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A Qualitative Study on Eight-Point Explicit Decoupled Group Method for Solving 3D-Hyperbolic Telegraph Equation

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Abstract

The numerical solution of hyperbolic partial differential equations (PDEs) is used in many applications in applied mathematics and engineering. These equations are frequently utilized in many different domains of science and mathematical engineering, such as structure vibration, electrical signal transmission and propagation, and random walk theory. This study will show the formulation of a novel accelerated version of the explicit decoupled group iterative approach for solving a three-dimensional second order hyperbolic telegraph equation. The proposed method stability quality is then investigated using certain new fundamental theorems. To confirm the findings, numerical experiments will be carried out.

Keywords: Three dimensional second order hyperbolic telegraph equation, Explicit decoupled group method, Preconditioning technique.

1. Introduction

In recent years, various numerical methods such as finite difference, finite element and collocation methods have been developed for solving one-, two- and three dimensional telegraph equations [1–8]. Improved techniques using explicit group methods derived from the standard and skewed (rotated) finite difference operators have been developed in solving such equations [9-12]. The group methods depend on rotated finite difference operator were shown to require less execution time requirements than the common point iterative methods based on the centered difference approximations [13-18]. In addition, several preconditioned strategies reported in the literature have been used for improving the convergence rate of explicit group methods derived from the standard and rotated finite difference operators in the solution of partial differential equations (PDEs) [19-23]. Therefore, in this study, the formulation of new group iterative methods based on the rotated seven-point formulas is presented in solving the following three dimensional telegraph equation,

$$\frac{\partial^2 u}{\partial t^2}+2\alpha\frac{\partial u}{\partial t}+\beta^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+F(x,y,z,t), \tag{1}$$

in the region $\Omega=\{(x,y,z,t):0<x,y,z<1,\ t>0\}$, where $\alpha(x,y,z,t)>0$ and $\beta(x,y,z,t)>0$. The initial and boundary conditions as follow,

$$\begin{aligned}
&u(x,y,z,0)=f_1(x,y,z); \quad \frac{\partial u}{\partial t}(x,y,z,0)=f_2(x,y,z)\\
&u(0,y,z,t)=f_3(y,z,t); \quad u(1,y,z,t)=f_4(y,z,t);\\
&u(x,0,z,t)=f_5(x,z,t); \quad u(x,1,z,t)=f_6(x,z,t);\\
&u(x,y,0,t)=f_7(x,y,t); \quad u(x,y,1,t)=f_8(x,y,t).
\end{aligned} \tag{2}$$

Mohanty [8], proposed a three-level implicit unconditionally stable scheme of second order accuracy. The most important feature of this scheme is that the system may be solved by an operator splitting technique using a tri-diagonal solver. The description of unconditionally stable explicit decoupled group relaxation methods derived from the rotated five-point difference approximation in solving equation (1) will be given in this work.

The main aim of this paper is to formulate a new accelerated explicit decoupled group iterative method in solving initial and boundary problems (1) and (2). The paper is organized in seven sections: Section 2 describes the formulation of rotated finite difference approximation in solving the 3D second order hyperbolic equation. In Section 3, A brief description of the derivation of Eight-Point EDG method will be given. In Section 4, the proposed accelerated version group iterative approximation method will be introduced. Section 5 presented the stability of the studied group methods. In Section 6, the numerical results are presented in order to show the efficiency of the proposed method. Finally, the conclusion is given in Section 7.

2. Formulation of Rotated Finite Difference Approximation

In order to discritized equation (1) by using finite difference approximations, we suppose that the domain Ω is discretized uniformly in the directions of x, y and z with a mesh size $h=\Delta x=\Delta y=\Delta z=1\backslash n$ where n is an arbitrary positive integer. The grid points are given by $(x_i,y_j,z_l,t_m)=(ih,jh,lh,mk)$, where $m=1,2,3,...$ and $h>0,\ k>0$ are the space and time steps, respectively. Let $U_{i,j,l}^{m}$ be the exact solution of the equation (1) and $u_{i,j,l}^{m}$ the approximation solution at the grid point. The centreds seven-point formula about the grid point can be written as follows [24]:

