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Subclasses of Univalent Functions Involving Modified Sigmoid Function

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Abstract

The authors obtained some geometric results on certain new classes of analytic functions involving sigmoid function defined by Fadipe-Joseph *et. al.* 2016 as $T_\gamma(\lambda, \beta, \alpha, \mu)$. Extreme point property, radius of starlikeness and convexity, convolution property and Fekete-Szego inequality for the class were proved.

Keywords: Sălăgean operator, modified sigmoid function, radius of starlikeness and convexity, convolution property and Fekete-Szego inequality.

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1. INTRODUCTION

The study of subclasses of analytic and univalent functions in geometric theory and applications is a link between geometry and analysis with wide range of interest among function theorists in recent time. Subclasses of analytic and univalent functions can be as many as researchers who are of interest in the area (see [1], [2], [3] and [4]). In this paper however, using Sălăgean differential operator involving modified sigmoid function $\gamma(s)$, we investigate some geometric properties of the class $T_\gamma(\lambda, \beta, \alpha, \mu)$ introduced in [1].

Let T_γ denote the class of functions of the form

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k; \;\; a_k \geq 0, \;\; \gamma(s) = \frac{2}{1+e^{-s}}. \tag{1.1}$$

and

$$g_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) b_k z^k; \;\; b_k \geq 0 \tag{1.2}$$

which are analytic and univalent in the unit disk $U = \{|z| : z \leq 1\}$.
Then $f_\gamma(z)$ and $g_\gamma(z)$ belong to class T_γ.
If for convenience, we set $T_\gamma = T_1$ we see that $T_1 = T$, is the usual class of the form $f(z) = z - \sum_{k=2}^{\infty} a_k z^k; \;\; a_k \geq 0$ which is analytic in the open unit disk U.
We define an identity function as

$$e_\gamma(z) = z \;\; ; \; a_k = 0 \;\; for \; all \;\; k \geq \; 2 \;\; but \;\; \gamma \; \neq 0 \tag{1.3}$$

1.1. Convolution or Hardamard Product:

Given two analytic function $f_\gamma(z)$ and $g_\gamma(z)$ in T_γ where $f_\gamma(z)$ and $g_\gamma(z)$ are given by (1.1) and (1.2) respectively.
Convolution of $f_\gamma(z)$ and $g_\gamma(z)$ are defined as

$$(f_\gamma * g_\gamma) = z - \sum_{k=2}^{\infty} \gamma(s) a_k b_k z^k = (g_\gamma * f_\gamma); \;\; a_k b_k \geq 0 \tag{1.4}$$

1.2. Starlikeness, Convexity and Close-to-convexity:

A function $f(z)$ defined by (1.1) is said to be starlike of order δ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta; \qquad z \in U \tag{1.5}$$

for some $\delta \;\; (0 \leq \delta \leq 1)$.

In the same way, a function $f(z)$ defined by (1.1) is said to be convex of order δ if and only if $zf'(z)$ is starlike of order δ. In other words, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta; \qquad z \in U \tag{1.6}$$

for some δ $(0 \leq \delta \leq 1)$.

Furthermore, a function $f(z)$ defined by (1.1) is said to be close-to-convex of order δ if

$$Re\left\{zf'(z)\right\} > \delta; \qquad z \in U \tag{1.7}$$

for some $\delta \;\; (0 \leq \delta \leq 1)$.

2. PRELIMINARY RESULTS

2.1. Sălăgean Differential Operator Involving Modified Sigmoid Function

Definition 2.1 *Sălăgean Differential Operator Involving Modified Sigmoid Function*

$$D^n f_\gamma(z) = \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^m(s)k^n a_k z^k; \quad m = n+1 \tag{2.1}$$

for details see [1].

Definition 2.2 *A function $f_\gamma \in T_\gamma$ defined by (1.1) belongs to the class $T_\gamma(\lambda, \beta, \alpha, \mu)$ if*

$$\left| \frac{\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} - \mu}{\left(\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} + \lambda\right) - 2\alpha\left(\frac{D^{n+1} f_\gamma(z)}{D^n f_\gamma(z)} - \mu\right)} \right| > \beta \quad (z \in U)$$

for $|z| < 1,\ 0 < \lambda \leq 1, \gamma(s) = \frac{2}{1+e^{-s}}$ $(i.e. \gamma \neq 0), 0 < \beta \leq 1, \frac{1}{2} \leq \alpha \leq 1,\ \mu \geq 1,$ $n \in N_0 = N \cup \{0\}$ *See [1]* .

