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New Oscillation Conditions for Second-Order Delay Difference Equations with Several Sub-Linear Neutral Terms

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Abstract

We derive oscillatory conditions for the second-order delay difference equation

$$\Delta\left(\varphi(\zeta)(\Delta v(\zeta))^{\xi}\right) + \mu(\zeta)x^{\nu}(\eta(\zeta)) = 0; \quad \zeta \geq \zeta_0,$$

where $v(\zeta) = x(\zeta) + \sum_{i=1}^{m} p_i(\zeta)x^{\lambda_i}(\kappa_i(\zeta))$. We investigate oscillatory behavior for the cases when $\xi > \nu$ and $\xi < \nu$. Many results presented in the literature are supplemented and improved by this new theorem. Finally, we give some examples to show our major findings.

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1. INTRODUCTION

In this article, we consider the neutral difference equation of the form

$$\Delta(\varphi(\zeta)(\Delta v(\zeta))^{\xi}) + \mu(\zeta)x^{\nu}(\eta(\zeta)) = 0, \quad \zeta \geq \zeta_0, \tag{1.1}$$

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where $v(\zeta) = x(\zeta) + \sum_{i=1}^{m} p_i x^{\lambda_i}(\kappa_i(\zeta))$. Here Δ is the forward difference operator given by $\Delta x(\zeta) = x(\zeta+1) - x(\zeta)$. We will assume throughout this work that:

(A_1) $\lambda_1, \lambda_2, \ldots, \lambda_m, \nu, \xi \in \left\{\frac{a}{b} : \text{a and b are odd positive integers}\right\}$;

(A_2) $\{\kappa_i(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ $(i = 1, 2, \ldots, m)$ and $\{\eta(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ are sequences of positive integers with $\kappa_i(\zeta) \le \zeta$ $(i = 1, 2, \ldots, m)$, $\eta(\zeta) \le \zeta$, $\lim_{\zeta\to\infty} \kappa_i(\zeta) = \lim_{\zeta\to\infty} \eta(\zeta) = \infty$, $i = 1, 2, \ldots, m$ and $\Delta\eta(\zeta) \ge 0$;

(A_3) $\{\varphi(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of positive real numbers;

(A_4) $\{\mu(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a sequence of non-negative real numbers and $\mu(\zeta) \not\equiv 0$ for sufficiently large ζ;

(A_5) $\lim_{\zeta\to\infty} R(\zeta) = \infty$, where $R(\zeta) = \sum_{s=\zeta_0}^{\zeta-1} \frac{1}{\varphi^{\frac{1}{\xi}}(s)}$;

(A_6) $\{p_i(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ are positive real sequences for $i = 1, 2, \ldots, m$.

A solution of (1.1) is said to be oscillatory if its terms are neither eventually positive nor eventually negative; otherwise, it is said to be a non-oscillatory solution. Equation (1.1) is called oscillatory if and only if every solution is oscillatory; otherwise, non-oscillatory.

The existence of solutions, asymptotic behavior, oscillation and non-oscillation for second-order difference equations were extensively analyzed in many research papers over the last three decades; see, for example, [4, 7, 8, 9, 10, 11, 17, 18, 20] and the cited references. Many areas of applied mathematics use neutral difference and differential equations, such as bifurcation analysis [5], stability theory [22, 23], the dynamical behavior of delayed network systems [24], circuit theory [6], population dynamics [14] and so on. As a result, these equations have piqued people's interest in recent decades. We can refer the monographs [1, 2, 3, 15, 19] for the general concept of difference equations.

We [21] determined oscillatory conditions for the second-order half-linear difference equation of advanced type

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\lambda}) + \mu(\zeta)x^{\lambda}(\zeta + \eta) = 0, \quad \zeta \ge \zeta_0,$$

under the condition that $\sum_{\zeta=\zeta_0}^{\infty} \frac{1}{\varphi^{\frac{1}{\lambda}}(\zeta)} < \infty$.

