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A New Subclass of Bi-Univalent Functions Associated with Balancing Polynomial

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Abstract

In this paper new subclass of bi-univalent function is introduced by using balancing polynomial. Then,coefficient estimate are determined for the $|a_2|$ and $|a_3|$ for Taylor- Maclaurin coefficient of function belonging to the new class.Finally Fekete-Szegö inequalities estimate for the functions in subclasses defined.

Keywords Balancing polynomial; bi-univalent; subordination; function;analytic function; Taylor-Maclaurin coefficient, Fekete-Szegö functional.

Mathemetics Subject Classification [MSC:]: 30C45;30C50.

1. INTRODUCTION

The notion of Balancing numbers $(B_n), n \geq 0$ introduced by, [1]. These number have been thoroughly examined during the past 20 years. Most new studies regarding the subject include the article [2], [3],[4], [5], [6],[7][8]. Generalization of Balancing number is accessible in a number of ways [9],[10],[11], [12], [13]

Definition 1.1. [11] Assume that $x \in C$ and $n \geq 2$, Balancing polynomial are defined with the subsequent recurrence relation

$$\mathcal{B}_n(x) = 6x\mathcal{B}_{n-1}(x) - \mathcal{B}_{n-2}(x) \tag{1.1}$$

where

$$\begin{aligned} \mathcal{B}_0(x) &= 0 \\ \mathcal{B}_1(x) &= 1 \end{aligned} \tag{1.2}$$

Using the recurrence relation provided by(1.1) It can be attained simply that

$$\begin{aligned} \mathcal{B}_2(x) &= 6x \\ \mathcal{B}_3(x) &= 36x^2 - 1 \end{aligned} \tag{1.3}$$

Lemma 1.2. *[12] The balancing polynomial's ordinary generating function is given by*

$$\mathcal{B}(x,z) = \sum_{n=0}^{\infty} \mathcal{B}_n z^n, = \frac{z}{1-6xz+z^2} \tag{1.4}$$

Let us denote by $\mathcal{A}$ the class of functions indicated by the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.5}$$

which are analytic in the open unit disc $\mathcal{D} = \{z : z \in Cand|z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Further,denoted by S the class of analytic normalized and univalent in function in D.

The Koebe-one quarter theorem,[14], ensure that the image of D under every univalent function $f \in A$ contain a disc of radius $\frac{1}{4}$. Thus every univalent function f has a f^{-1} satisfying the $f^{-1}(f(z)) = z, z \in D$ and $f^{-1}(f(w)) = w, (|w| < r_0(f), r_0(f) \geq \frac{1}{4})$.

The inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{1.6}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in D. Let $\sum$ denote the class of bi-univalent functions defined in D. Some examples of functions in the class $\sum$ are

$$\frac{z}{z-1}, \frac{1}{2}\log\frac{1+z}{1-z}, -\log(1-z)$$

Koebe function is a member of of S but not in the class $\sum$

The $\sum$ was first studied by [15] and showed that $|a_2| \leq 1.51$. Later,[16], conjecture that $|a_2| \leq \sqrt{2}$. After that, [17], showed that $\max|a_2| = \frac{4}{3}$

For two analytic functionf_1 and f_2 such that $f_1(0) = f_2(0)$, we say that f_1 is subordinate to f_2 in U and write $f_1(z) \prec f_1(z), z \in U$, if there exist a Schwartz function $v(z)$ with $v(0) = 0$ and $|v(z)| \leq |z|, z \in U$ such that $f_1(z) = f_2(v(z)), z \in U$. Furthermore, if the function f_2 is univalent in U, then we have the following equivalence

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2 \text{ and } f_1(U) \subset f_2(U)$$

In a recent study, using Balancing polynomials [18], the authors defined the class $\mathcal{B}_{\sum}(\alpha; \mathcal{B}(x, z))$ and examined the initial coefficients of the function belonging to the class $\mathcal{B}_{\sum}(\alpha; \mathcal{B}(x, z))$ as follows:

