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Adaptive Finite Element Scheme for One-dimensional Singularly Perturbed Parabolic Problems

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Abstract

In the present work, robust and efficient adaptive finite element techique has been proposed for solving singularly perturbed parabolic equation in one-dimension. Quasilinearization technique has been considered to tackle with the nonlinearity arising in the problem. Implicit Euler method has been used for temporal semi-discretization. Exponentially fitted splines have been used for spatial discretization on piecewise uniform Shishkin mesh. Streamline upwind/Petrov-Galerkin (SUPG) stabilization technique has been taken into the consideration for the considered problem. The proposed adaptive numerical scheme has been shown absolute stable. In the end, numerical tests have been performed and it has been shown that the proposed adaptive technique is quite efficient in capturing sharp boundary layers as singular perturbation parameter becomes sufficiently small.

Keywords : Adaptive finite element technique, quasilinearization, singularly perturbed problem, Streamline upwind/Petrov-Galerkin (SUPG) technique, boundary layer, shishkin mesh, implicit Euler method, exponentially fitted splines

1 INTRODUCTION

Partial differential equations are used to model wide variety of phenomena such as diffusion [11, 15], heat, sound, etc. Singularly perturbed parabolic differential equations occur frequently during the analysis of biological systems, heat transfer process, mass transfer process, etc. From literature, it can be easily seen that the classical discretization methods fail to give good approximations to exact solutions for problems exhibiting sharp boundary layers until very large number of mesh points are considered in the domain. Various numerical methods have been proposed for solving singularly perturbed parabolic differential equations on layer adapted meshes e.g. Bakhalov mesh, Shishkin mesh, Logarithmic mesh, etc. Ramos et al. [13] developed a third-order convergent numerical strategy for solving non-linear singularly perturbed problems using non-standard algorithm on adaptive Shishkin mesh and through numerical tests shown that the proposed scheme is very efficient in capturing sharp boundary layers. Natesan et al. [10] proposed parameter uniform convergent numerical technique for approximating singularly perturbed turning point problems on piecewise uniform Shishkin mesh and discussed the error estimates, efficiency and accuracy of the proposed scheme. Kaushik et al. [7] gave a parameter uniform finite element numerical scheme for singularly perturbed problems on modified graded mesh. The authors derived some error estimates and compared their numerical results with those existing in the literature. Clavero et al. [4] proposed uniform convergent numerical scheme for convection-diffusion parabolic problems on nonuniform mesh. Ramos [12] developed exponentially fitted numerical scheme for singularly perturbed linear parabolic convection-diffusion-reaction problems on uniform mesh.

In [5], Jiwari et al. developed two numerical schemes i.e. based on cubic trigonometric B-splines functions and modified cubic trigonometric B-splines functions for approximating non-linear parabolic problems.

In this paper, firstly a singularly perturbed parabolic differential equation in 1D, representing a linear model in fluid mechanics, has been considered. Implicit Euler scheme has been used for time semi-discretization. Spatial discretization has been carried out using finite element technique and Streamline upwind/Petrov-Galerkin (SUPG) method on layer adapted Shishkin mesh. Exponentially fitted splines have been considered to obtain uniform convergent scheme. Numerical tests have been carried out and it has been shown that the proposed adaptive schemes are efficient in capturing sharp boundary layers arising in the solution as the singular perturbation parameter ϵ becomes small. Further, one-dimensional non-linear singularly perturbed parabolic differential equation has been taken into consideration. Quasilinearization process has been used to deal with the nonlinearity occurring in the problem. Time semi-discretization has been done using implicit Euler method. Further, spatial discretization has been carried out using finite element method and SUPG technique. An adaptive numerical schemes have been obtained using layer adapted Shishkin mesh based on exponentially fitted splines. The region of absolute stability has been discussed. At the end, numerical tests have been carried out and it has been shown that the proposed adaptive schemes works efficiently for solving SPP.

The paper is organized as follows: In Section 2, a singularly perturbed linear parabolic differential equation in one-dimension has been presented. Section 3 deals with the construction of numerical scheme using temporal and spatial discretization. Streamline upwind/Petrov-Galerkin (SUPG) stabilization technique has been discussed for the considered problem. In Section 4, effectivity of proposed adaptive numerical scheme has been tested using various numerical examples. It has been shown that sharp boundary layers appear in the solution as $\epsilon \rightarrow 0$. In Section 5, one-dimensional non-linear singularly perturbed parabolic differential equation has been discussed. In Section 6, adaptive numerical scheme has been proposed using finite element method and SUPG technique. Section 7 deals with the stability analysis. Section 8 includes numerical tests which have been performed to test the effectivity of proposed adaptive numerical scheme. In the last, conclusion has been presented.

2 SINGULARLY PERTURBED LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION

Consider the time dependent singularly perturbed linear parabolic partial differential equation

$$\frac{\partial w}{\partial t} - \epsilon \frac{\partial^2 w}{\partial x^2} + a(x,t)w_x + b(x,t)w = f(x,t), \quad 0 < x < 1, 0 < t \leq 1, \tag{2.1}$$

with boundary conditions

$$w(0,t) = f_1(t) \text{ for } 0 < t \leq 1,$$
$$w(1,t) = f_2(t) \text{ for } 0 < t \leq 1,$$

and initial condition as

$$w(x,0) = w_0(x) \text{ for } \quad 0 \leq x \leq 1,$$

whose solution domain is D = $[0 \leq x \leq 1] \times [0 \leq t \leq 1]$. ϵ is singular perturbation parameter ($0 < \epsilon \ll 1$). Take Ω =($0 < x < 1$). Assume that $a(x,t)$, $b(x,t)$ and $f(x,t)$ are continuous functions satisfying $a(x,t) > \alpha > 0$ and $b(x,t) \geq b \geq 0$.

3 CONSTRUCTION OF ADAPTIVE NUMERICAL SCHEME

3.1 Temporal discretization

To find the weak formulation of Eq.(2.1), first we will discretize the equation w.r.t. time. Temporal semi-discretization has been performed using implicit Euler method with uniform step size Δt. After performing semi-discretization with respect to time, the above equation reduces to

$$w^{j+1} = \Delta t \left[\epsilon (w_{xx})^{j+1} - a(x,t)(w_x)|^{j+1} - b(x,t)w|^{j+1} + f(x,t)|^{j+1}\right] + w^j, j \geq 0, \; x \in \Omega, \tag{3.1.1}$$

with boundary conditions

$$w^{j+1}(0) = f_1(t_{j+1}),$$
$$w^{j+1}(1) = f_2(t_{j+1}), \; j \geq 0,$$

where $\Delta t = \frac{1}{M}$, M is the no. of time-steps and w^{j+1} is solution of the above equation at $(j+1)$-th time level i.e. $t_{j+1} = (j+1)\Delta t$.