Gopalakrishnan et al. [12] studied oscillatory properties for the non-canonical second-order difference equation of the delay and advanced type

$$\Delta(\varphi(\zeta)\Delta x(\zeta)) + \mu(\zeta)x(\zeta+\eta) = 0; \quad \zeta \geq \zeta_0.$$

Gopalakrishnan et al. [13] established single-condition criteria for the second-order half-linear advanced difference equation of non-canonical type

$$\Delta(\varphi(\zeta)(\Delta x(\zeta))^{\lambda}) + \mu(\zeta)x^{\lambda}(\zeta+\eta) = 0, \quad \zeta \geq \zeta_0$$

Our aim in this work is to derive oscillatory condition of all solutions of the equation (1.1) under the cases $\xi > \nu$ and $\xi < \nu$.

In the following sections, we presume that all functional inequalities are satisfied; eventually, that is, for all ζ large enough.

2. PRELIMINARY RESULTS

We use the following notations, for any positive decreasing sequence $\{\psi(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ which is converges to zero:

$$P(\zeta) = \left(1 - \sum_{i=1}^{m} \lambda_i p_i(\zeta) - \frac{1}{\psi(\zeta)} \sum_{i=1}^{m} (1-\lambda_i) p_i(\zeta)\right) \geq 0;$$

$$\begin{aligned}
Q_1(\zeta) &= \mu(\zeta)P^{\nu}(\eta(\zeta));\\
Q_2(\zeta) &= \mu(\zeta)P^{\nu}(\eta(\zeta))\psi^{\nu-1}(\eta(\zeta));\\
Q_3(\zeta) &= \mu(\zeta)P^{\nu}(\eta(\zeta))R^{\nu-1}(\eta(\zeta),\zeta_1);\\
Q_4(\zeta) &= \mu(\zeta)P^{\nu}(\eta(\zeta))R^{\nu}(\eta(\zeta),\zeta_1);\\
U(\zeta) &= \sum_{s=\zeta}^{\infty} \mu(s)x^{\nu}(\eta(s)) \geq 0.
\end{aligned}$$

The following lemmas are very useful in proving our results.

Lemma 2.1. *[16] Let a and b be nonnegative real numbers. Then*

$$a^{\lambda}b^{1-\lambda} \leq \lambda a + (1-\lambda)\, b \textit{ for } o < \lambda \leq 1.$$

The equality holds if and only if $a = b$.

Lemma 2.2. *Suppose that $\{x(\zeta)\}$ is an eventually positive solution of* (1.1). *Then*

$$v(\zeta) > 0,\ \Delta v(\zeta) > 0,\ \textit{and}\ \Delta(\varphi(\zeta)(\Delta v(\zeta))^{\xi}) \leq 0,\ \textit{eventually}. \tag{2.1}$$

Proof. By assumption, $v(\zeta) > 0$ and we can find an integer $\zeta_1 \geq \zeta_0$ with

$$x(\zeta) > 0,\ x(\kappa_i(\zeta)) > 0,\ \text{and } x(\eta(\zeta)) > 0$$

for all $\zeta \geq \zeta_1$ and $i = 1, 2, \ldots, m$. From (1.1) it follows that

$$\Delta(\varphi(\zeta)(\Delta v(\zeta))^{\xi}) = -\mu(\zeta)x^{\nu}(\eta(\zeta)) \leq 0, \quad \zeta \geq \zeta_1.$$

Therefore, $\{\varphi(\zeta)(\Delta v(\zeta))^{\xi}\}$ is nonincreasing for $\zeta \geq \zeta_1$. Assume that $\varphi(\zeta)(\Delta v(\zeta))^{\xi} < 0$ for $\zeta \geq \zeta_2 \geq \zeta_1$. Hence

$$\varphi(\zeta)(\Delta v(\zeta))^{\xi} \leq \varphi(\zeta_2)(\Delta v(\zeta_2))^{\xi} \text{ for all } \zeta \geq \zeta_2,$$

that is,

$$\Delta v(\zeta) \leq \left(\frac{\varphi(\zeta_2)}{\varphi(\zeta)}\right)^{\frac{1}{\xi}} \Delta v(\zeta) \text{ for } \zeta \geq \zeta_2.$$