$$\begin{aligned}\frac{u_{i,j,l,m+1}-2u_{i,j,l,m}+u_{i,j,l,m-1}}{\Delta t^2}+2\alpha\frac{u_{i,j,l,m+1}-u_{i,j,l,m-1}}{2\Delta t}&=\frac{1}{2}\Big[\frac{u_{i-1,j,l,m+1}-2u_{i,j,l,m+1}+u_{i+1,j,l,m+1}}{\Delta x^2}+\frac{u_{i-1,j,l,m}-2u_{i,j,l,m}+u_{i+1,j,l,m}}{\Delta x^2}\Big]\\&+\frac{1}{2}\Big[\frac{u_{i,j-1,l,m+1}-2u_{i,j,l,m+1}+u_{i,j+1,l,m+1}}{\Delta y^2}+\frac{u_{i,j-1,l,m}-2u_{i,j,l,m}+u_{i,j+1,l,m}}{\Delta y^2}\Big]\\&+\frac{1}{2}\Big[\frac{u_{i,j-1,l-1,m+1}-2u_{i,j,l,m+1}+u_{i,j,l+1,m+1}}{\Delta z^2}+\frac{u_{i,j,l-1,m}-2u_{i,j,l,m}+u_{i,j,l+1,m}}{\Delta z^2}\Big]\\&-\frac{\beta^2}{2}(u_{i,j,l,m+1}+u_{i,j,l,m})+F_{i,j,l,m+\frac{1}{2}}\end{aligned}\tag{3}$$

where $x=i\,\Delta x,\ y=j\,\Delta y,\ z=l\,\Delta z,\ t=m\,\Delta t;\ (i,j,l=0,1,2,...,n-1;m=0,1,2,...)$. The above equation (3) is called standard point formula and after simplification it can be written as

$$\begin{aligned}&(1+3r+a+b/2)u_{i,j,l,m+1}-\frac{r}{2}[u_{i+1,j,l,m+1}+u_{i-1,j,l,m+1}+u_{i,j+1,l,m+1}+u_{i,j-1,l,m+1}+u_{i,j,l+1,m+1}+u_{i,j,l-1,m+1}]=(r/2)[u_{i+1,j,l,m}\\&+u_{i-1,j,l,m}+u_{i,j+1,l,m}+u_{i,j-1,l,m}+u_{i,j,l+1,m}+u_{i,j,l-1,m}+(2-3r-b/2)u_{i,j,l,m}+(a-1)u_{i,j,l,m-1}+\Delta t^2F_{i,j,l,m+\frac{1}{2}}\end{aligned}\tag{4}$$

where

$$r=\frac{\Delta t^2}{h^2},\quad a=\alpha\,\Delta t,\quad b=\beta^2\Delta t^2\ \cdot$$

By rotating the x-y and z axis clockwise 45^0, with respect to the standard mesh, we obtained the following rotated seven-point finite difference approximation for equation (1) as

$$\begin{aligned}\frac{u_{i,j,l,m+1}-2u_{i,j,l,m}+u_{i,j,l,m-1}}{\Delta t^2}+2\alpha\frac{u_{i,j,l,m+1}-u_{i,j,l,m-1}}{2\Delta t}&=\frac{1}{4}\Big[\frac{u_{i-1,j+1,l,m+1}-2u_{i,j,l,m+1}+u_{i+1,j-1,l,m+1}}{\Delta x^2}+\frac{u_{i-1,j+1,l,m}-2u_{i,j,l,m}+u_{i+1,j-1,l,m}}{\Delta x^2}\Big]\\&+\frac{1}{4}\Big[\frac{u_{i-1,j-1,l,m+1}-2u_{i,j,l,m+1}+u_{i+1,j+1,l,m+1}}{\Delta y^2}+\frac{u_{i-1,j-1,l,m}-2u_{i,j,l,m}+u_{i+1,j+1,l,m}}{\Delta y^2}\Big]\\&+\frac{1}{4}\Big[\frac{u_{i+1,j-1,l+1,m+1}-2u_{i,j,l,m+1}+u_{i-1,j,l-1,m+1}}{\Delta z^2}+\frac{u_{i+1,j,l+1,m}-2u_{i,j,l,m}+u_{i-1,j,l-1,m}}{\Delta z^2}\Big]\\&-\frac{\beta^2}{2}(u_{i,j,l,m+1}+u_{i,j,l,m})+F_{i,j,l,m+\frac{1}{2}}\end{aligned}\tag{5}$$