3.2 Shishkin mesh methodology

Static grid refinement techniques result in generation of mesh where in grid is more finely spaced in some regions as compared to other regions but the shape is maintained over the time. Static adaptation results in improving the accuracy and efficiency of the approximations on a discrete mesh. Since the solutions to the considered singularly perturbed problems exhibit sharp boundary layers as ϵ approaches 0, therefore, in the present work, static grid adaption approach has been considered for grid refinement under SUPG framework while approximating the solutions of linear and non-linear singularly perturbed partial differential equation. From asymptotic analysis available in literature, it has been observed that the problem under consideration displays sharp boundary layers near x = 1 as $\epsilon \rightarrow 0$. In order to resolve these sharp boundary layers, a piecewise uniform mesh, called Shishkin mesh, has been considered so that more mesh points can be generated in the boundary layer region as $\epsilon \rightarrow 0$. Suppose that number of mesh elements be N where $N \geq 4$ is a positive even integer. Shishkin mesh is obtained by dividing spatial domain Ω into two subintervals [0, 1-τ] and [$1-\tau$,1], where τ is transition parameter defined as

$$\tau = \min\{\tfrac{1}{2}, C\epsilon \log N\},$$

where C is a constant chosen appropriately.
We discretize each of the two subintervals [0, 1-τ] and [$1-\tau$, 1] into $\frac{N}{2}$ equal mesh elements with mesh spacing defined by

$$h = \begin{cases} h_1 = 2\frac{(1-\tau)}{N}, & \text{if } i = 1, 2, ..., \frac{N}{2}, \\ h_2 = 2\frac{\tau}{N}, & \text{if } i = \frac{N}{2}+1, ..., N. \end{cases}$$

Therefore, mesh points $\{x_i\}_{i=1}^{N}$ are given by

$$x_i = \begin{cases} h_1 i, & i = 0, 1, 2, ...\frac{N}{2}, \\ (1-\tau) + h_2(i - \frac{N}{2}), & i = \frac{N}{2}+1, ..., N. \end{cases} .$$

Hence, the piecewise uniform discretization of the spatial domain $\overline{\Omega}$ is defined as:

$$\overline{\Omega}_h = \bigcup_{e=1}^{N} \Omega_e^h,$$

where Ω_e^h represents linear element [x_e,x_{e+1}] and N is the number of elements.

3.3 Exponentially fitted finite element method

Assume that

$$S = \{w \in (H^1(\Omega); L^2(T)): w(x,0) = w_0(x), w(0,t) = f_1(t), w(1,t) = f_2(t)\},$$

and

$$V = \{v \in (H^1(\Omega); L^2(\text{T})): v(0,t) = 0, v(1,t) = 0\},$$

be solution space and test space.
The variational formulation of Eq.(3.1.1) is defined as follows:

Find $w \in S$ such that

$$\int_\Omega w^{j+1} v dx = \Delta t \left[\int_\Omega \epsilon (w_{xx})|^{j+1} - a(x,t) w_x|^{j+1} - b(x,t) w|^{j+1} + f(x,t)|^{j+1} \right] v dx + \int_\Omega w^j v dx, \qquad \forall\, v \in V. \qquad (3.3.1)$$

Assume that S_h and V_h be finite dimensional subspaces of S and V. In the literature, it has been shown that classical methods fail to give satisfactory results for singularly perturbed problems as singular perturbation parameter ϵ tends to zero, unless large number of mesh points are to be chosen or some other special treatment has to be incorporated. In order to overcome this difficulty and to produce uniform convergent scheme, exponentially fitted splines have been used as test functions satisfying the condition

$$\epsilon(\psi_i^e)_{xx}(x, t_{j+1}) + \alpha(\psi_i^e)_x(x, t_{j+1}) = 0.$$

Spatial discretization of Eq.(3.3.1) has been done using standard Galerkin FEM. Let $\{\phi_i^e(x)\}_{i=1}^N$ be standard basis of S_h comprising of standard linear interpolation functions defined on each element Ω_e^h. Therefore, discrete approximation to exact solution $w(x,t)$ is given as:

$$w_h(t) = \sum_{i=1}^N w_i(t)\phi_i(x).$$

By substituting the above finite element discretization into the element-level finite dimensional discrete version of Eq.(3.3.1) we get element-level linear system of equations. After assembly of element-level matrices and on solving the global system we get

$$[A]\{w\} = \{f\},$$

where $w = w^{j+1}$ is finite element solution at $(j+1)$th-time level.

3.4 Streamline upwind/Petrov-Galerkin (SUPG) method

Since the classical Galerkin formulation fails to give satisfactory results for singularly perturbed problems. To enhance the precision and stability of the Galerkin discretization, SUPG technique proposed by Brooks and Hughes [6] has been preferred by many researchers for convection dominated problems. For approximating the sols. of SPP, we have also considered the SUPG technique. The SUPG technique eliminates unwanted oscillations in the region of boundary layer. The weak formulation of Eq.(2.1) is defined as:

Find $w \in H^1(D)$

$$a(w,v) = (F,v) \qquad \forall\, v \in H_0^1(D).$$

Here $a(w,v) = \epsilon(\nabla w, \nabla v) + (\mathbf{a}\nabla w, v)$ and $F = f - bw$.

The SUPG technique is defined as:

Find $w_h \in L^2(D)$ such that

$$a_h(w_h, v_h) + (R_h(w_h), \tau \mathbf{a} \nabla_h v_h) = (F, v_h) \qquad \forall\, v_h \in V_h,$$

where $R_h(w) = -\epsilon \triangle_h w + \mathbf{a}\nabla_h w - F$, τ is the nonnegative stabilization parameter and V_h is the finite dimensional solution subspace. We choose

$$\tau|_K \equiv \tau_K = \begin{cases} \tau_0 h_K, & \text{if } Pe_K > 1, \\ \tau_1 h_K^2/\epsilon, & \text{if } Pe_K \le 1, \end{cases}$$

as proposed by Roos [14] where $Pe_K = \frac{|\mathbf{a}| h_K}{2\epsilon}$, K is the element of the mesh, h_K is the characteristic dimension of K, τ_0 and τ_1 are positive constants.

4 NUMERICAL RESULTS

In this Section, some numerical results have been presented to test the efficiency of proposed adaptive numerical scheme.

Example 1. Consider Eq. (2.1) for the specific values of $a(x,t) = 1$ and $b(x,t) = 0$

subject to initial condition

$$w(x,0) = \exp(-0)(C_1 + C_2 x - \exp(-(1-x)/\epsilon)), \quad 0 < x < 1,$$

and boundary conditions

$$w(0,t) = \exp(-t)(C_1 + C_2.0 - \exp(-(1-0)/\epsilon)), \quad t > 0,$$
$$w(1,t) = \exp(-t)(C_1 + C_2.1 - \exp(-(1-1)/\epsilon)), \quad t > 0.$$

The source function $f(x,t)$ is so chosen to satisfy the analytical solution given by

$$w(x,t) = \exp(-t)(C_1 + C_2 x - \exp(-(1-x)/\epsilon)),$$

where $C_1 = \exp(-1/\epsilon)$ and $C_2 = 1 - C_1$. Since for the present problem the analytical solution is known, some error estimates have been presented in L_∞-norm defined by

$$\| w(x,t) - w_h(x,t) \|_{L_\infty} = \max_{1 \le i \le N} | w(x_i,t) - w_h(x_i,t) |,$$

$$E_\epsilon^{N,\delta t} = \max_{0 \le i \le N, 0 \le j \le M} |w(x_i,t_j) - w_h(x_i,t_j)|,$$

where N represents the number of mesh points in the spatial direction. In Table 1 and 2, maximum absolute errors have been tabulated at time level $t = 1$ and have been compared with uniform convergent scheme proposed by Clavero et al. [4] for different values of ϵ.