Taking summation from ζ_2 to $\zeta - 1$, we have

$$v(\zeta) \leq v(\zeta_2) + (\varphi(\zeta_2))^{\frac{1}{\xi}} \Delta v(\zeta_2)(R(\zeta) - R(\zeta_2)) \to -\infty \text{ as } \zeta \to \infty,$$

which is a contradiction to $v(\zeta) > 0$. Therefore $\varphi(\zeta)(\Delta v(\zeta))^{\xi} > 0$ for all $\zeta \geq \zeta_2$ and hence $\Delta v(\zeta) > 0$, and hence the proof. □

Lemma 2.3. *Suppose that* $\{x(\zeta)\}$ *is an eventually positive solution of* (1.1). *Then there exists* $\zeta \geq \zeta_0$ *with*

$$v(\zeta) \geq (\varphi(\zeta))^{\frac{1}{\xi}} \Delta v(\zeta) R(\zeta, \zeta_1), \quad \zeta \geq \zeta_1$$

and

$$\Delta\left(\frac{v(\zeta)}{R(\zeta, \zeta_1)}\right) \leq 0, \quad \zeta \geq \zeta_1$$

where $R(\zeta, \zeta_1) = R(\zeta) - R(\zeta_1)$.

Proof. Applying procedure as we followed in the proof of Lemma 2.2, we get (2.1) for all $\zeta \geq \zeta_1 \geq \zeta_0$. Since $\{\varphi(\zeta)(\Delta v(\zeta))^{\xi}\}$ is decreasing, we have

$$v(\zeta) \geq \sum_{s=\zeta_1}^{\zeta-1} \varphi^{\frac{1}{\xi}}(s) \frac{\Delta v(s)}{\varphi^{\frac{1}{\xi}}(s)} \geq \varphi^{\frac{1}{\xi}}(\zeta) \Delta v(\zeta) \sum_{s=\zeta_1}^{\zeta-1} \frac{1}{\varphi^{\frac{1}{\xi}}(s)} = \varphi^{\frac{1}{\xi}}(\zeta) \Delta v(\zeta) R(\zeta, \zeta_1).$$

Using the above inequality, we have

$$\Delta\left(\frac{v(\zeta)}{R(\zeta, \zeta_1)}\right) = \frac{\varphi^{\frac{1}{\xi}}(\zeta)\Delta v(\zeta)R(\zeta, \zeta_1) - v(\zeta)}{\varphi^{\frac{1}{\xi}}(\zeta)R(\zeta, \zeta_1)R(\zeta + 1, \zeta_2)} \leq 0.$$

We conclude that $\left\{\frac{v(\zeta)}{R(\zeta, \zeta_1)}\right\}$ is decreasing for $\zeta \geq \zeta_1$ and hence the proof.

□

Lemma 2.4. *Suppose that* $\{x(\zeta)\}$ *is an eventually positive solution of* (1.1). *Then*

$$x(\zeta) \geq P(\zeta)v(\zeta), \text{ eventually.} \tag{2.2}$$

Proof. Lemma 2.2 implies that there exists $\zeta_1 \geq \zeta_0$ such that $v(\zeta) > 0$, for all $\zeta \geq \zeta_1$. Now,

$$\begin{aligned} x(\zeta) &= v(\zeta) - \sum_{i=1}^{m} p_i(\zeta)x^{\lambda_i}(\kappa_i(\zeta)) \\ &\geq v(\zeta) - \sum_{i=1}^{m} p_i(\zeta)v^{\lambda_i}(\kappa_i(\zeta)) \\ &\geq v(\zeta) - \sum_{i=1}^{m} p_i(\zeta)v^{\lambda_i}(\zeta). \end{aligned}$$

Using Lemma 2.1, we get

$$\begin{aligned} x(\zeta) &\geq v(\zeta) - \sum_{i=1}^{m} p_i(\zeta)(\lambda_i v(\zeta) + (1-\lambda_i)) \\ &= \left(1 - \sum_{i=1}^{m} \lambda_i p_i(\zeta)\right) v(\zeta) - \sum_{i=1}^{m}(1-\lambda_i)p_i(\zeta). \end{aligned} \tag{2.3}$$