Definition 1.3. [18] A function f is named to be in the class $\mathcal{B}_{\sum}(\alpha; \mathcal{B}(x, z))$ if the following subordinations

$$1 + \frac{z^{2-\alpha} f''(z)}{(zf'(z))^{1-\alpha}} \prec \frac{\mathcal{B}(x, z)}{z} = \frac{z}{1 - 6xz + z^2} = K(x, z)$$

$$1 + \frac{w^{2-\alpha} g''(z)}{(wg'(w))^{1-\alpha}} \prec \frac{\mathcal{B}(x, w)}{w} = \frac{w}{1 - 6xw + w^2} = K(x, w)$$

$z, w \in D$, $0 \leq \alpha \leq 1$ and $g(w) = f^{-1}(z)$ is defined by recurrence the relation (1.1)

Theorem 1.4. *[18] If $f \in \mathcal{B}_{\sum}(\alpha; \mathcal{B}(x, z))$ and $x \in \mathbb{C} \setminus \left\{ \mp \frac{1}{\sqrt[3]{2|1-2\alpha|}} \right\}$ then*

$$|a_2| \leq \frac{3|x|\sqrt{6|x|}}{\sqrt{1 - 18x^2(2\alpha - 1)}}, \tag{1.7}$$

$$|a_3| \leq |x|(9|x| + 1). \tag{1.8}$$

The following theorem gives the Fekete-Szeg ö type inequality for the function in $\mathcal{B}_{\sum}(\alpha; \mathcal{B}(x, z))$

Theorem 1.5. *[18] If $f \in \mathcal{B}_{\sum}(\alpha; \mathcal{B}(x, z))$ and $x \in \mathbb{C} \setminus \left\{ \mp \frac{1}{\sqrt[3]{2|1-2\alpha|}} \right\}$, then*

$$|a_3 - \delta a_2^2| \leq \begin{cases} |x|, & \textit{for } |1 - \delta| \leq \frac{|1+18x^2(2\alpha-1)|}{54x^2} \\ \frac{54|x|^3|1-\delta|}{|1+18x^2(2\alpha-1)|}, & \textit{for } |1 - \delta| \geq \frac{|1+18x^2(2\alpha-1)|}{54x^2} \end{cases} \tag{1.9}$$

Over the past few decades, they have been useful in number theory, combinatorics, numerical analysis, and other fields, theory and applications of Fibonacci, Lucas, Chebyshev, LucasLehmer, LucasBalancing polynomials, Gregory numbers, telephone numbers has become increasingly important in contemporary science. Now a days, these kinds of polynomials have been looked into by numerous authors in,[19],[20], [21], [22], [23], [24], [25], [26],[27],[28],[29]

2. COEFFICIENT BOUND OF THE CLASS $\mathcal{N}^{\lambda}_{\sum}(\mathcal{B}(x,z))$ AND THE FEKETE-SZEGö INEQUALITIES

Consider in the next section analytic bi-univalent function class $\mathcal{N}^{\lambda}_{\sum}(\mathcal{B}(x,z))$ deals with the Lucas balancing polynomials to obtain the estimates of the coefficients $|a_2|$,$|a_3|$ and Fekete-Szegö functional problem [30]

Definition 2.1. Let $\lambda \in [0,1]$ and $x \in (\frac{1}{2}, 1]$. A function $f \in \sum$ given by(1.5) is said to be in the class $\mathcal{N}^{\lambda}_{\sum}(\mathcal{B}(x,z))$ if the following subordination are satisfied

$$\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big)+(1-\lambda)\Big(\frac{zf'(z)}{f(z)}\Big) \prec \mathcal{B}(x,z)=\frac{z}{1-6xz+z^2}=K(x,z) \quad (2.1)$$

and

$$\lambda\Big(1+\frac{wg''(w)}{g'(w)}\Big)+(1-\lambda)\Big(\frac{wg'(w)}{g(w)}\Big) \prec \mathcal{B}(x,w)=\frac{w}{1-6xw+w^2}=K(x,w) \quad (2.2)$$

where the function $g(w) = f^{-1}(w)$ is defined by(1.6).