Table 1: Maximum pointwise errors $E_\epsilon^{N,\Delta t}$ for Example 1 on piecewise uniform Shishkin mesh

	Proposed Sch	Clavero [4]	Proposed Sch	Clavero [4]	Proposed Sch	Clavero [4]
ϵ	N=16	N=16	N=32	N=32	N=64	N=64
2^{-0}	2.132E-5	1.307E-3	1.755E-5	7.907E-4	1.662E-5	3.698E-4
2^{-2}	1.000E-3	1.739E-2	4.733E-4	9.684E-3	3.523E-4	5.105E-3
2^{-4}	6.700E-3	4.013E-2	3.000E-3	2.555E-2	1.500E-3	1.586E-2
2^{-6}	6.700E-3	5.966E-2	3.100E-3	3.737E-2	1.700E-3	2.179E-2
2^{-8}	6.700E-3	6.879E-2	3.100E-3	4.540E-2	1.700E-3	2.711E-2
2^{-10}	6.700E-3	7.147E-2	3.100E-3	4.833E-2	1.700E-3	2.963E-2
2^{-12}	6.700E-3	7.217E-2	3.100E-3	4.912E-2	1.700E-3	3.038E-2
2^{-14}	6.700E-3	7.235E-2	3.100E-3	4.932E-2	1.700E-3	3.055E-2
2^{-16}	6.700E-3	7.239E-2	3.100E-3	4.937E-2	1.700E-3	3.063E-2
2^{-18}	6.700E-3	7.240E-2	3.100E-3	4.939E-2	1.700E-3	3.064E-2
2^{-20}	6.700E-3	7.241E-2	3.100E-3	4.939E-2	1.700E-3	3.064E-2
2^{-22}	6.700E-3	7.241E-2	3.100E-3	4.939E-2	1.700E-3	3.064E-2
2^{-24}	6.700E-3	7.241E-2	3.100E-3	4.939E-2	1.700E-3	3.064E-2
2^{-26}	6.700E-3	7.241E-2	3.100E-3	4.939E-2	1.700E-3	3.064E-2

Table 2: Maximum pointwise errors $E_\epsilon^{N,\Delta t}$ for Example 1 on piecewise uniform Shishkin mesh

	Proposed Scheme	Clavero [4]	Proposed Scheme	Clavero [4]
ϵ	N=128	N=128	N=256	N=256
2^{-0}	1.639E-5	1.889E-4	1.633E-5	9.551E-5
2^{-2}	3.226E-4	2.622E-3	3.152E-4	1.328E-3
2^{-4}	1.000E-3	9.560E-3	9.000E-4	5.599E-3
2^{-6}	1.300E-3	1.238E-2	1.200E-3	6.970E-3
2^{-8}	1.300E-3	1.527E-2	1.300E-3	8.339E-3
2^{-10}	1.400E-3	1.708E-2	1.300E-3	9.466E-2
2^{-12}	1.400E-3	1.772E-2	1.300E-3	9.976E-3
2^{-14}	1.400E-3	1.789E-2	1.300E-3	1.013E-2
2^{-16}	1.400E-3	1.793E-2	1.300E-3	1.013E-2
2^{-18}	1.400E-3	1.794E-2	1.300E-3	1.017E-2
2^{-20}	1.400E-3	1.795E-2	1.300E-3	1.018E-2
2^{-22}	1.400E-3	1.795E-2	1.300E-3	1.018E-2
2^{-24}	1.400E-3	1.795E-2	1.300E-3	1.018E-2
2^{-26}	1.400E-3	1.795E-2	1.300E-3	1.018E-2

In Fig. 1, comparison of numerical solution has been done for the Example 1 using FEM and SUPG method on piecewise uniform Shishkin mesh. From the solution plot, it can be verified that the both SUPG method and FEM works efficiently for very small values of ϵ. For the other solution plots, FEM has been taken into consideration.

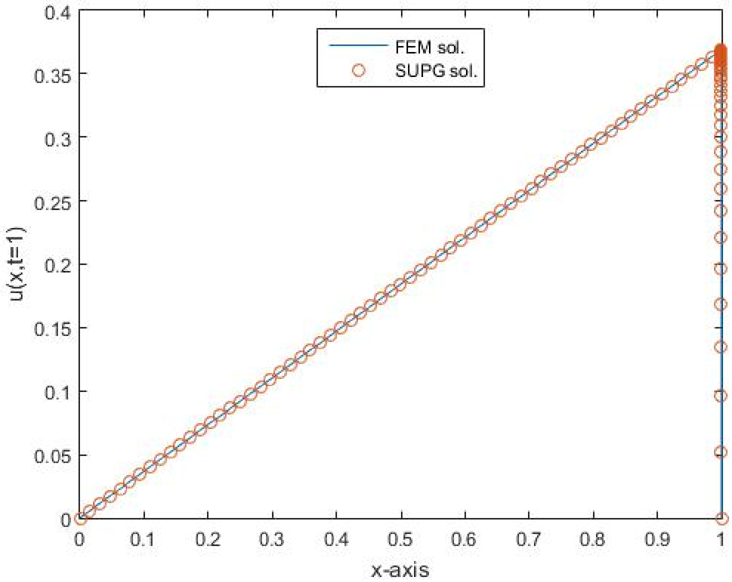
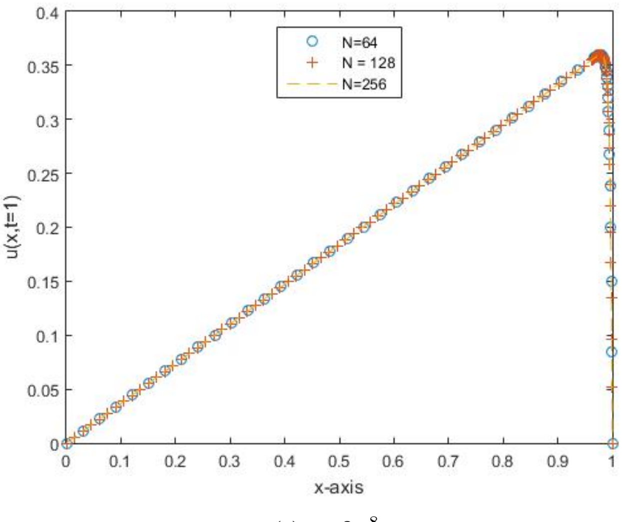
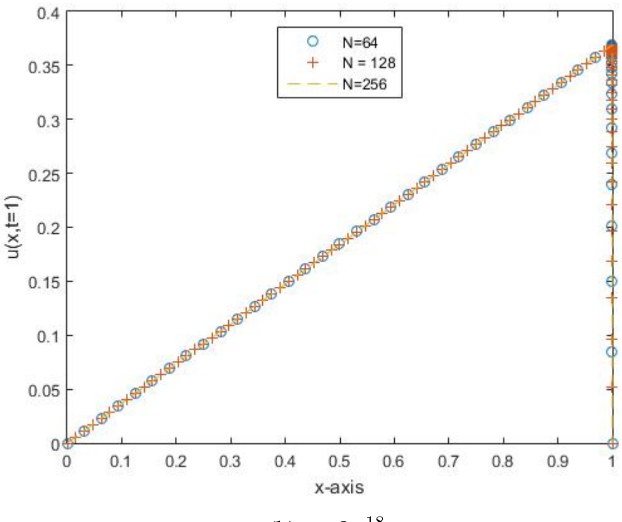


Figure 1: Comparison of numerical solution profile for $\epsilon = 2^{-16}$ using FEM and SUPG method

In Fig. 2, grid validation of the code has been tested for different values of singular perturbation parameter $\epsilon = 2^{-8}$ and 2^{-18} at time level $t = 1$. From solution plots, it can be seen that a grid of 64 elements is sufficient enough to capture the boundary layers even for very small values of ϵ like $\epsilon = 2^{-18}$. From grid validation tests, we analyze that as we move from grid of 16 elements to grid of 256 elements, the boundary layers are captured very nicely. For further numerical results to follow, grid of 128 elements has been considered.