Furthermore, by rotating the *x*-, *y*-axis clockwise $45^\circ$and rotating the *z*-axis clockwise $315^\circ$with respect to the standard mesh [24], the following scheme obtained

$$\begin{aligned}\frac{u_{i,j,l,m+1}-2u_{i,j,l,m}+u_{i,j,l,m-1}}{\Delta t^2}+2\alpha\frac{u_{i,j,l,m+1}-u_{i,j,l,m-1}}{2\Delta t}&=\frac{1}{4}\Big[\frac{u_{i-1,j+1,l,m+1}-2u_{i,j,l,m+1}+u_{i+1,j-1,l,m+1}}{\Delta x^2}+\frac{u_{i-1,j+1,l,m}-2u_{i,j,l,m}+u_{i+1,j-1,l,m}}{\Delta x^2}\Big]\\&+\frac{1}{4}\Big[\frac{u_{i-1,j-1,l,m+1}-2u_{i,j,l,m+1}+u_{i+1,j+1,l,m+1}}{\Delta y^2}+\frac{u_{i-1,j-1,l,m}-2u_{i,j,l,m}+u_{i+1,j+1,l,m}}{\Delta y^2}\Big]\\&+\frac{1}{4}\Big[\frac{u_{i-1,j,l+1,m+1}-2u_{i,j,l,m+1}+u_{i+1,j,l-1,m+1}}{\Delta z^2}+\frac{u_{i-1,j,l+1,m}-2u_{i,j,l,m}+u_{i+1,j,l-1,m}}{\Delta z^2}\Big]\\&-\frac{\beta^2}{2}(u_{i,j,l,m+1}+u_{i,j,l,m})+F_{i,j,l,m+\frac{1}{2}}\end{aligned}\tag{6}$$

After simplification of Eqs. (5) and (6), and assuming that $h = \Delta x = \Delta y = \Delta z$, the following rotated formulas obtained

$$\begin{aligned}(1+3r/2+a+b/2)u_{i,j,l,m+1} - \frac{r}{4}[u_{i-1,j+1,l,m+1} + u_{i+1,j-1,l,m+1} + u_{i-1,j-1,l,m+1} + u_{i+1,j+1,l,m+1} + u_{i+1,j,l,m+1} + u_{i-1,j,l-1,m+1}] + \\ = \frac{r}{4}[u_{i-1,j+1,l,m} + u_{i+1,j-1,l,m} + u_{i-1,j-1,l,m} + u_{i+1,j+1,l,m} + u_{i+1,j,l+1,m} + u_{i-1,j,l-1,m}] + (2-3r/2-b/2)u_{i,j,l,m} \\ + (a-1)u_{i,j,l,m} + \Delta t^2 F_{i,j,l,m+\frac{1}{2}}\end{aligned} \tag{7}$$

$$\begin{aligned}(1+3r/2+a+b/2)u_{i,j,l,m+1} - \frac{r}{4}[u_{i-1,j+1,l,m+1} + u_{i+1,j-1,l,m+1} + u_{i-1,j-1,l,m+1} + u_{i+1,j+1,l,m+1} + u_{i-1,j,l+1,m+1} + u_{i+1,j,l-1,m+1}] \\ = \frac{r}{4}[u_{i-1,j+1,l,m} + u_{i+1,j-1,l,m} + u_{i-1,j-1,l,m} + u_{i+1,j+1,l,m} + u_{i-1,j,l+1,m} + u_{i+1,j,l-1,m}] + (2-3r/2-b/2)u_{i,j,l,m} \\ + (a-1)u_{i,j,l,m-1} + \Delta t^2 F_{i,j,l,m+\frac{1}{2}}\end{aligned} \tag{8}$$

The rotated seven-point difference scheme can be constructed by dividing the grid points into two types of points on the x-, y-and z-space of the solution domain. Eq.(7)is used to compute the points at odd y-direction j =1, 3, 5, ..., while Eq.(8)is used to compute the points at even y-direction j =2, 4, 6, ... (as shown in Fig.1). Iterations can be generated involving one type of points only and when convergence is achieved; the solution at the remaining points will be evaluated directly using Eq.(3). Similarly, the process stopped when the desired time level is reached.