Example 2.2. A bi-univalent function $f \in \sum$ is said to be in the class $\mathcal{N}^{0}_{\sum}(\mathcal{B}(x,z)) = \mathcal{S}^{*}_{\sum}(x,z)$, if the following subordination conditions holds:

$$\frac{zf'(z)}{f(z)} \prec \mathcal{B}(x,z) \quad (2.3)$$

and

$$\frac{wg'(w)}{g(w)} \prec \mathcal{B}(x,w) \quad (2.4)$$

where the function $g = f^{-1}$ is defined by (1.6).

Example 2.3. A bi-univalent function $f \in \sum$ is said to be in the class $\mathcal{N}^{1}_{\sum}(\mathcal{B}(x,z)) = \mathcal{K}_{\sum}(x,z)$, if the following subordination conditions holds:

$$1+\frac{zf''(z)}{f'(z)} \prec \mathcal{B}(x,z) \quad (2.5)$$

and

$$1+\frac{wg''(w)}{g'(w)} \prec \mathcal{B}(x,w) \quad (2.6)$$

where the function $g = f^{-1}$ is defined by (1.6).

Let Ω be the class of all analytic function $w \in \mathbb{D}$ which satisfy that $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. We start by going over the following lemma, which is crucial to proving the main outcome. Subsequently, we will present the coefficient estimates for the class $\mathcal{N}^{\lambda}_{\sum}(\mathcal{B}(x,z))$ given in definition (2.1)

Lemma 2.4. *[14] Let* $w \in \Omega$ *with* $w(z) = \sum_{n=1}^{\infty} w_n z^n$, $z \in \mathbb{D}$. *Then*

$$|w_1| \leq 1,\ |w_n| \leq 1 - |w|^2 \textit{ for } n \in \mathbb{N} \setminus \{1\}.$$

Theorem 2.5. *If* $f \in N_{\sum}^{\lambda}(B(x,z))$ *and* $x \in \mathbb{C} \setminus \left\{ \mp \frac{1}{3\sqrt{(1+\lambda)|1-(1+\lambda)|}} \right\}$, *then*

$$|a_2| \leq \frac{6|x|\sqrt{6x}}{\sqrt{(1+\lambda)((1+\lambda) - 36\lambda x^2)}} \tag{2.7}$$

$$|a_3| \leq |x|\left(\frac{36|x|}{(1+\lambda)^2} + \frac{3}{(1+2\lambda)}\right) \tag{2.8}$$

Proof. Let $f \in N_{\sum}^{\lambda}(B(x,z))$, then from the definition (1.3) , the subordination(2.1) and (2.2)satisfy.Thus ther exist an analytic function κ in D with $\kappa(0) = 0, |\kappa(z)| < 1$,

$$|\kappa_i| < 1 \tag{2.9}$$

such that

$$\lambda\Big(1 + \frac{zf''(z)}{f'(z)}\Big) + (1-\lambda)\Big(\frac{zf'(z)}{f(z)}\Big) = K(x, \kappa(z)) \tag{2.10}$$

Also there exist an analytic function Φ in D with $\Phi(0) = 0, |\Phi(z)| < 1$

$$|\Phi_i| < 1 \tag{2.11}$$

such that

$$\lambda\Big(1 + \frac{wg''(w)}{g'(w)}\Big) + (1-\lambda)\Big(\frac{wg'(w)}{g(w)}\Big) = K(x, \Phi(w)) \tag{2.12}$$

where $i \in N$ and the analytic function κ and Φ have the form

$$\kappa(z) = \kappa_1 z + \kappa_2 z^3 + \kappa_3 z^3 + \cdots$$
$$\Phi(z) = \Phi_1 z + \Phi_2 z^2 + \Phi_3 z^3 + \cdots$$