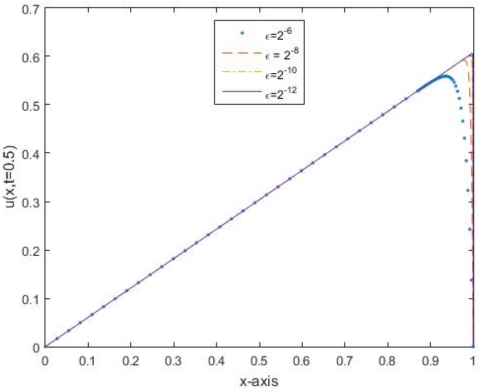
(a) $\epsilon = 2^{-8}$



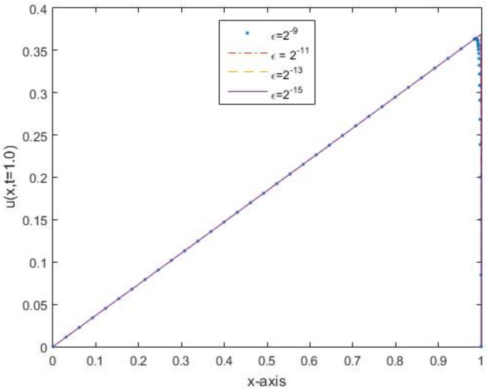
(b) $\epsilon = 2^{-18}$

Figure 2: Grid validation test for $\epsilon = 2^{-8}, 2^{-18}$

In Fig. 3, the effect of singular perturbation parameter ϵ on the solution has been depicted as $\epsilon \to 0$. The solution plots have been drawn for various values of ϵ at time levels $t = 0.5$ and $t = 1$. In both the plots, it is very clear that as singular perturbation parameter becomes smaller and smaller, sharper boundary layers appear in the solution and the proposed numerical scheme is efficient enough to capture these boundary layers.



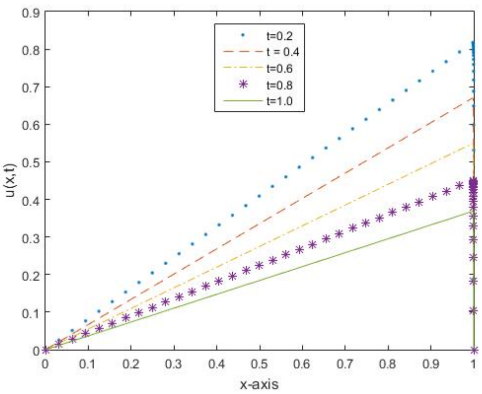
(a) Effect of ϵ on the sol. at $t = 0.5$



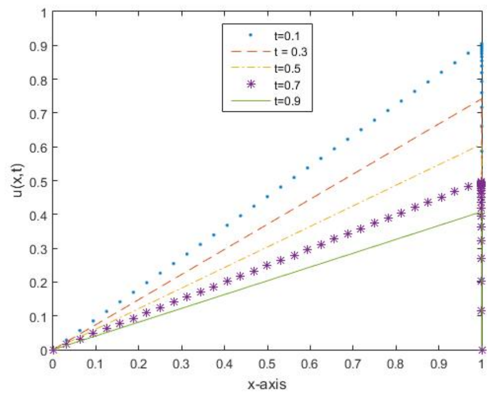
(b) Effect of ϵ on the sol. at $t = 1.0$

Figure 3: ϵ - effect at different time levels

In Fig. 4(a), finite element solution of singularly perturbed problem has been plotted for $\epsilon = 2^{-12}$ at different time levels $t = 0.2$, 0.4, 0.6, 0.8 and 1. Further, the solution has been plotted for $\epsilon = 2^{-16}$ at various time levels $t = 0.1$, 0.3, 0.5, 0.7 and 0.9 in Fig. 4(b). Both the plots depict the efficiency and robustness of the proposed method in capturing very sharp boundary layers.



(a) $\epsilon = 2^{-12}$



(b) $\epsilon = 2^{-16}$

Figure 4: Time-effect for $\epsilon = 2^{-12}$, 2^{-16}

Fig. 5 presents numerical solution profile of Example 1. for $\epsilon = 2^{-26}$ over the whole time domain [0, 1].

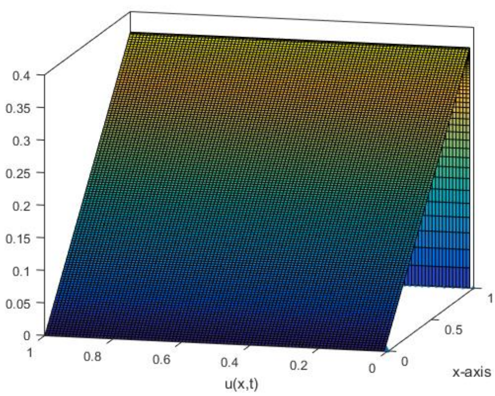


Figure 5: The numerical solution profile for $\epsilon = 2^{-26}$

5 NON-LINEAR SINGULARLY PERTURBED PARABOLIC PARTIAL DIFFERENTIAL EQUATION

Consider time dependent singularly perturbed non-linear parabolic partial differential equation in one-dimension

$$\frac{\partial u}{\partial t} - \epsilon\frac{\partial^2 u}{\partial x^2} + a(x,t,u)u_x = f(x,t,u), \quad 0 < x < 1, t > 0, \tag{5.1}$$

with boundary conditions

$$u(0,t) = f(t), t \geq 0,$$
$$u(1,t) = g(t), t \geq 0,$$

and initial condition as

$$u(x,0) = u_0(x) \text{ for } 0 \leq x \leq 1,$$

over the domain $D = [0 \leq x \leq 1] \times [t \geq 0]$, where ϵ is small singular perturbation parameter ($0 < \epsilon \ll 1$) and $\Omega = [0 < x < 1]$. Assuming $a(x,t,u)$ and $f(x,t,u)$ as continuous functions.

6 PROPOSED NUMERICAL STRATEGY

6.1 Time semi-discretization

In the first step, we discretize the time derivative using implicit Euler method with constant step size Δt. Using time-discretization, we get the following semi-discretized non-linear differential problem:

$$\frac{(u^{j+1} - u^j)}{\Delta t} = \epsilon u_{xx}^{j+1} - a(x,t,u)u_x|^{j+1} + f(x,t,u)|^{j+1}, \tag{6.1.1}$$

$$u^0 = u(x,0) = u_0(x), \qquad 0 < x < 1, \tag{6.1.2}$$

together with the boundary conditions

$$\begin{aligned} u(0,t_{j+1}) &= u^{j+1}(0) = f(t_{j+1}), \\ u(1,t_{j+1}) &= u^{j+1}(1) = g(t_{j+1}), \ \ j > 0, \end{aligned} \tag{6.1.3}$$

where u^{j+1} denote the solution of the above differential equation at $(j+1)$-th time level. Next, before proceeding to spatial discretization of Eq. (6.1.1), we will use quasilinearization process to linearize the equation.

6.2 Quasilinearization process and its convergence

Quasilinearization is a very well known and established process used for approximating and obtaining the solutions of nonlinear problems. Lot of researchers have used different quasilinearization techniques. For instance, Yakar and Koksal [16] proposed quasilinearization strategy for approximating non-linear problems using casual operators. Motsa et al. [9] proposed successive linearization technique for approximating non-linear problems and then the obtained reduced system of equations have been solved using Chebyshev spectral collocation method. Mittal and Jiwari [8]developed numerical scheme based on differential quadrature method for solving nonlinear generalizations of Fisher and Burgers' equation using quasilinearization technique proposed by Bellman and Kalaba [2].

