n}\mu_i\right|^{\frac{2}{n}} \\ &= 2\sum_{i=1}^{n}(a_{h_i}+b_{h_i})d_h^2(v_i) + n(n-1)\left|\prod_{i=1}^{n}\mu_i\right|^{\frac{2}{n}}.\end{aligned}$$

Since, $n(n-1) \geq 2m$ and $d_h(v) \geq \delta_h(G)$ for every connected graph, it follows by using

Remark 2.2 that $E_M^{h^2}(G) \geq 2m\delta^2(G) + 2m\left(det(M^h(G))\right)^{\frac{2}{n}}$

$$= \sqrt{2m\left[\delta^2(G) + (det(M^h(G)))^{\frac{2}{n}}\right]}.$$

□

3.1 Bound for the spectral radius of the Maximum hub degree matrix

Theorem 3.4. *Let G be a graph of order n, size m and minimum hub degree $\delta_h(G)$. If $\mu_1(G)$ is the largest maximum hub degree eigenvalue of G, then $\mu_1(G) \geq \frac{2m\delta_h(G)}{n}$.*

Proof. Let G be a graph of order n and let μ_1 be the largest maximum hub degree eigenvalue

of G. Then from [13] we have $\mu_1 \geq max_{X\neq 0}\left\{\frac{X^tAX}{X^tX}\right\}$, where X is any nonzero vector and

X^t is its transpose and A is a matrix. By Setting $X = J = (1,1,...,1)^t$, we have

$$\mu_1 \geq \frac{J^tM^h(G)J}{J^tJ} = \frac{\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}\right)}{n}$$

Since, $a_{ij} = max\{d_h(v_i), d_h(v_j)\}$, if $v_iv_j \in E(G)$ and 0, otherwise. It follows that

$$\mu_1 \geq \frac{2}{n}\sum_{i<j}^{n} max\{d_h(v_i), d_h(v_j)\} = \frac{2}{n}\sum_{i=1}^{n}(a_{h_i}+b_{h_i})d_h(v_i).$$

since $d_h(v) \geq \delta_h(G)$, for every $v \in V(G)$ and by remark, $\sum_{i=1}^{n}(a_{h_i}+b_{h_i}) = m$, then,

$\mu_1 \geq \frac{2m\delta_h(G)}{n}$. □

3.2 Bound for the Maximum hub degree energy involving the spectral radius

Theorem 3.5. *Let G be a connected graph of order n, size m and maximum hub degree $\Delta_h(G)$. Then*

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$$E_M^h(G) \leq \frac{2m\delta_h(G)}{n} + \frac{m}{n}\sqrt{2n^2\Delta_h^2(G) - 4m\delta_h^2(G)}$$

Proof. Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choosing $a_i = 1$ and $b_i = |\mu_i|$, we get

$$\left(\sum_{i=2}^{n} |\mu_i|\right)^2 \leq \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n} \mu_i^2\right)$$

Hence by theorem 2.9 and remark 2.2, we have

$$\begin{aligned}\left(E_M^h(G) - |\mu_1|\right)^2 &\leq (n-1)\left(2\sum_{i=1}^{n}(a_{h_i} + b_{h_i})d_h^2(v_i) - \mu_1^2\right)\\ &\leq (n-1)(2m\Delta_h^2(G) - \mu_1^2)\end{aligned}$$

Therefore, $E_M^h(G) \leq \mu_1 + \sqrt{(n-1)(2m\Delta^2(G) - \mu_1^2)}$.

By using that $n - 1 \leq m$, for every connected graph, and by theorem 3.4 we have

$$E_M^h(G) \leq \frac{2m\delta_h(G)}{n} + \sqrt{m\left(2m\Delta_h^2(G) - \left(\frac{2m\delta_h(G)}{n}\right)^2\right)}$$

$$E_M^h(G) \leq \frac{2m\delta_h(G)}{n} + \frac{m}{n}\sqrt{2n^2\Delta_h^2(G) - 4m\delta_h^2(G)}.$$ □

Open Problems:

1. Characterize all graphs G for which $E_M^h(G) = E(G)$.
2. Characterize all graphs which are hyperenergetic and hypoenergetic under the
Maximum hub degree energy.

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