Using the fact that $\{v(\zeta)\}$ is positive increasing sequence, and $\{\psi(\zeta)\}$ is positive decreasing sequence which tends to zero, we find an integer $\zeta_2 \geq \zeta_1$ such that

$$v(\zeta) \geq \psi(\zeta) \text{ for } \zeta \geq \zeta_2. \tag{2.4}$$

Applying (2.4) in (2.3), we get

$$x(\zeta) \geq P(\zeta)v(\zeta), \quad \zeta \geq \zeta_2.$$

The proof is now complete. □

Lemma 2.5. *Suppose that* $\{x(\zeta)\}$ *is an eventually positive solution of* (1.1). *Then there exists* $\zeta^* \geq \zeta_0$ *and* $\theta > 0$ *such that*

$$0 < v(\zeta) < \theta R(\zeta) \tag{2.5}$$

and

$$R(\zeta, \zeta^*) \left[\sum_{s=\zeta}^{\infty} \mu(s)x^{\nu}(\eta(s))\right]^{\frac{1}{\xi}} \leq v(\zeta). \tag{2.6}$$

hold for all $\zeta \geq \zeta^*$.

Proof. There exists an integer $\zeta_1 \geq \zeta_0$ with $x(\zeta) > 0$, $x(\kappa_i(\zeta)) > 0$, and $x(\eta(\zeta)) > 0$ for all $\zeta \geq \zeta_1$ and for all $i = 1, 2, \ldots, m$. So there exists a $\zeta_2 \geq \zeta_1$ such that Lemma 2.2 holds and $\{v(\zeta)\}$ satifies (2.1) for $\zeta \geq \zeta_2$. From $\varphi(\zeta)(\Delta v(\zeta))^{\xi} > 0$ and being nonincreasing, we have

$$\Delta v(\zeta) \leq \left(\frac{\varphi(\zeta_2)}{\varphi(\zeta)}\right)^{\frac{1}{\xi}} \Delta v(\zeta_2), \quad \zeta \geq \zeta_2.$$

Sum the inequality from ζ_2 to $\zeta - 1$, we obtain

$$v(\zeta) \leq v(\zeta_2) + (\varphi(\zeta_2))^{\frac{1}{\xi}} \Delta v(\zeta_2) R(\zeta, \zeta_2).$$

Since $R(\zeta, \zeta_2) \to \infty$ as $\zeta \to \infty$ and $\varphi(\zeta)(\Delta v(\zeta))^{\xi}$ is positive and decreasing, there exists a $\theta > 0$ and $\zeta^* \geq \zeta_2$ such that (2.5) holds. On the other hand, $\lim_{\zeta\to\infty} \varphi(\zeta)(\Delta v(\zeta))^{\xi}$ exists, and summing (1.1) from ζ to $l - 1$ implies

$$\varphi(l)(\Delta v(l))^{\xi} - \varphi(\zeta)(\Delta v(\zeta))^{\xi} = \sum_{s=\zeta}^{l-1} \mu(s) x^{\nu}(\eta(s)).$$

Taking limit $l \to \infty$, we have

$$\varphi(\zeta)(\Delta v(\zeta))^{\xi} \geq \sum_{s=\zeta}^{\infty} \mu(s) x^{\nu}(\eta(s)), \tag{2.7}$$

that is,

$$\Delta v(\zeta) \geq \left(\frac{1}{\varphi(\zeta)} \sum_{s=\zeta}^{\infty} \mu(s) x^{\nu}(\eta(s))\right)^{\frac{1}{\xi}}.$$

Therefore,

$$v(\zeta) \geq \sum_{u=\zeta_1}^{\zeta-1} \left[\frac{1}{\varphi(u)} \sum_{s=u}^{\infty} \mu(s) x^{\nu}(\eta(s))\right]^{\frac{1}{\xi}} \geq R(\zeta, \zeta_1) \sum_{s=\zeta}^{\infty} \mu(s) x^{\nu}(\eta(s))$$

and hence the proof.

