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Bounds of Balanced Laplacian Energy of a Complete Bipartite Graph

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Abstract

Let G be a signed connected graph with order n and size m. The signed laplacian is defined by $\bar{L} = \bar{D} - W$, where $\bar{D}$ is signed degree matrix and W is a symmetric matrix with zero diagonal entries. The signed laplacian is a symmetric positive semidefinite. Let $\mu_1 \geq \mu_2 \geq \ldots \mu_{n-1} \geq \mu_n = 0$ be the eigen values of the laplacian matrix. The signed laplacian energy is defined as $\bar{L}E(G) = \sum_{i=0}^{n} \left|\mu_i - \frac{2m}{n}\right|$. In this paper, we defined balanced laplacian energy of a complete bipartite graph and its upper bounds are attained.

Keywords: Complete bipartite graph, Balanced graph, Signed Laplacian matrix , Signed laplacian Energy of a graph.
AMS Classification: 05C50, 05C69

I INTRODUCTION

A Signed graph is a graph with the additional structure that edges are given a sign of either +1 or -1. Formally, a signed graph is a pair $\Sigma = (\Gamma,\sigma)$ consisting of an underlying graph $\Gamma = (V,E)$ and a signature $\sigma : E \rightarrow \{+1,-1\}$[19,20,21].

Define the adjacency matrix $A(\sum) = (a_{ij})_{nxn}$ as

$$a_{ij} = \begin{cases} \sigma_{ij} \ if \ v_i \ \ is \ \ adjacent \ \ to \ v_j \\ 0 \ \ otherwise \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, of A, assumed in non increasing order, are the eigenvalues of the graph G. The energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [1,2,3,4,12,13].

I.Gutman and B.zhou defined the Laplacian energy of a graph G in the year 2006[5,6,8,10,11]. Let G be a finite , simple and connected graph with order n and size m . The Laplacian matrix of the graph G denoted by L(G) = D(G) - A(G) is a square matrix of order n, where D(G) is the diagonal matrix of vertex degrees of the graph G and A(G) is the adjacency matrix. Let $\mu_1, \mu_2, \ldots, \mu_n$ form the Laplacian spectrum of its Laplacian matrix G then the Laplacian energy LE(G) of G is defined as $LE(G) = \sum_{i=0}^{n} \left| \mu_i - \frac{2m}{n} \right|$ [14,15,16,17,18].

Definitions and Examples:

Definition 1.1

A Signed graph is just an ordinary graph with each of its edges labelled with either + or a – .

Definition 1.2

Given a signed graph G = (V, W) (where W is a symmetric matrix with zero diagonal entries), the underlying graph of G is the graph with the vertex set V and the set of (undirected) edges $E = \{ (v_i, v_j) / w_{ij} \neq 0 \}$.

Definition 1.3

Let G be a graph with order n and size m. The Laplacian matrix of the graph G is denoted by $L = (L_{ij})$ is a square matrix defined by

$$L_{ij} = \begin{cases} -1 \text{ if } v_i \text{ is adjacent to } v_j \\ 0 \text{ if } v_i \text{ is not adjacent to } v_j \\ d_i \text{ if } v_i = v_j \end{cases}$$

Where d_i is the degree of the vertex v_i .

Definition 1.4

If (V,W) is a signed graph where W is a (mxm) symmetric matrix with zero diagonal entries and with the other entries w_{ij} ε R be arbitrary. The degree of any vertex v_i is defined as $d_i = d(v_i) = \sum_{i=0}^{n} W_{ij}$ and $\overline{D}$ signed degree matrix where $\overline{D}$ = diag ((v_1), (v_2), ... (v_m)).

Definition 1.5

The Signed Laplacian is defined by $\overline{L} = \overline{D} - W$, where $\overline{D}$ is the signed degree matrix. The signed Laplacian is symmetric positive semidefinite.

Definition 1.6

Let G = (V, W) be a signed graph whose underlying graph is connected. Then G is balanced if there is a partition of its vertex set V into two clusters V_1 and V_2 such that all the positive edges connect vertices within V_1 or V_2 and all the negative edges connect vertices between V_1 and V_2. If the signed graph has even number of negative edges then it is called a Balanced Signed graph.