Hence from the functions $K(x, \kappa(z))$ and $K(x, \Phi(w))$ are of the form

$$\begin{aligned} K(x, \kappa(z)) = \mathcal{B}_1(x) + \mathcal{B}_2(x)\kappa_1 z + [\mathcal{B}_2(x)\kappa_2 + \mathcal{B}_3(x)\kappa_1^2]z^2 + \\ [\mathcal{B}_2(x)\kappa_3 + 2\mathcal{B}_3(x)\kappa_1\kappa_2 + 2\mathcal{B}_3(x)\kappa_1^3]z^3 + \cdots \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} K(x, \kappa(w)) = \mathcal{B}_1(x) + \mathcal{B}_2(x)\Phi_1 w + [\mathcal{B}_2(x)\Phi_2 + \mathcal{B}_3(x)\Phi_1^2]w^2 + \\ [\mathcal{B}_2(x)\Phi_3 + 2\mathcal{B}_3(x)\Phi_1\Phi_2 + 2\mathcal{B}_3(x)\Phi_1^3]w^3 + \cdots \end{aligned} \tag{2.14}$$

so comparing the corresponding coefficients in (2.10) by (2.13) and (2.12) by (2.14) we obtained that

$$2a_2 = \mathcal{B}_2(x)\kappa_1 \tag{2.15}$$

$$-(1+\lambda)2a_2 = \mathcal{B}_2(x)\Phi_1 \tag{2.16}$$

$$2(1+2\lambda)a_3 - (1+3\lambda)a_2^2 = \mathcal{B}_2(x)\kappa_2 + \mathcal{B}_2(x)\kappa_1^2, \tag{2.17}$$

$$(3+5\lambda)a_2^2 - 2(1+2\lambda)a_3 = \mathcal{B}_2(x)\Phi_2 + \mathcal{B}_2(x)\Phi_1^2 \tag{2.18}$$

From (2.15)and (2.16)

$$\kappa_1 = -\Phi_1 \tag{2.19}$$

and

$$\frac{2(1+\lambda)^2 a_2^2}{\mathcal{B}_2^2(x)} = (\kappa_1^2 + \Phi_1^2) \tag{2.20}$$

also from the summation of the equation (2.17) and (2.18)we easily obtained

$$(2+2\lambda)a_2^2 = \mathcal{B}_2(x)(\kappa_1 + \Phi_2) + \mathcal{B}_3(x)(\kappa_1^2 + \Phi_1^2) \tag{2.21}$$

By using (2.20)in (2.21) we have

$$a_2^2 = \frac{\mathcal{B}_2^3(x)(\kappa_2 + \Phi_2)}{\mathcal{B}_2^2(x) - (1-\lambda)\mathcal{B}_3(x)} \tag{2.22}$$

considering the relation (1.2)and (1.3) and using them in (2.22), we get

$$a_2^2 = \frac{108x^3(\kappa_2 + \Phi_2)}{(1+\lambda)((1+\lambda) - 36\lambda x^2)} \tag{2.23}$$

Using (2.9) and (2.11) together with the triangle's inequality in in the (2.23) it follows

$$|a_2| = \frac{6|x|\sqrt{6x}}{\sqrt{(1+\lambda)((1+\lambda) - 36\lambda x^2)}} \tag{2.24}$$

Also if we subtract from (2.17) from (2.18), considering (2.19), we have

$$(4+8\lambda)a_3 - (8\lambda+4)a_2^2 = \mathcal{B}_x(\kappa_2 - \Phi_2)$$

then

$$a_3 = \frac{\mathcal{B}_2(x)(\kappa_2 - \Phi_2)}{(4+8\lambda)} + a_2^2 \tag{2.25}$$

This equation combine with (2.20)leads to

$$a_3 = \frac{B_2^2(x)(\kappa_1^2 + \Phi_1^2)}{2(1+\lambda)^2} + \frac{B_2(x)(\kappa_2 + \Phi_2)}{(4+8\lambda)} \tag{2.26}$$