Lemma 2.1 *[1]*
Let $f_\gamma(z) \in T_\gamma$ *defined as* $f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k$ $(a_k \geq 0$ *and* $\gamma \neq 0)$. *If* $f_\gamma \in T_\gamma(\lambda, \beta, \alpha, \mu)$*, then*

$$\sum_{k=2}^{\infty} \gamma(s)k^n \left[\gamma(s)k[1 - \beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_k|$$
$$< \gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu.$$

3. MAIN RESULTS

Theorem 3.1 *Extreme points for class $T_\gamma(\lambda, \beta, \alpha, \mu)$.*
If a function $f_\gamma(z)$ defined by (1.1) belongs to the class $T_\gamma(\lambda, \beta, \alpha, \mu)$.
Let $f_1(z) = z$ and

$$f_\gamma(z) = \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k, \quad k \geq 2.$$

The function $f_\gamma \in T_\gamma(\lambda, \beta, \alpha, \mu)$ if and only if it can be expressed in the form

$$f_\gamma(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z) \tag{3.1}$$

where $\Psi_k \geq 0$ *and* $\sum_{k=1}^{\infty} \Psi_k = 1$.

Proof. Let $f_\gamma(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z)$, $\Psi_k \geq 0$, $k = 1, 2, \ldots$ with

$$\sum_{k=1}^{\infty} \Psi_k = 1$$

.

$$f_\gamma(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z)$$

$$= \Psi_1 f_1(z) + \sum_{k=2}^{\infty} \Psi_k f_k(z)$$

$$= \Psi_1(z) + \sum_{k=2}^{\infty} \Psi_k \left\{ z - \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k \right\}$$

that is,

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \Psi_k \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k.$$

Then,

$$f_\gamma(z) = \sum_{k=2}^{\infty} \Psi_k \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}$$

$$\times \frac{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}$$

So that

$$\sum_{k=2}^{\infty} \Psi_k = 1 - \Psi_1 \leq 1.$$

In other words,

$$f_\gamma(z) = \Psi_1 + \sum_{k=2}^{\infty} \Psi_k = 1 \Rightarrow 1 - \Psi_1 \leq 1.$$

So by Lemma 2.1, $f_\gamma(z) \in T_\gamma(\lambda, \beta, \alpha, \mu)$.
Conversely, assume that the function $f_\gamma(z)$ defined by (1.1) belongs to the class $T_\gamma(\lambda, \beta, \alpha, \mu)$, then by Lemma 2.1,

$$a_k \leq \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} \quad (k \geq 2).$$

Setting

$$\Psi_k \leq \frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]a_k}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}$$

and

$$\Psi_1 = 1-\sum_{k=2}^{\infty}\Psi_k.$$

So that

$$\sum_{k=1}^{\infty}\Psi_k f_k = \frac{\{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu\}\left\{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]\right\}}{\left\{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]\right\}\{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu\}}.$$

We thus notice that we can express $f_k(z)$ in the form (3.1).
Therefore, $f_\gamma(z)=\sum_{k=1}^{\infty}\Psi_k f_k$ which completes the proof.

Fekete-Szegŏ inequality for the class $T_\gamma(\lambda,\beta,\alpha,\mu)$

In this section, we use the values of a_2 and a_3 given by Lemma 2.1 to establish the Fekete-Szegŏ inequality for functions $f_\gamma(z)$ belonging to the class $T_\gamma(\lambda,\beta,\alpha,\mu)$.

Theorem 3.2 *Fekete-Szegŏ inequality for class* $T_\gamma(\lambda,\beta,\alpha,\mu)$.
If a function $f(z)\in T_\gamma$ *defined by (1.1) belongs to the class* $T_\gamma(\lambda,\beta,\alpha,\mu)$ *and* $\mu\in R$. *Then,*

$$|a_3-\sigma a_2^2| \leq \frac{B_3[B_2^2-\sigma B_3 B_1]}{B_1 B_2^2}.$$

Proof: From Lemma 2.1,

$$a_2 \leq \frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)2^n\left[2\gamma(s)[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} \qquad and$$

$$a_3 \leq \frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)3^n\left[3\gamma(s)[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}.$$

So that

$$a_3-\sigma a_2^2 = \frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)3^n\left[3\gamma(s)[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}$$

$$-\sigma\left\{\frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)2^n\left[2\gamma(s)[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}\right\}^2.$$

Set

$$B_1 = \left\{\gamma(s)3^n\left[3\gamma(s)[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]\right\}; \ \ and$$

$$B_2 = \left\{\gamma(s)2^n\left[2\gamma(s)[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]\right\}$$

$$B_3 = \left\{(\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu)\right\}.$$

Thus,

$$|a_3 - \sigma a_2^2| \ \leq \ \frac{B_3 B_2^2 - \sigma B_3^2 B_1}{B_1 B_2^2} = \frac{B_3[B_2^2 - \sigma B_3 B_1]}{B_1 B_2^2}$$

which completes the proof.