In the present work, we have used the quasilinearization technique proposed by Bellman and Kalaba [2] to obtain linear approximates of the non-linear differential equations. It has been shown that the sequence of approximate solutions of these linearized problems converges monotonically and quadratically to the solution of the non-linear equation (5.1). The semi-discretized equation (6.1.1) is linearized around a nominal solution satisfying the given boundary conditions. Let $u_r(x)$ be nominal solution of the above problem (6.1.1-6.1.3). Applying quasilinearization process for a fixed time level t_{j+1}, we get a sequence of solutions $\{u_r^{j+1}(x)\}_{r=0}^{r=\infty}$ of linear equations given by the following recurrence relation:

$$u_{r+1}^0 = u_0(x). \tag{6.2.1}$$

We write the Eq.(6.1.1) as

$$\begin{aligned} \frac{u^{j+1} - u^j}{\Delta t} &= \epsilon u_{xx}^{j+1} - a(x,t,u)u_x|^{j+1} + f(x,t,u)|^{j+1} \\ &= \epsilon u_{xx}^{j+1} + G(u,u_x,x,t)|^{j+1}. \end{aligned} \tag{6.2.2}$$

Applying quasilinearization technique [2], we get

$$\frac{u_{r+1}^{j+1}-u_{r+1}^{j}}{\Delta t} = \epsilon(u_{xx})_{r+1}^{j+1} + G(u,u_x,x,t)|_r^{j+1} + (u_{r+1}^{j+1}-u_r^{j+1})\left(\frac{\partial G(u,u_x,x,t)}{\partial u}\right)|_r^{j+1}$$

$$+((u_x)_{r+1}^{j+1}-(u_x)_r^{j+1})\left(\frac{\partial G(u,u_x,x,t)}{\partial u_x}\right)|_r^{j+1}. \tag{6.2.3}$$

Also, the boundary conditions reduce to

$$u_{r+1}^{j+1}(0)=f((j+1)\Delta t),\ u_{r+1}^{j+1}(1)=g((j+1)\Delta t), \qquad j\geq 0, \tag{6.2.4}$$

where $r = 0,1,2,...$ is iteration index and $u_0^{j+1}(x)$ is initial guess. Further, in order to prove the convergence of the quasilinearization process, for the sake of convenience, we consider

$$\epsilon u_{xx}^{j+1} = H(u^{j+1}), \qquad 0< x<1,$$

$$u^{j+1}(0) = f(t_{j+1})\ ,\ u^{j+1}(1) = g(t_{j+1}).$$

Using quasilinearization process, we obtain a sequence of linear differential equations determined by the following recurrence relation:

$$\epsilon(u_{xx})_{r+1}^{j+1} = H(u_r^{j+1}) + (u_{r+1}^{j+1}-u_r^{j+1})\frac{\partial H(u_r^{j+1})}{\partial u}, \qquad 0<x<1, \tag{6.2.5}$$

$$u_{r+1}^{j+1}(0) = f(t_{j+1}) \text{ and } u_{r+1}^{j+1}(1) = g(t_{j+1}),$$

where we assume that $u_{r=0}^{j+1}$ is the initial guess which satisfy the boundary conditions also. Rewriting the above relation at previous iteration step, we get

$$\epsilon(u_{xx})_r^{j+1} = H(u_{r-1}^{j+1}) + (u_r^{j+1}-u_{r-1}^{j+1})\frac{\partial H(u_{r-1}^{j+1})}{\partial u}, \qquad 0<x<1. \tag{6.2.6}$$

Subtracting Eq.(6.2.6) from Eq.(6.2.5), we get

$$\epsilon((u_{xx})_{r+1}^{j+1}-(u_{xx})_r^{j+1}) = H(u_r^{j+1}) - H(u_{r-1}^{j+1}) + (u_{r+1}^{j+1}-u_r^{j+1})\frac{\partial H(u_r^{j+1})}{\partial u}$$

$$-(u_r^{j+1}-u_{r-1}^{j+1})\frac{\partial H(u_{r-1}^{j+1})}{\partial u}, \qquad 0<x<1.$$

The above differential equation is of second order in (u_{r+1}^{j+1} - u_r^{j+1}). Converting it into integral equation by using Green's function, we obtain

$$\epsilon(u_{r+1}^{j+1}-u_r^{j+1}) = \int_0^1 G(x,s)[H(u_r^{j+1})-H(u_{r-1}^{j+1}) + (u_{r+1}^{j+1}-u_r^{j+1})\frac{\partial H(u_r^{j+1})}{\partial u}$$

$$-(u_r^{j+1}-u_{r-1}^{j+1})\frac{\partial H(u_{r-1}^{j+1})}{\partial u}]ds, \qquad 0<x<1, \tag{6.2.7}$$

where Green's function $G(x,s)$ is given by

$$G(x,s) = \begin{cases} x(s-1), & 0\leq x\leq s\leq 1, \\ (x-1)s, & 0\leq s\leq x\leq 1, \end{cases}$$

and

$$\max_{x,s} G(x,s) = \frac{1}{4}. \tag{6.2.8}$$

Using Mean Value Theorem,

$$H(u_r^{j+1}) - H(u_{r-1}^{j+1}) = (u_r^{j+1}-u_{r-1}^{j+1})\frac{\partial H(u_{r-1}^{j+1})}{\partial u} + \frac{(u_r^{j+1}-u_{r-1}^{j+1})^2}{2}\frac{\partial^2 H(\theta)}{\partial u^2}, \tag{6.2.9}$$

where $u_{r-1}^{j+1}\leq\theta\leq u_r^{j+1}$.

Using Eq.(6.2.9), Eq.(6.2.7) becomes

$$\epsilon\left(u_{r+1}^{j+1}-u_{r}^{j+1}\right)=\int_0^1 G(x,s)\left[\frac{(u_r^{j+1}-u_{r-1}^{j+1})^2}{2}\frac{\partial^2 H(\theta)}{\partial u^2}+(u_{r+1}^{j+1}-u_r^{j+1})\frac{\partial H(u_r^{j+1})}{\partial u}\right]ds,$$

$$0<x<1. \tag{6.2.10}$$

Taking

$$\max_{|u_{r-1}^{j+1}|\le 1}\left|\frac{\partial^2 H(\theta)}{\partial u^2}\right|=P,\quad \max_{|u_r^{j+1}|\le 1}\left|\frac{\partial H(u_r^{j+1})}{\partial u}\right|=Q. \tag{6.2.11}$$

Taking maximum norm over the spatial domain and using equations (6.2.8) and (6.2.11) in (6.2.10), we get

$$\left|\left|(u_{r+1}^{j+1}-u_r^{j+1})\right|\right|\le\frac{1}{4\epsilon}\int_0^1\left[\frac{P}{2}\left(u_r^{j+1}-u_{r-1}^{j+1}\right)^2+Q\left|\left|(u_{r+1}^{j+1}-u_r^{j+1})\right|\right|\right]ds. \tag{6.2.12}$$

Simplifying this inequality, we get

$$\left|\left|(u_{r+1}^{j+1}-u_r^{j+1})\right|\right|_\Omega\le\frac{P}{8\epsilon-2Q}\left|\left|(u_r^{j+1}-u_{r-1}^{j+1})\right|\right|_\Omega^2,\le C\left|\left|(u_r^{j+1}-u_{r-1}^{j+1})\right|\right|_\Omega^2, \tag{6.2.13}$$

where $C=\frac{P}{8\epsilon-2Q}$.