Definition 1.7

Let $\mu_1, \mu_2, \ldots \mu_n$ be the eigenvalues of $\bar{L}E$ (G), which are called Signed Laplacian eigenvalues of G. The Signed Laplacian energy $\bar{L}$ E(G) of G is defined $\bar{L}$ E(G) = $\sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, where $\frac{2m}{n}$ is the average degree of the graph G.

Definition 1.8

A **Complete bipartite graph** is a graph whose vertices can be partitioned into two vertex subsets V_1 and V_2 such that no edge has both end vertices in the same subset and every possible edge that could connect n vertices in different subsets is part of the graph. That is, it is a bipartite graph (V_1 ,V_2 , E) such that any two vertices v_1, v_2 where $v_1 \in V_1$, $v_2 \in V_2$ is an edge in E.

A Complete bipartite graph with partitions of size $|V_1|$ = m and $|V_2|$ = n, is denoted by $K_{m,n}$. A Complete bipartite graph $K_{n,n}$ is a Circulant graph of order 2n and size n^2 .

Preliminaries:

In this section, we listed some previously known results that are needed in the subsequent sections.

Lemma 1: A Bipartite graph G = (V,E) is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G has one end in V_1 and another end in V_2. If $|V_1| = |V_2|$, then we say that G is balanced.

Lemma2: A Signed graph is balanced if every cycle has an even number of negative edges.

Lemma 3: An all negative signed graph is balanced if and only if it is bipartite.

Lemma 4: A Complete bipartite graph $K_{m,n}$ is balanced if m = n.

Lemma 5: Let G be a complete bipartite graph of order 2n with two partite sets V_1 and V_2 and $|V_1| = |V_2| = n$ if and if only if

$d(x) + d(y) = 2n$ for all $x \in V_1$ and $y \in V_2$.

Lemma 6: A Signing of a graph is an assignment of weight + 1 or -1 to each of its edges . A cycle C of a signed bipartite graph G is balanced if the sum of the weights of the edges in C is congruent to 0 (mod 4).

Lemma 7: Let G be a bipartite graph of order n with size m . Then $\mu_1 \geq \frac{4m}{n}$, with equality holds if and only if G is regular.

Lemma 8: Let G be a r- regular graph of order n. Then $\mu_i = r - \lambda_{n-i+1}, 1 \leq i \leq n$.

Lemma 9: Let G be a graph and $G \ncong K_n$. Then $\mu_{n-1} \leq \delta$, where δ is the minimum degree of G.

2. BALANCED LAPLACIAN ENERGY OF A COMPLETE BIPARTITE GRAPH

EXAMPLE 2.1

Let G be a complete bipartite graph $K_{n,n}$ of order 2n with size n^2. The complete bipartite graph contains two different vertex subsets. Each set contains n vertices. Each set belongs to one partition. The two partitions are connected by negative edges. Each vertex in one partition is connected to all other vertices in the second partition by means of a negative edge. So this graph contains an even number of negative edges . Therefore this graph G is called a balanced complete bipartite graph.

Consider the complete bipartite graphs $K_{n,n}$

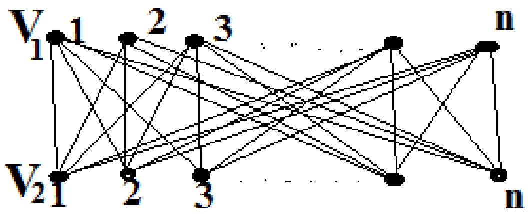

Figure 1: Complete bipartite graph $K_{n,n}$

From the figure 1, we obtained the balanced signed adjacency matrix of the graph as follows

$$A(K_{n,n}) = \begin{pmatrix} 0 & -1 & -1 & \cdots\cdots & -1 & -1 \\ -1 & 0 & -1 & \cdots\cdots & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots\cdots & -1 & 0 \end{pmatrix}_{2nX2n}$$

$$D(K_{n,n}) = \begin{pmatrix} n & 0 & 0 & \cdots\cdots & 0 \\ 0 & n & 0 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots & n \end{pmatrix}_{2nX2n}$$

$\bar{L}(K_{n,n}) = D(K_{n,n}) - A(K_{n,n})$

$$\bar{L}(K_{n,n}) = \begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & n & 1 & \cdots & 1 \\ \cdots & & & & \\ \cdots & & & & \\ 1 & 1 & 1 & \cdots & n \end{pmatrix}_{2nX2n}$$

The characteristic equation of $\bar{L}(K_{n,n})$ is $(\mu - n + 1)^{n-1} (\mu - 3n + 1) = 0$.