Radius Properties for class $T_\gamma(\lambda,\beta,\alpha,\mu\,)$**.**

We now obtain the radii of starlikeness, convexity and close to convexity in this section as follows:

Theorem 3.3 *(Starlikeness): Let the function* $f_\gamma(z)$ *defined by (1.1) be in the class* $T_\gamma(\lambda,\beta,\alpha,\mu)$*; then* $f_\gamma(z)$ *is starlike of order* σ $(0 \leq \delta < 1)$ *in* $|z| < r_1$*, where*

$$r_1 = inf_k \left\{\frac{(1-\delta)\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]\gamma(s)(k-\delta)}\right\}^{\frac{1}{k-1}} ; \ k \geq 2 \tag{3.2}$$

The result is sharp for

$$f_\gamma(z) = z - \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k \ , \ \ k \geq 2.$$

Proof: It suffices to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta.$
That is,

$$\left|\frac{zf_\gamma'(z)}{f_\gamma} - 1\right| = \left|\frac{z - \sum_{k=2}^{\infty} k\gamma(s)a_k z^k - z + \sum_{k=2}^{\infty}\gamma(s)a_k z^k}{z - \sum_{k=2}^{\infty}\gamma(s)a_k z^k}\right|$$

$$\left|\frac{-\sum_{k=2}^{\infty}\gamma(s)(k-1)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty}\gamma(s)a_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty}\gamma(s)(k-1)a_k|z|^{k-1}}{(1 - \sum_{k=2}^{\infty}\gamma(s)a_k|z|^{k-1})} < 1 - \delta$$

$$\sum_{k=2}^{\infty}\gamma(s)(k-1)a_k|z|^{k-1} \leq (1-\delta)(1-\sum_{k=2}^{\infty}\gamma(s)a_k|z|^{k-1})$$

$$\sum_{k=2}^{\infty}\gamma(s)(k-1)a_k|z|^{k-1} \leq (1-\delta)-\sum_{k=2}^{\infty}\gamma(s)(1-\delta)a_k|z|^{k-1}$$

$$\sum_{k=2}^{\infty}\gamma(s)(k-1+1-\delta)a_k|z|^{k-1} \leq (1-\delta)$$

$$\sum_{k=2}^{\infty}\gamma(s)(k+\delta)a_k|z|^{k-1} \leq (1-\delta)$$

$$\sum_{k=2}^{\infty}\gamma(s)\left(\frac{k-\delta}{1-\delta}\right)a_k|z|^{k-1} \leq 1 \ .$$

Hence by Lemma 2.1, the above inequality holds if

$$\sum_{k=2}^{\infty}\gamma(s)\frac{k-\delta}{1-\delta}a_k|z|^{k-1} \leq 1 \ .$$

It follows that

$$\sum_{k=2}^{\infty}\gamma(s)\frac{(k-\delta)|z|^{k-1}}{(1-\delta)} \leq \frac{1}{a_k}$$

$$\frac{\gamma(s)(k-\delta)|z|^{k-1}}{(1-\delta)} \leq \frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}; \ \ (k=2,3,\ldots)$$

By multiplying both sides by the inverse of $\frac{\gamma(s)(k-\delta)}{(1-\delta)}$, we have that

$$|z|^{k-1} \leq \frac{(1-\delta)\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\gamma(s)(k-\delta)}.$$

We find the $(k-1)th$ root of both sides, so that

$$|z| \leq \left\{\frac{(1-\delta)\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\{\gamma^2(s)(1-\beta)+\gamma(s)\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\gamma(s)\mu\}\gamma(s)(k-\delta)}\right\}^{\frac{1}{k-1}} \quad where\ |z|<r_1.$$

Thus,

$$r_1 = inf_k\left\{\frac{(1-\delta)\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu]\gamma(s)(k-\delta)}\right\}^{\frac{1}{k-1}} \ ; \ k\geq 2$$

which completes the proof.