□

3. SUFFICIENT CONDITIONS FOR OSCILLATION

Theorem 3.1. *If*

$$\sum_{\zeta=\zeta_0}^{\infty} Q_1(\zeta) = \infty \tag{3.1}$$

holds, then (1.1) *is oscillatory.*

Proof. Assume that $\{x(\zeta)\}$ is an eventually positive solution of (1.1). Then, we can find an integer $\zeta_1 \geq \zeta_0$ with $x(\zeta) > 0$, $x(\kappa_i(\zeta)) > 0$, and $x(\eta(\zeta)) > 0$ for all $\zeta \geq \zeta_1$, $i = 1, 2, \ldots, m$ and Lemmas 2.2 and 2.4 hold for $\zeta \geq \zeta_1$. From (1.1), we have

$$\Delta(\varphi(\zeta)(\Delta v(\zeta))^{\xi}) + \mu(\zeta)P^{\nu}(\eta(\zeta))v^{\nu}(\eta(\zeta)) \leq 0 \tag{3.2}$$

for all $\zeta \geq \zeta_1$. Applying (2.1), we conclude that $\lim_{\zeta\to\infty} \varphi(\zeta)(\Delta v(\zeta))^{\xi}$ exists and there exists an integer $\zeta_2 \geq \zeta_1$ and a positive number c with $v(\zeta) \geq c$ for $\zeta \geq \zeta_2$. Summing (3.2) from ζ_2 to $\zeta - 1$, we have

$$c^{\nu} \sum_{s=\zeta_2}^{\zeta-1} \mu(s)P^{\nu}(\eta(s)) \leq \varphi(\zeta_2)(\Delta v(\zeta_2))^{\xi} - \varphi(\zeta))(\Delta v(\zeta))^{\xi} < \infty \text{ as } \zeta \to \infty,$$

which contradicts (3.1).

The proof is similar in the case where $\{x(\zeta)\}$ is an eventually negative solution. □

Remark 3.2. Theorem 3.1 is true for any ν and ξ.

Now, we derive criteria for oscillation to (1.1) when $\nu > 1$.

Theorem 3.3. *If*

$$\sum_{\zeta=\zeta_0}^{\infty} Q_2(\zeta) = \infty \tag{3.3}$$

holds, then (1.1) *is oscillatory.*

Proof. Using the procedure followed in the Theorem 3.1, we get (3.2). Using (2.4) in (3.2), we see that

$$\Delta(\varphi(\zeta)(\Delta v(\zeta))^{\xi}) + \mu(\zeta)P^{\nu}(\eta(\zeta))\psi^{\nu-1}(\eta(\zeta))v(\eta(\zeta)) \leq 0. \tag{3.4}$$

The remainder of the proof is identical to that of Theorem 3.1, thus proved. □

Next, we derive an oscillation criteria for the equation (1.1) where $0 < \nu < 1$.

Theorem 3.4. *If*

$$\sum_{\zeta=\zeta_0}^{\infty} Q_3(\zeta) = \infty \tag{3.5}$$

holds, then (1.1) *is oscillatory.*

Proof. We arrived (3.2) by using the process followed in the proof of Theorem 3.1. From (3.2), we have

$$\Delta(\varphi(\zeta)(\Delta v(\zeta))^{\xi}) + \mu(\zeta)P^{\nu}(\eta(\zeta))R^{\nu-1}((\eta(\zeta),\zeta_1)\frac{v^{\nu-1}(\eta(\zeta))}{R^{\nu-1}(\eta(\zeta),\zeta_1)} \leq 0 \tag{3.6}$$

for $\zeta \geq \zeta_2 \geq \zeta_1$. Since $\left\{\frac{v(\zeta)}{R(\zeta,\zeta_1)}\right\}$ is a non-increasing sequence, there is a constant β such that

$$\frac{v(\eta(\zeta))}{R(\eta(\zeta),\zeta_1)} \leq \beta \text{ for } \zeta \geq \zeta_2. \tag{3.7}$$

Using (3.7) and $\nu < 1$ in (3.6), we have

$$\Delta(\varphi(\zeta)(\Delta v(\zeta))^{\xi}) + \frac{\mu(\zeta)P^{\nu}(\eta(\zeta))R^{\nu-1}(\eta(\zeta),\zeta_1)}{\beta^{\nu-1}}v(\eta(\zeta)) \leq 0.$$

The remainder of the part is same to Theorem 3.3, thus proved. □

In the next theorem, we suppose that there is a constant ν_1, the ratio of odd positive integers with $0 < \nu < \nu_1 < \xi$.