Thus, with appropriate choice of initial iterative approximation u_0^{j+1}, the quasilinearization process converges quadratically. Therefore, in order to approximate the sol. of the problem (6.1.1-6.1.3), it is sufficient to approximate the sol. of its linearized version (6.2.3-6.2.4). The finite element formulation and SUPG formulation of the linearized problem (6.2.3-6.2.4) can be carried out on Shishkin mesh in similar way as discussed in Subsections 3.3 and 3.4 respectively.

7 ABSOLUTE STABILITY

Consider the problem given by (5.1). Its approximate linear differential equation is given by equation (6.2.3). Applying Galerkin finite element or SUPG technique to the linearized version in the spatial direction and using boundary conditions, we get a system of first order differential equations in time variable and can be written as

$$[A_1]\left\{\frac{\partial\widetilde{U}}{\partial t}\right\}=[B_1]\{\widetilde{U}\}+S, \tag{7.1}$$

with initial condition

$$\widetilde{U}(0)=[\phi_2,\phi_3,...,\phi_N]^T, \tag{7.2}$$

where $[A_1]$, $[B_1]$ are $(N-1)\times(N-1)$ tridiagonal matrices, $\widetilde{U}$= $\{u_2^{r+1}$, u_3^{r+1},..., $u_N^{r+1}\}$ and S are column vectors of order $(N-1)$.
On simplifying the system (7.1), we get

$$\left\{\frac{\partial\widetilde{U}}{\partial t}\right\}=[C]\{\widetilde{U}\}+S_1, \tag{7.3}$$

where $C=[A_1]^{-1}[B_1]$ and $S_1=[A_1]^{-1}S$.
Let $\{\lambda_s, s=1,2,...,N-1\}$ be eigen values of the matrix C where $\lambda_s=\lambda_{sR}+i\lambda_{sI}$. The proposed numerical scheme will be stable if all the eigenvalues of matrix C lie outside the unit circle

$$(1-\Delta t\lambda_{sR})^2+(\Delta t\lambda_{sI})^2=1.$$

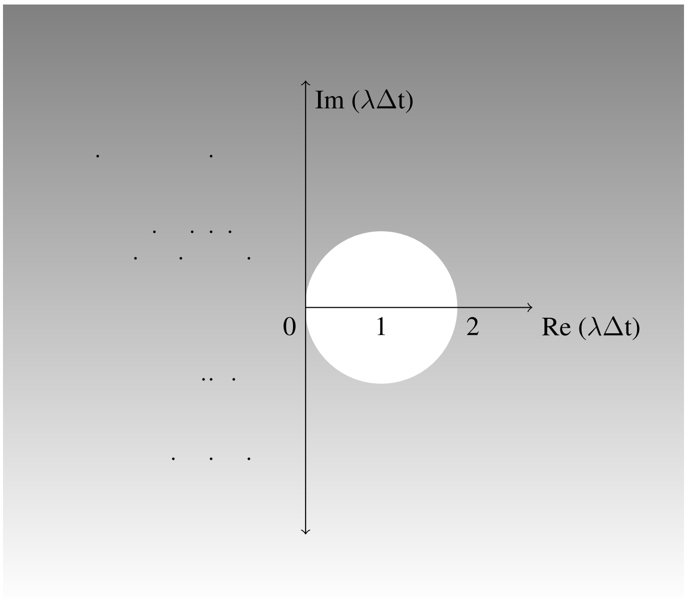


Figure 6: Stability region for non-linear singularly perturbed problem

8 NUMERICAL RESULTS

In this Section, we have presented some numerical results to check the efficiency and robustness of the developed adaptive schemes.

Example 2. Consider Eq. (5.1) for $a(x,t,u) = 0$, $f(x,t,u) = (-bu^2 - cu)$ together with the initial condition

$$u(x,0) = (d.x+1)^{-1}, \qquad 0 < x < 1,$$

and boundary conditions

$$u(0,t) = (c.t+1)^{-1}, \qquad t > 0,$$
$$u(1,t) = (c.t+d.1+1)^{-1}, \quad t > 0.$$

The analytic solution of the above problem is given by

$$u(x,t) = (c.t+d.x+1)^{-1},$$

where $d = \sqrt{b/2\epsilon}$, $b = 0.01$ and $c = 0.01$. The errors have been presented in the L_∞-norm defined by

$$\| u(x,t) - u_h(x,t) \|_\infty = \max_{1\le i\le N, 1\le j\le M} | u(x_i,t_j) - u_h(x_i,t_j) |,$$

where M is the number of time-steps.

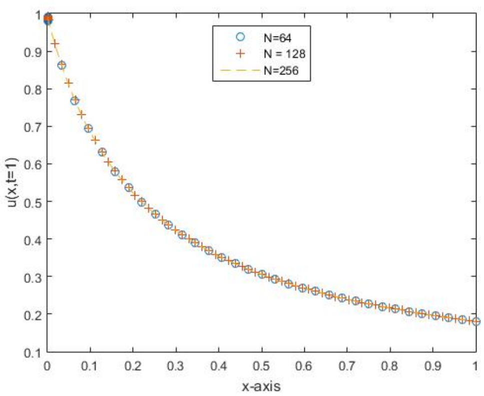
The problem has been solved using both the proposed numerical schemes FEM i.e. finite element method and stabilized SUPG technique. Since the solution obtained using both the schemes differs minutely, therefore, the results have been presented for stabilized SUPG technique. The region of absolute stability, which is outside the above mentioned unit circle, for the problem under consideration is shown in Fig. 6. From Fig. 6, it can easily be seen that all eigenvalues of matrix $[C]$ lie in the region of absolute stability which proves the stability of the proposed numerical scheme.

In Table 3, maximum absolute errors have been tabulated using SUPG method for different values of ϵ at time level $t = 1$.

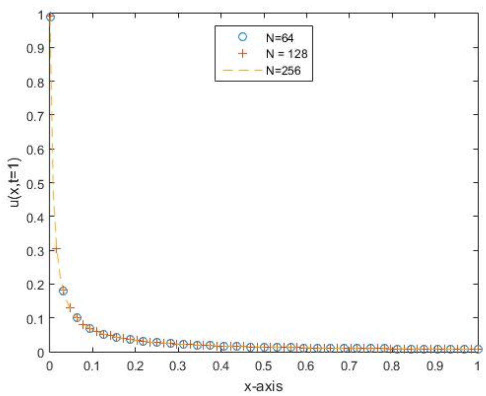
Table 3: Maximum SUPG pointwise errors $E_\epsilon^{N,\Delta t}$ for the Example 2 on piecewise uniform Shishkin mesh

	Proposed Sch.	Proposed Sch.	Proposed Sch.	Proposed Sch.	Proposed Sch.
ϵ	N=16	N=32	N=64	N=128	N=256
2^{-0}	1.416E-5	1.415E-5	1.415E-5	1.415E-5	1.415E-5
2^{-2}	3.473E-5	3.475E-5	3.472E-5	3.472E-5	3.471E-5
2^{-4}	4.634E-5	4.502E-5	4.485E-5	4.483E-5	4.482E-5
2^{-6}	5.533E-5	4.782E-5	4.571E-5	4.522E-5	4.509E-5
2^{-8}	7.726E-5	5.704E-5	4.796E-5	4.574E-5	4.524E-5
2^{-10}	1.254E-4	8.090E-5	5.679E-5	4.785E-5	4.571E-5
2^{-12}	2.232E-4	1.354E-4	8.373E-5	5.578E-5	4.719E-5
2^{-14}	3.950E-4	2.048E-4	1.420E-4	8.615E-5	5.477E-5
2^{-16}	6.071E-4	2.518E-4	2.197E-4	1.456E-4	8.664E-5
2^{-18}	8.025E-4	3.849E-4	2.535E-4	2.288E-4	1.475E-4
2^{-20}	9.444E-4	5.075E-4	2.319E-4	2.704E-4	2.339E-4
2^{-22}	1.000E-3	5.965E-4	3.054E-4	1.850E-4	2.804E-4
2^{-24}	1.100E-3	6.515E-4	3.588E-4	1.783E-4	2.005E-4

In Fig. 7, grid validation of the code has been done for different values of singular perturbation parameter $\epsilon = 2^{-12}, 2^{-22}$ at time level *t*=1. From solution plots, it can be seen that for very small values of ϵ like 2^{-22}, grid of 64 elements is sufficient enough to capture the boundary layers. From grid validation tests, we analyze that as we move from grid of 16 elements to grid of 256 elements, the boundary layers are captured very nicely. For further numerical results to follow, grid of 256 elements has been considered.