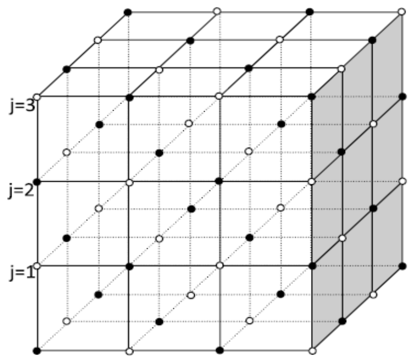

Fig. 1 Solution domain with rotated points

3. FORMULATION OF EIGHT-POINT EDG METHOD

The EDG method is derived based on the rotated seven-point formulas. By applying the Eqs.(7) and (8) to any group of eight points (cube) on a discretised solution domain Ω with Eq.(7)is applied to the points in odd y-direction and Eq.(8)is applied to the points in even y-direction, will result in an (8 ×8)system of equation as following

$$\begin{pmatrix} \ell_1 & -\ell_2 & -\ell_2 & 0 & 0 & 0 & 0 & 0 \\ -\ell_2 & \ell_1 & 0 & -\ell_2 & 0 & 0 & 0 & 0 \\ -\ell_2 & 0 & \ell_1 & -\ell_2 & 0 & 0 & 0 & 0 \\ 0 & -\ell_2 & -\ell_2 & \ell_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ell_1 & -\ell_2 & -\ell_2 & 0 \\ 0 & 0 & 0 & 0 & -\ell_2 & \ell_1 & 0 & -\ell_2 \\ 0 & 0 & 0 & 0 & -\ell_2 & 0 & \ell_1 & -\ell_2 \\ 0 & 0 & 0 & 0 & 0 & -\ell_2 & -\ell_2 & \ell_1 \end{pmatrix} \begin{pmatrix} u_{i,j,l,m+1} \\ u_{i+1,j+1,l,m+1} \\ u_{i+1,j,l+1,m+1} \\ u_{i,j+1,l+1,m+1} \\ u_{i,j,l+1,m+1} \\ u_{i+1,j+1,l+1,m+1} \\ u_{i+1,j,l,m+1} \\ u_{i,j+1,l,m+1} \end{pmatrix} = \begin{pmatrix} rhs_{i,j,l} \\ rhs_{i+1,j+1,l} \\ rhs_{i+1,j,l} \\ rhs_{i,j+1,l+1} \\ rhs_{i,j,l+1} \\ rhs_{i+1,j+1,l+1} \\ rhs_{i+1,j,l} \\ rhs_{i,j+1,l} \end{pmatrix} \quad (9)$$

where

$\ell_1 = 1 + 3r/2 + a + b/2; \quad \ell_2 = r/4;$

$$\begin{aligned} rhs_{i,j,l} = {} & \ell_2 (u_{i-1,j+1,l,m+1} + u_{i+1,j-1,l,m+1} + u_{i-1,j-1,l,m+1} + u_{i-1,j,l-1,m+1} + u_{i-1,j+1,l,m} + u_{i+1,j-1,l,m}) \\ & + \ell_2 (u_{i-1,j-1,l,m} + u_{i+1,j+1,l,m+1} + u_{i+1,j,l+1,m} + u_{i-1,j,l-1,m}) + (2 - 3r/2 - b/2) u_{i,j,l,m} \\ & + (a-1) u_{i,j,l,m-1} + \Delta t^2 F_{i,j,l,m+1/2} \end{aligned}$$

$$\begin{aligned} rhs_{i+1,j+1,l} = {} & \ell_2 (u_{i,j+2,l,m+1} + u_{i+2,j,l,m+1} + u_{i+2,j+2,l,m+1} + u_{i+2,j+1,l-1,m+1} + u_{i,j+2,l,m} + u_{i+2,j,l,m}) \\ & + \ell_2 (u_{i,j,l,m} + u_{i+2,j+2,l,m} + u_{i,j+1,l+1,m} + u_{i+2,j+1,l-1,m}) + (2 - 3r/2 - b/2) u_{i+1,j+1,l,m} \\ & + (a-1) u_{i+1,j+1,l,m-1} + \Delta t^2 F_{i+1,j+1,l,m+1/2} \end{aligned}$$

$$\begin{aligned} rhs_{i+1,j,l+1} = {} & \ell_2 (u_{i+2,j-1,l+1,m+1} + u_{i,j-1,l+1,m+1} + u_{i+2,j+1,l+1,m+1} + u_{i+2,j,l+2,m+1} + u_{i,j+1,l+1,m} + u_{i+2,j-1,l+1,m}) \\ & + \ell_2 (u_{i,j-1,l+1,m} + u_{i+2,j+1,l+1,m+1} + u_{i+2,j,l+2,m} + u_{i,j,l,m}) + (2 - 3r/2 - b/2) u_{i+1,j,l+1,m} \\ & + (a-1) u_{i+1,j,l+1,m-1} + \Delta t^2 F_{i+1,j,l+1,m+1/2} \end{aligned}$$