Theorem 3.5. *If*

$$\sum_{\zeta=\zeta_0}^{\infty} Q_4(\zeta) = \infty \tag{3.8}$$

holds, then (1.1) *is oscillatory.*

Proof. Suppose that $\{x(\zeta)\}$ is an eventually positive solution of (1.1). Now, we can find an integer $\zeta_1 \geq \zeta_0$ with $x(\zeta) > 0$, $x(\kappa_i(\zeta)) > 0$, $i = 1, 2, \ldots, m$, $x(\eta(\zeta)) > 0$ for all $\zeta \geq \zeta_1$, and Lemmas 2.2 and 2.5 hold for $\zeta \geq \zeta_1$. Hence,

$$v(\zeta) \geq R(\zeta,\zeta_1)U^{\frac{1}{\xi}}(\zeta) \geq 0 \text{ for } \zeta \geq \zeta_1. \tag{3.9}$$

Using (2.2), (2.5), $\nu - \nu_1 < 0$, and (3.9), we have

$$\begin{aligned}
x^{\nu}(\zeta) &\geq P^{\nu}(\zeta)v^{\nu-\nu_1}(\zeta)v^{\nu_1}(\zeta) \\
&\geq P^{\nu}(\zeta)(\theta R(\zeta,\zeta_1))^{\nu-\nu_1}v^{\nu_1}(\zeta) \\
&\geq P^{\nu}(\zeta)(\theta R(\zeta,\zeta_1))^{\nu-\nu_1}R^{\nu_1}(\zeta,\zeta_1)U^{\frac{\nu_1}{\xi}}(\zeta) \\
&= P^{\nu}(\zeta)\theta^{\nu-\nu_1}R^{\nu}(\zeta,\zeta_1)U^{\frac{\nu_1}{\xi}}(\zeta) \text{ for } \zeta \geq \zeta_2.
\end{aligned}$$

Since $\Delta U(\zeta) = -\mu(\zeta)x^{\nu}(\eta(\zeta)) \leq 0, \quad \zeta \geq \zeta_2$, that is $\{U(\zeta)\}$ is a non-increasing sequence, then

$$\begin{aligned} x^{\nu}(\eta(\zeta)) &\geq P^{\nu}(\eta(\zeta))\theta^{\nu-\nu_1}R^{\nu}(\eta(\zeta),\zeta_1)U^{\frac{\nu_1}{\xi}}(\eta(\zeta)) \\ &\geq P^{\nu}(\eta(\zeta))\theta^{\nu-\nu_1}R^{\nu}(\eta(\zeta),\zeta_1)U^{\frac{\nu_1}{\xi}}(\zeta). \end{aligned} \tag{3.10}$$

Therefore,

$$\Delta\left(U^{1-\frac{\nu_1}{\xi}}\right) \leq \left(1-\frac{\nu_1}{\xi}\right)U^{-\frac{\nu_1}{\xi}}(\zeta)\Delta U(\zeta). \tag{3.11}$$

Summing (3.11) from ζ_2 to $\zeta-1$ and from $U(\zeta) > 0$, we have

$$\begin{aligned} \infty > U^{1-\frac{\nu_1}{\xi}}(\zeta_2) &\geq \left(1-\frac{\nu_1}{\xi}\right)\left[-\sum_{s=\zeta_2}^{\zeta-1}U^{-\frac{\nu_1}{\xi}}(s)\Delta U(s)\right] \\ &= \left(1-\frac{\nu_1}{\xi}\right)\sum_{s=\zeta_2}^{\zeta-1}U^{-\frac{\nu_1}{\xi}}(s)\mu(s)x^{\nu}(\eta(s)) \\ &\geq \frac{\left(1-\frac{\nu_1}{\xi}\right)}{\theta^{(\nu_1-\nu)}}\sum_{s=\zeta_2}^{\zeta-1}\mu(s)P^{\nu}(\eta(s))R^{\nu}(\eta(s),\zeta_1), \end{aligned}$$

which contradicts (3.8) as $\zeta \to \infty$ and hence proved. □

In the next theorem, we suppose that there is a constant ν_2, the ratio of odd positive integers with $\lambda < \nu_2 < \nu$.