Theorem 3.4 *(Convexity): Let the function $f_\gamma(z)$ defined by (1.1) be in the class $T_\gamma(\lambda,\beta,\alpha,\mu)$; then $f_\gamma(z)$ is convex of order δ $(0 \leq \delta < 1)$ in $|z| < r_2$, where*

$$r_2 = inf_k \left\{ \frac{(1-\delta)\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{k(k-\delta)\gamma(s)[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]} \right\}^{\frac{1}{k-1}} \quad ; \; k \geq 2 \tag{3.3}$$

The result is sharp for

$$f_\gamma(z) = z - \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k \;, \quad k \geq 2.$$

Proof: It suffices to show that $\left|\frac{zf''_\gamma(z)}{f'_\gamma(z)}\right| < 1-\delta, \; |z| < r_2.$

$$Since \; \left|\frac{zf''_\gamma(z)}{f'_\gamma(z)}\right| = \left|\frac{\sum_{k=2}^{\infty}\gamma(s)k(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}k\gamma(s)a_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty}\gamma(s)k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}\gamma(s)ka_k|z|^{k-1}} < 1-\delta$$

To prove the Theorem, we must show that

$$\frac{\sum_{k=2}^{\infty}\gamma(s)k(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}\gamma(s)ka_k|z|^{k-1}} \leq 1-\delta$$

$$\sum_{k=2}^{\infty}\gamma(s)k(k-\delta)a_k|z|^{k-1} \leq 1-\delta.$$

And by Lemma 2.1, we obtain

$$|z|^{k-1} \leq \frac{(1-\delta)\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{k(k-\delta)\gamma(s)[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]}$$

or

$$r_2 = inf_k \left\{ \frac{(1-\delta)\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{k(k-\delta)\gamma(s)[\gamma(s)\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]} \right\}^{\frac{1}{k-1}}$$

which completes the proof.

Theorem 3.5 *(Close-to-convex): Let the function $f_\gamma(z)$ defined by (1.1) be in the class $T_\gamma(\lambda,\beta,\alpha,\mu)$. Then $f_\gamma(z)$ is closed-to-convex of order δ $(0 \le \delta < 1)$ in $|z| < r_3$, where*

$$r_3 \le \left\{ \frac{(1-\delta)\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{k\gamma(s)[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]} \right\}^{\frac{1}{k-1}} ;\ k \ge 2. \tag{3.4}$$

The result is sharp

$$f_\gamma(z) = z - \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} z^k ,\quad k \ge 2.$$

Proof: It suffices to show that $|f'_\gamma(z) - 1| = 1 - \delta,\ \ (0 \le \delta < 1)$ for $|z| < r_3$.
Thus,

$$|f'_\gamma(z) - 1| = \left| 1 - \sum_{k=2}^{\infty} k\gamma(s)a_k z^{k-1} - 1 \right| = \left| -\sum_{k=2}^{\infty} k\gamma(s)a_k z^{k-1} \right| \le \left| \sum_{k=2}^{\infty} k\gamma(s)a_k z^{k-1} \right|.$$

$$Since\ \ |f'_\gamma(z) - 1| \le \sum_{k=2}^{\infty} \gamma(s)ka_k|z^{k-1}| \le 1-\delta\ \ if\ we\ divide\ both\ by\ (1-\delta),\ then,$$

$$\sum_{k=2}^{\infty} \gamma(s)\left(\frac{k}{1-\delta}\right) a_k \left|z^{k-1}\right| \le 1$$

$$\sum_{k=2}^{\infty} \frac{k\gamma(s)\left|z^{k-1}\right|}{(1-\delta} \le \frac{1}{a_k} \tag{3.5}$$

By coefficient estimates of $f_\gamma(z) \in T_\gamma(\lambda,\beta,\alpha,\mu)$ given by Lemma 2.1 above, (3.5) holds if

$$\sum_{k=2}^{\infty} \frac{k\gamma(s)|z|^{k-1}}{(1-\delta)} \le \frac{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu};\ k \ge 2.$$

$$|z| \le \left\{ \frac{(1-\delta)\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{k\gamma(s)[\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu]} \right\}^{\frac{1}{k-1}} ;\ k \ge 2.$$

Hence,

$$r_3 \le \left\{ \frac{(1-\delta)\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}{k\left\{\gamma^2(s)(1-\beta) + \gamma(s)\beta[2\alpha(\gamma(s)-\mu) - \lambda] - \gamma(s)\mu\right\}} \right\}^{\frac{1}{k-1}} ;\ k \ge 2.$$

Neighbourhood Property for class $T_\gamma(\lambda,\beta,\alpha,\mu)$.

We define the (n,ς)-neighborhood of a function $f_\gamma \in T_\gamma$ by

$$N_{(n,\varsigma)}(e) = \left\{ g_\gamma : g_\gamma \in T_\gamma, g_\gamma(z) = z - \sum_{k=t+1}^{\infty} \gamma(s) b_k z^k \ \ and \ \ \sum_{k=2}^{\infty} \gamma(s) k |b_k| \leq \varsigma \right\} \tag{3.6}$$

And for the identity function $e_\gamma(z)$ defined by (1.3), we have the following result.