Utilizing the triangle inequality (2.9),(2.11)and (2.22)from (2.26) it follows

$$|a_3| \leq |x|\left(\frac{36|x|}{(1+\lambda)^2} + \frac{3}{(2\lambda+1)}\right) \tag{2.27}$$

for the special choice of the parameter λ we obtain the following: □

Corollary 2.6. *I $f \in \mathcal{N}_{\sum}^{\lambda}(\mathcal{B}(x,z))$, then our results coincides with the results Theorem 1 in .*

Corollary 2.7. *If $f \in \mathcal{N}_{\sum}^{\lambda}(\mathcal{B}(x,z))$,and $\lambda = 1$, $x \in \mathbb{C} \setminus \left\{ \mp \frac{\sqrt{2}}{6} \right\}$ then we obtain .*

$$|a_2| \leq \frac{3|x|\sqrt{6|x|}}{\sqrt{1-18x^2}}$$

$$|a_3| \leq |x|(9|x|+1)$$

Theorem 2.8. *If $f \in N_{\sum}^{\lambda}(B(x,z))$ and $x \in \mathbb{C} \setminus \left\{ \mp \frac{1}{3\sqrt{(1+\lambda)|1-(1+\lambda)|}} \right\}$, then*

$$|a_3 - \varphi a_2^2| \leq \begin{cases} |x| & for\ |1-\varphi| \leq \frac{|(1+\lambda)((1+\lambda)-36\lambda x^2)|}{216x^3} \\ \frac{216|x^3||1-\varphi|}{|(1+\lambda)((1+\lambda)-36\lambda x^2)|} & for |1-\varphi| \geq \frac{|(1+\lambda)((1+\lambda)-36\lambda x^2)|}{216x^3} \end{cases} \tag{2.28}$$

Proof. If $f \in N_{\sum}^{\lambda}(B(x,z))$ has the form (1.5) from the equations (2.22) and (2.25) we get

$$\begin{aligned} a_3 - \varphi a_2^2 &= a_2^2 \frac{\mathcal{B}_2(x)(\kappa_2 - \Phi_2)}{(8\lambda+4)} - \varphi a_2^2 \\ &= (1-\varphi)a_2^2 + \frac{\mathcal{B}_2(x)(\kappa_2 - \Phi_2)}{(8\lambda+4)} \end{aligned} \tag{2.29}$$

$$= \mathcal{B}_2(x)\left\{ \left[\hbar(\varphi) + \frac{1}{(8\lambda+4)} \right] \kappa_2 + \left[\hbar(\varphi) + \frac{1}{(8\lambda+4)} \right] \Phi_2 \right\}$$

where $\hbar(\varphi) = \frac{(1-\varphi)\mathcal{B}_2^2(x)}{2(1+\lambda)[\mathcal{B}_2^2(x)-(1+\lambda)\mathcal{B}_3(x)]}$. Using the inequalities (2.9), (2.11) and applying lemma (2.4) we obtained the desired results. □

Corollary 2.9. *I $f \in \mathcal{N}_{\sum}^{\lambda}(\mathcal{B}(x,z))$, then our results coincides with the results Theorem 1 in .*

Corollary 2.10. *If $f \in \mathcal{N}_{\sum}^{\lambda}(\mathcal{B}(x,z))$,and $\lambda = 1$, $x \in \mathbb{C} \setminus \left\{ \mp \frac{\sqrt{2}}{6} \right\}$ then we obtain .*

$$|a_3 - \varphi a_2^2| \leq \begin{cases} |x|, & for |1-(\varphi)| \leq \frac{|1+18x^2|}{54x^2} \\ \frac{216|x^3||1-\varphi|}{|(1+\lambda)((1+\lambda)-36\lambda x^2)|} & for |1-(\varphi)| \geq \frac{|1+18x^2|}{54x^2} \end{cases}$$

3. CONCLUSION

In this research, we introduced a new subclass of bi-univalent functions defined in the open unit disc using balancing polynomials. Then we investigated $|a_2|$ and $|a_3|$ for the initial two functions' Taylor-Maclaurin coefficients that belong to the new class and Fekete-Szegö type inequality for defined subclass.

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