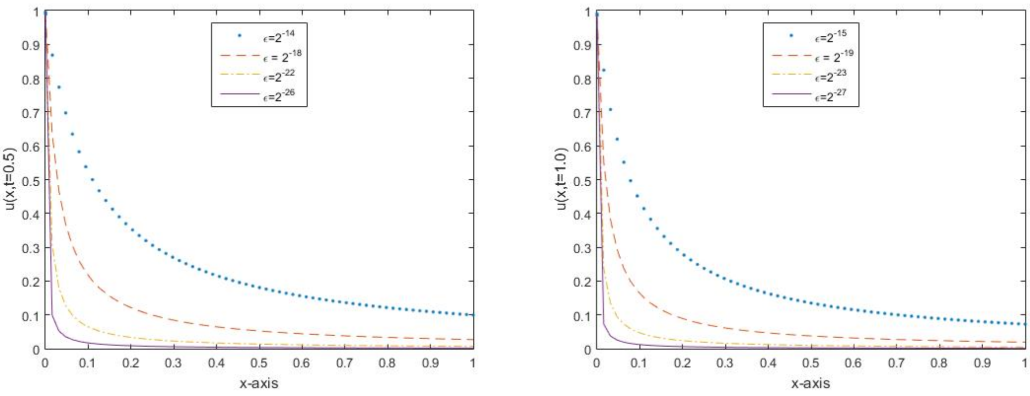
(a) $\epsilon = 2^{-12}$



(b) $\epsilon = 2^{-22}$

Figure 7: Grid validation test for $\epsilon = 2^{-12}, 2^{-22}$

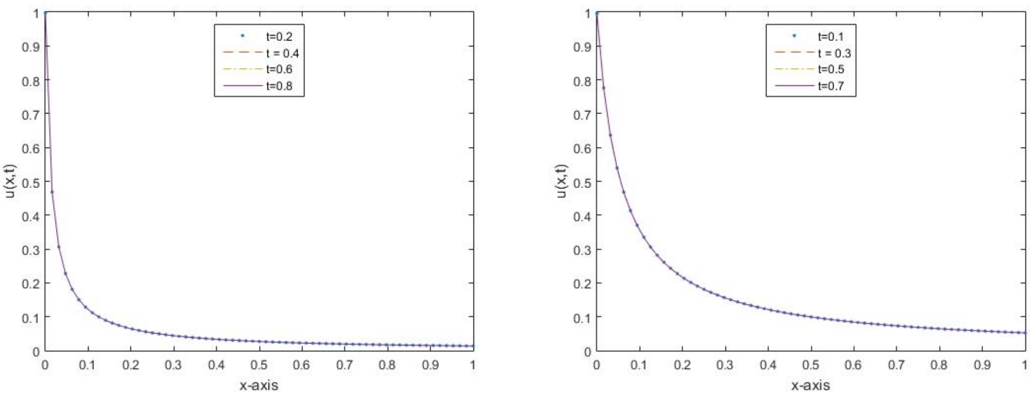
In Fig. 8, the behavior of singular perturbation parameter ϵ has been depicted as $\epsilon \to 0$. The solution plots have been drawn for various values of ϵ at time levels $t = 0.5$ and $t = 1$. In both the plots, it is very clear that as singular perturbation parameter gets smaller and smaller, sharper boundary layers appear in the solution and the proposed numerical scheme is efficient enough to capture these boundary layers.



(a) $t = 0.5$ (b) $t = 1.0$

Figure 8: ϵ - effect at $t = 0.5$, 1

In Fig. 9(a), finite element solution of singularly perturbed problem has been plotted for $\epsilon = 2^{-20}$ at different time levels $t = 0.2$, 0.4, 0.6 and 0.8. Further, the solution has been plotted for $\epsilon = 2^{-16}$ at various time levels $t = 0.1$, 0.3, 0.5 and 0.7 in Fig. 9(b).



(a) $\epsilon = 2^{-20}$ (b) $\epsilon = 2^{-16}$

Figure 9: Temporal-effect on the FEM sol.

In Fig. 10, finite element solution profile of Example 2 is shown for $\epsilon = 2^{-25}$ over the continuous time-domain [0,1].

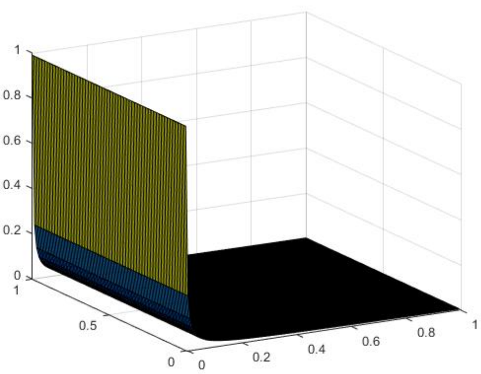


Figure 10: FEM solution profile for $\epsilon = 2^{-25}$ over the time-domain [0,1]

Further, the SUPG solution has been plotted on piecewise uniform Shishkin mesh for different values of ϵ in Fig. 11.

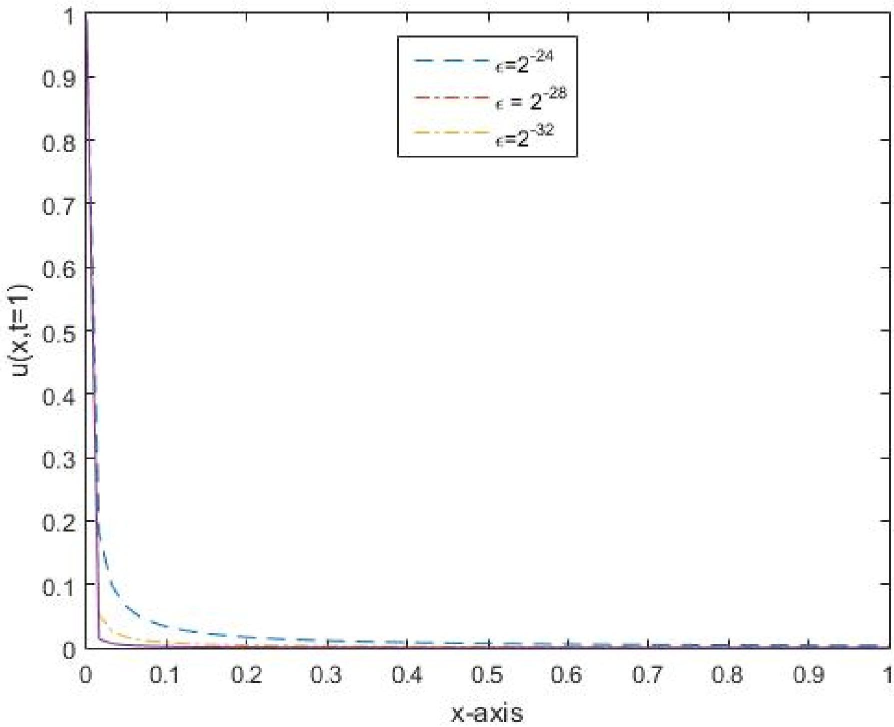


Figure 11: SUPG solution at t=1 for different values of ϵ

Example 3. Consider the following singularly perturbed non-linear partial differential equation defined by

$$u_t - \epsilon u_{xx} = ku(1-u), \qquad 0 < x < 1.$$

The exact solution of the considered problem for ϵ =1 is given by

$$u(x,t) = \frac{1}{\left(1 + exp\left(\sqrt{\frac{k}{6}}x - \frac{5k}{6}t\right)\right)^2}.$$

In Table 4, numerical results obtained using the proposed scheme have been compared with those cited in [3] for ϵ =1.