Let T_γ denotes the class of functions $f_\gamma(z)$ defined by (1.1) which is analytic in the open unit disk

$U = \{|z| : \ z \in C \ \ and \ \ |z| < 1\}$.

Theorem 3.6 *Neighbourhood Property for class* $T_\gamma(\lambda,\beta,\alpha,\mu)$.

Let

$$\varsigma = \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]} \tag{3.7}$$

Then $T_\gamma(\lambda,\beta,\alpha,\mu) \subset N_{(n,\varsigma)}e(z)$.

Proof: Suppose $f_\gamma \in T_\gamma(\lambda,\beta,\alpha,\mu)$, then from Lemma 2.1, we have that

$$\sum_{k=2}^{\infty} \gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_k|$$
$$< \gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu$$

and thus for $k = 2$, we have that

$$\gamma(s)2^n \left[\gamma(s)2[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_2|$$
$$\leq \sum_{k=2}^{\infty} \gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right] |a_k|$$
$$< \gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu$$

$$\sum_{k=2}^{\infty} \gamma(s)|a_k| \leq \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s)-\mu) - \lambda] - \mu}{\gamma(s)k^n \left[\gamma(s)k[1-\beta(1-2\alpha)] - \beta\left(\lambda + 2\alpha\mu + \frac{1}{\beta}\right)\right]}. \tag{3.8}$$

But $|z| < r$ which implies that

$$|f'(z)| \leq 1 - |z| \sum_{k=2}^{\infty} k\gamma(s)a_k \leq 1 - r \sum_{k=2}^{\infty} k\gamma(s)a_k \tag{3.9}$$

It thus follows that

$$|f'_{\gamma}(z)| \leq 1-r\sum_{k=2}^{\infty} k\gamma(s)a_k \leq k\frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]k^n}$$

$$\Rightarrow \sum_{k=2}^{\infty} k\gamma(s)a_k \leq \sum_{k=2}^{\infty}\gamma(s)a_k$$

$$\leq \frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]k^n} = \varsigma \;\; by \quad equation \quad (3.6)$$

Thus, $f_{\gamma} \in N_{(n,\varsigma)}(e)$.

Theorem 3.7 *Convolution Property for class $T_{\gamma}(\lambda,\beta,\alpha,\mu)$.*
Let $f_{\gamma}(z)$ and $g_{\gamma}(z)$ defined by (1.1) and (1.2)) respectively be members of $T_{\gamma}(\lambda,\beta,\alpha,\mu)$ such that $h_{\gamma}(z) = z - \sum_{k=2}^{\infty}\gamma(s)a_k b_k z^k$ where $a_k b_k \geq 0$ as defined by (1.4).
Then, $h_{\gamma}(z)$ is in the subclass of $T_{\gamma}(\lambda,\beta,\alpha,\mu)$ where

$$\frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}$$

$$\leq \frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_2}$$

.

Proof: From Lemma 2.1,

$$\sum_{k=2}^{\infty}\frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}a_k \leq 1 \tag{3.10}$$

Similarly,

$$\sum_{k=2}^{\infty}\frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}b_k \leq 1 \tag{3.11}$$

We however need to determine the largest μ_2 such that

$$\sum_{k=2}^{\infty}\frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_2}a_k b_k \leq 1.$$

By Cauchy-Schwarz inequality,

$$\sum_{k=2}^{\infty}|a_k b_k| \leq \sqrt{\sum_{k=2}^{\infty}|a_k|^2}\sqrt{\sum_{k=2}^{\infty}|b_k|^2}.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty}\frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}\sqrt{a_k b_k} \leq 1.$$

It suffices to show what

$$\frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_2}a_k b_k$$

$$\leq \sum_{k=2}^{\infty}\frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}\sqrt{a_k b_k} \leq 1 \qquad (3.12)$$

equivalently,

$$\sqrt{a_k b_k} \leq$$

$$\frac{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}\times$$

$$\frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_2}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]} \leq$$

$$\frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_2}.$$

But from (3.12) we have

$$\sqrt{a_k b_k} \leq \frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}.$$

Consequently, we show that

$$\frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right]}$$

$$\leq \frac{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1}{\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_2}.$$

Equivalently,

$$\frac{[\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1][\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_2]}{\gamma(s)k^n\left[\gamma(s)k[1-\beta(1-2\alpha)]-\beta\left(\lambda+2\alpha\mu+\frac{1}{\beta}\right)\right][\gamma(s)(1-\beta)+\beta[2\alpha(\gamma(s)-\mu)-\lambda]-\mu_1]} \leq 1.$$

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