Theorem 3.6. *Assume that* $\Delta\varphi(\zeta) \geq 0$. *Suppose that*

$$\sum_{\zeta=\zeta_0}^{\infty}\left[\frac{1}{\varphi(\eta(\zeta))}\sum_{s=\zeta+1}^{\infty}Q_1(s)\right]^{\frac{1}{\xi}} = \infty. \tag{3.12}$$

Then (1.1) *is oscillatory.*

Proof. Assume that $\{x(\zeta)\}$ is an eventually positive solution of (1.1). Then, we can find an integer $\zeta_1 \geq \zeta_0$ with $x(\zeta) > 0$, $x(\kappa_i(\zeta)) > 0$, $(i = 1, 2, \ldots, m)$, $x(\eta(\zeta)) > 0$ and Lemmas 2.2 and 2.4 hold for $\zeta \geq \zeta_1$. We conclude that $\{v(\zeta)\}$ satisfies (2.1), $\{v(\zeta)\}$ is increasing and $x(\zeta) \geq P(\zeta)v(\zeta)$ for all $\zeta \geq \zeta_1$. So,

$$x^{\nu}(\zeta) \geq P^{\nu}(\zeta)v^{\nu}(\zeta) \geq P^{\nu}(\zeta)v^{\nu-\nu_2}(\zeta)v^{\nu_2}(\zeta) \geq P^{\nu}(\zeta)v^{\nu-\nu_2}(\zeta_1)v^{\nu_2}(\zeta)$$

implies that

$$x^{\nu}(\eta(\zeta)) \geq P^{\nu}(\eta(\zeta))v^{\nu-\nu_2}(\zeta_1)v^{\nu_2}(\eta(\zeta)) \text{ for } \zeta \geq \zeta_2 \geq \zeta_1. \tag{3.13}$$

Using (2.7) and (3.13), we have

$$\varphi(\zeta)(\Delta v(\zeta))^{\xi} \geq v^{\nu-\nu_2}(\zeta_1)\left[\sum_{s=\zeta}^{\infty}\mu(s)P^{\nu}(\eta(\zeta))\right]v^{\nu_2}(\eta(\zeta)) \tag{3.14}$$

for $\zeta \geq \zeta_2$. From $\{\varphi(\zeta)(\Delta v(\zeta))^{\xi}\}$ being nonincreasing and $\eta(\zeta) \leq \zeta$, we have

$$\varphi(\eta(\zeta))(\Delta v(\eta(\zeta)))^{\frac{1}{\xi}} \geq \varphi(\zeta)(\Delta v(\zeta))^{\frac{1}{\xi}}.$$

Using the increasing nature of $\{v(\zeta)\}$ and the decreasing nature of $\{\varphi(\zeta)(\Delta v(\zeta))^{\xi}\}$ in (3.14)

$$\varphi(\zeta)(\Delta v(\eta(\zeta)))^{\xi} \geq v^{\nu-\nu_2}(\zeta_1)\left[\sum_{s=\zeta+1}^{\infty}\mu(s)P^{\nu}(\eta(\zeta))\right]v^{\nu_2}(\eta(\zeta+1))$$

or

$$\frac{\Delta v(\eta(\zeta))}{v^{\frac{\nu_2}{\xi}}(\eta(\zeta+1))} \geq \left[\frac{v^{\nu-\nu_2}(\zeta_1)}{\varphi(\eta(\zeta))}\sum_{s=\zeta+1}^{\infty}\mu(s)P^{\nu}(\eta(s))\right]^{\frac{1}{\xi}}$$

for $\zeta \geq \zeta_2$.

Sum the above inequality from ζ_2 to $\zeta - 1$, we get

$$\sum_{u=\zeta_2}^{\zeta-1}\frac{\Delta v(\eta(u))}{v^{\frac{\nu_2}{\xi}}(\eta(u+1))} \geq v^{\nu-\nu_2}(\zeta_1)\sum_{u=\zeta_2}^{\zeta-1}\left[\frac{1}{\varphi(\eta(u))}\sum_{s=u+1}^{\infty}\mu(s)P^{\nu}(\eta(s))\right]^{\frac{1}{\xi}}, \text{ for } \zeta \geq \zeta_2.$$