Table 4: Numerical results for Example 3 for $k = 6$, ϵ = 1 and Δt = 0.001, $T = 0.01$

x	Proposed Scheme	Semi-implicit method	Exact sol
0.25	0.2027	0.2027	0.2026
0.50	0.1516	0.1516	0.1516
0.75	0.1101	0.1101	0.1101

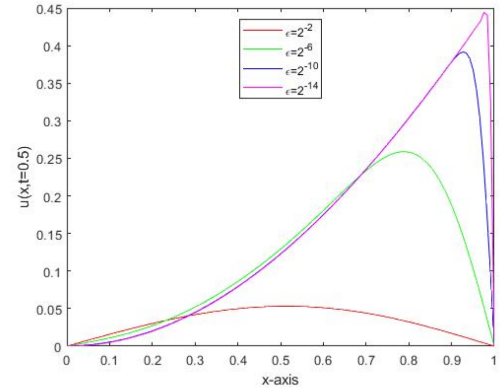
For the case $\epsilon \ll 1$, numerical results could not be found. In order to test the effectivity of the proposed adaptive numerical scheme, the considered problem has been solved subject to the initial condition

$$u(x,0) = 1 - cos(x), \qquad 0 \le x \le 1,$$

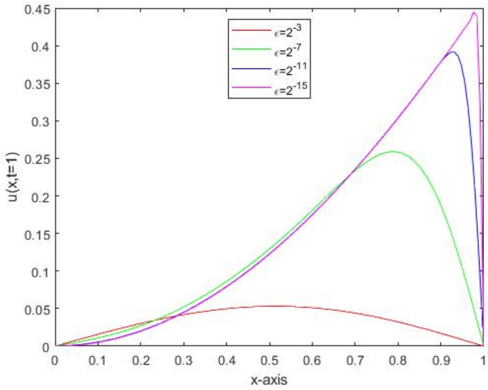
and boundary conditions

$$u(0,t) = 0, \qquad t > 0,$$
$$u(1,t) = 0, \qquad t > 0.$$

In Fig. 12(a) and (b), ϵ-effect has been shown on the numerical solution at time t = 0.5 and t = 1. From the solution plots, it can be noticed that the proposed scheme is efficient enough in capturing sudden sharp changes in the solution.



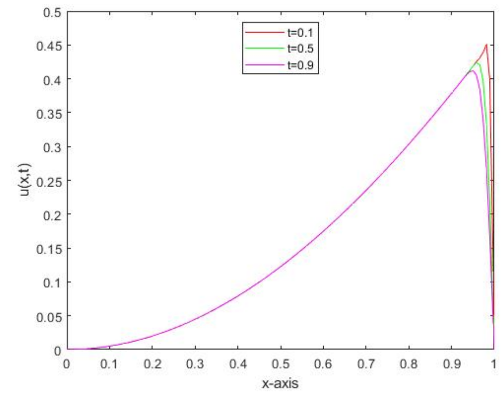
(a) $k = 0.0001$, $t = 0.5$



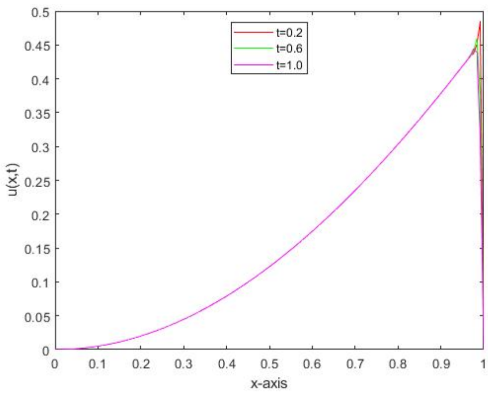
(b) $k = 0.0001$, $t = 1$

Figure 12: ϵ-effect on the FEM sol.

Further, in Fig. 13(a) and (b), numerical solution has been plotted for $\epsilon = 2^{-12}$ and 2^{-15} at different time levels. It can be observed from the solution plots that the sharp boundary layers appear near the boundary $x = 1$ as singular perturbation parameter ϵ becomes small.



(a) $k = 0.0001$, $\epsilon = 2^{-12}$



(b) $k = 0.0001$, $\epsilon = 2^{-15}$

Figure 13: Time-effect on the sol. for $\epsilon = 2^{-12}$ and 2^{-15}

In Fig. 14, behavior of the numerical solution has been shown for $\epsilon = 2^{-14}$ over the time-domain [0,1].

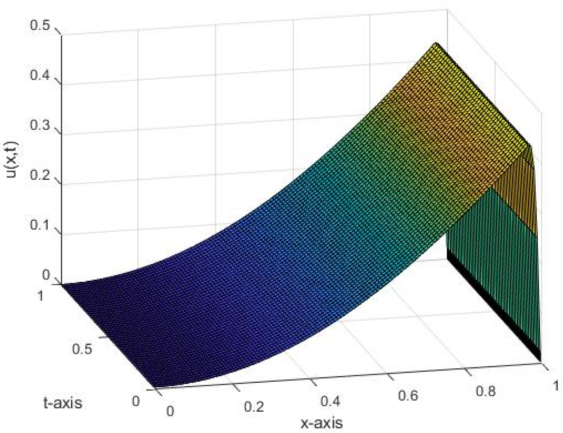


Figure 14: FEM solution profile for $k = 0.0001$, $\epsilon = 2^{-14}$ over the time-domain [0,1]

9 CONCLUSION

In this paper, two numerical techniques, i.e. finite element technique and stabilized SUPG technique, on adaptive piecewise uniform Shishkin mesh have been proposed for approximating one-dimensional singularly perturbed problems. The considered problems are time-dependent. Time semi-discretization has been performed using implicit Euler's method. Piecewise uniform Shishkin mesh has been considered for spatial discretization. Spatial discretization has been carried out using both the proposed finite element and stabilized SUPG methods. Exponentially fitted splines have been considered as a basis for the test spaces. Streamline upwind/Petrov-Galerkin (SUPG) stabilization strategy removes the unwanted oscillations in the boundary layer region. For non-linear singularly perturbed problems, quasilinearization strategy has been invoked to handle the non-linear terms. Stability of the proposed schemes have also been discussed. Numerical experiments have been performed and it has been shown that the proposed schemes approximates the solution very effectively on adaptive Shishkin grids. It has been observed that both the schemes works very well for solving the singularly perturbed problems and the numerical solutions obtained by using both the proposed schemes agree with the exact solutions for both linear and non-linear problems.

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