Average degree is $\frac{2m}{n} = \frac{2n^2}{2n} = n$

Balanced Laplacian Energy of complete bipartite graph $K_{n,n}$ is

$\bar{L}E(K_{n,n}) = |n - 1 - n|(n-1) \text{ times} + |3n - 1 - n|$

$= n-1 + 2n-1$

$= 3n - 2.$

3. OBSERVATIONS:

3.1. The lower and upper bounds on the energy of a bipartite graph G

$$E(G) \geq \sqrt{4m + n(n-2)(detA)^{2/n}} \quad (1)$$

$$E(G) \leq \sqrt{2mn - 4m + 2n(detA)^{2/n}} \quad (2)$$

3.2. Let G be a bipartite graph of order n with size m. Then the inequality (1) holds. Equality in (1) is attained if only if $G \cong nK_1$ (or)

$G \cong K_{p,q} \cup (n-p-q)K_1$, $p+q \leq n$, $1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor$.

3.3. Let G be a bipartite graph of even order n with m edges. Then the inequality (2) holds. Equality in (2) is attained if and only if G ≅ nK_1 or

G ≅ $\frac{n}{2} K_2$.

3.4. Koolen and Moulton found the bound for the energy of bipartite graphs, in terms of n and m (n ≤ 2m)

$$E(G) \leq \frac{4m}{n} + \sqrt{(n-2)(2m - \frac{8m^2}{n^2})} \quad (3)$$

Equality holds if and only if G ≅ mK_2 (n = 2m) or G ≅ $K_{v,v}$ (n = 2v).

3.5. Let G be a bipartite graph of order n with m edges. Then $\lambda_1 = \frac{2m}{n}$,

$|\lambda_2| = ... = |\lambda_{n-1}|$, $\lambda_n = -\frac{2m}{n}$ if and only if G ≅ nK_1 or G ≅ mK_2 (n = 2m) or

G ≅ $K_{v,v}$ (n = 2v).

4. UPPER BOUND OF BALANCED LAPLACIAN ENERGY OF COMPLETE BIPARTITE GRAPH.

Theorem 4.1:

Let G be a complete bipartite graph of order 2n with m = n^2 edges.

Then $\bar{L}E(G) \leq 2[\, n + \sqrt{(n-1)(M-n^2)}\,]$

where $M = m + \frac{M_1}{2} - \frac{2m^2}{n}$. Equality holds in (3) if and only if

G ≅ $K_1 \cup K_2$ or G ≅ nK_1 or G ≅ mK_2 (n = 2m) or G ≅ $K_{v,v}$ (n = 2v).

Proof:

We defined $\gamma_i = \mu_i - \frac{2m}{n}$, Since $2m = \sum_{i=1}^{n} d_i$, $M_1(G) = \sum_{i=1}^{n} d_i^2$ and

$\sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i(d_i + 1)$.

Now, $\bar{L}\,E(G) = \mu_1 + \sum_{i=2}^{n-1} |\gamma_i|$

$\leq \mu_1 + \sqrt{(n-2)\sum_{i=2}^{n-1} \gamma_i^2}$

$$= \frac{2m}{n} + \gamma_1 + \sqrt{(n-2)\left(2M - \frac{4m^2}{n^2} - \gamma_i^2\right)}.$$

Since G is a complete bipartite graph $K_{n,n}$ with order 2n and size $m = n^2$

and by Lemma 7, $\gamma_1 \geq \frac{2m}{n}$ and by lemma 8, $\mu_{n-1} \leq \delta \leq \frac{2m}{n}$.

Similarly, for $\mu_i \leq \frac{2m}{n}$, we get

$$\bar{L}E(G) \leq 2[\, n + \sqrt{(n-1)(M-n^2)}\,].$$

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