This implies that

$$\frac{1}{1-\frac{\nu_2}{\xi}}\left[\frac{1}{v^{\frac{\nu_2}{\xi}-1}(\eta(\zeta))} - \frac{1}{v^{\frac{\nu_2}{\xi}-1}(\eta(\zeta_2))}\right] \geq v^{\frac{\nu-\nu_2}{\xi}}(\zeta_1)\sum_{u=\zeta_2}^{\zeta-1}\left[\frac{1}{\varphi(\eta(u))}\sum_{s=u+1}^{\infty}\mu(s)P^{\nu}(\eta(s))\right]^{\frac{1}{\xi}}.$$

Since $\xi < \nu_2$, we have

$$\frac{1}{\frac{\nu_2}{\xi}-1}v^{1-\frac{\nu_2}{\xi}}(\eta(\zeta_2)) \geq v^{\frac{\nu-\nu_2}{\xi}}(\zeta_1)\sum_{u=\zeta_2}^{\zeta-1}\left[\frac{1}{\varphi(\eta(u))}\sum_{s=u+1}^{\infty}\mu(s)P^{\nu}(\eta(s))\right]^{\frac{1}{\xi}},$$

which implies that

$$\sum_{u=\zeta_2}^{\zeta-1}\left[\frac{1}{\varphi(\eta(u))}\sum_{s=u+1}^{\infty}\mu(s)P^{\nu}(\eta(s))\right]^{\frac{1}{\xi}} < \infty.$$

This contradicts (3.12) as $\zeta \to \infty$. This contradiction shows that $\{x(\zeta)\}$ cannot be an eventually positive solution. We omit the case where $\{x(\zeta)\}$ is eventually negative solution. □

Finally, we offer some instances to illustrate the usefulness and viability of our main results.

4. EXAMPLE

Example 4.1. *Consider the second-order difference equation*

$$\Delta\left(\zeta\left(\Delta\left(x(\zeta)+\frac{1}{\zeta}x^{\frac{1}{3}}(\zeta-1)+\frac{1}{\zeta^2}x^{\frac{1}{5}}(\zeta-2)\right)\right)^3\right)+\zeta^6x^5(\zeta-1)=0; \quad \zeta\geq 2, \tag{4.1}$$

where $\varphi(\zeta)=\zeta$, $\mu(\zeta)=\zeta^6$, $\eta(\zeta)=\zeta-1$, $\kappa_1(\zeta)=\zeta-1$, $\kappa_2(\zeta)=\zeta-2$, $p_1(\zeta)=\frac{1}{\zeta}$, $p_2(\zeta)=\frac{1}{\zeta^2}$. *Take* $\psi(\zeta)=\frac{1}{\zeta^6}$. *Now, we can easily verify all assumptions of the Theorem 3.1. Then by Theorem 3.1,* (3.1) *is oscillatory.*

Example 4.2. *Let us investigate the second-order difference equation*

$$\Delta\left(\frac{1}{\zeta^{\frac{1}{5}}}\left(\Delta\left(x(\zeta)+\frac{1}{\zeta}x^{\frac{1}{3}}(\zeta-1)+\frac{1}{\zeta^2}x^{\frac{1}{5}}(\zeta-2)\right)\right)^{\frac{1}{5}}\right)+\zeta^{\frac{2}{5}}x^{\frac{1}{3}}(\zeta-2)=0; \quad \zeta\geq 2. \tag{4.2}$$

Here, we have $\varphi(\zeta)=\frac{1}{\zeta^{\frac{1}{5}}}$, $\mu(\zeta)=\zeta^{\frac{2}{5}}$, $p_1(\zeta)=\frac{1}{\zeta}$, $p_2(\zeta)=\frac{1}{\zeta^2}$, $\lambda_1=\frac{1}{3}$, $\lambda_2=\frac{1}{5}$, $\xi=\frac{1}{5}$, $\nu=\frac{1}{3}$, $\kappa_1(\zeta)=\zeta-1$, $\kappa_2(\zeta)=\zeta-2$ *and* $\eta(\zeta)=\zeta-2$. *Set* $\psi(\zeta)=\frac{1}{\zeta}$. *We can easily verify all assumption of the Theorem 3.4. Hence, by Theorem 3.4,* (3.2) *is oscillatory.*

5. CONCLUSION

We explored the oscillatory behavior of solutions to (1.1) in this study and derived various sufficient conditions for oscillation to (1.1). Examples are given to demonstrate the importance of our results.

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