$$\begin{aligned} rhs_{i,j+1,l+1} = {} & \ell_2 (u_{i-1,j+2,l+1,m+1} + u_{i-1,j,l+1,m+1} + u_{i+1,j+2,l+1,m+1} + u_{i-1,j+1,l+2,m+1} + u_{i-1,j+2,l+1,m} + u_{i+1,j,l+1,m}) \\ & + \ell_2 (u_{i-1,j,l+1,m} + u_{i+1,j+2,l+1,m} + u_{i-1,j+1,l+2,m} + u_{i+1,j+1,l,m}) + (2 - 3r/2 - b/2) u_{i,j+1,l+1,m} \\ & + (a-1) u_{i,j+1,l+1,m-1} + \Delta t^2 F_{i,j+1,l+1,m+1/2} \end{aligned}$$

$$\begin{aligned} rhs_{i,j,l+1} = {} & \ell_2 (u_{i-1,j+1,l+1,m+1} + u_{i+1,j-1,l+1,m+1} + u_{i-1,j-1,l+1,m+1} + u_{i-1,j,l+2,m+1} + u_{i-1,j+1,l+1,m} + u_{i+1,j-1,l+1,m}) \\ & + \ell_2 (u_{i-1,j-1,l+1,m} + u_{i+1,j+1,l+1,m} + u_{i-1,j,l+2,m} + u_{i+1,j,l,m}) + (2 - 3r/2 - b/2) u_{i,j,l+1,m} \\ & + (a-1) u_{i,j,l+1,m-1} + \Delta t^2 F_{i,j,l+1,m+1/2} \end{aligned}$$

$$\begin{aligned} rhs_{i+1,j+1,l+1} = {} & \ell_2 (u_{i,j+2,l+1,m+1} + u_{i+2,j,l+1,m+1} + u_{i+2,j+2,l+1,m+1} + u_{i+2,j+1,l+2,m+1} + u_{i,j+2,l+1,m} + u_{i+2,j,l+1,m}) \\ & + \ell_2 (u_{i,j,l+1,m} + u_{i+2,j+2,l+1,m} + u_{i+2,j+1,l+2,m} + u_{i,j+1,l,m}) + (2 - 3r/2 - b/2) u_{i+1,j+1,l+1,m} \\ & + (a-1) u_{i+1,j+1,l+1,m-1} + \Delta t^2 F_{i+1,j+1,l+1,m+1/2} \end{aligned}$$

$$
\begin{aligned}
rhs_{i+1,j,l} &= \ell_2(u_{i+2,j-1,l,m+1}+u_{i,j-1,l,m+1}+u_{i+2,j+1,l,m+1}+u_{i+2,j,l-1,m+1}+u_{i,j+1,l,m}+u_{i+2,j-1,l,m}) \\
&\quad +\ell_2(u_{i,j-1,l,m}+u_{i+2,j+1,l,m}+u_{i,j,l+1,m}+u_{i+2,j,l-1,m})+(2-3r/2-b/2)u_{i+1,j,l,m} \\
&\quad +(a-1)u_{i+1,j,l,m-1}+\Delta t^2 F_{i+1,j,l,m+1/2} \\
rhs_{i,j+1,l} &= \ell_2(u_{i-1,j+2,l,m+1}+u_{i-1,j,l,m+1}+u_{i+1,j+2,l,m+1}+u_{i-1,j+1,l-1,m+1}+u_{i-1,j+2,l,m}+u_{i+1,j,l,m}) \\
&\quad +\ell_2(u_{i-1,j,l,m}+u_{i+1,j+2,l,m}+u_{i+1,j+1,l+1,m}+u_{i-1,j+1,l-1,m})+(2-3r/2-b/2)u_{i,j+1,l,m} \\
&\quad +(a-1)u_{i,j+1,l,m-1}+\Delta t^2 F_{i,j+1,l,m+1/2}
\end{aligned}
$$

This system may then be decoupled into two (4 ×4) systems of equations

$$
\begin{pmatrix} u_{i,j,l,m+1} \\ u_{i+1,j+1,l,m+1} \\ u_{i+1,j,l+1,m+1} \\ u_{i,j+1,l+1,m+1} \end{pmatrix} = A \begin{pmatrix} q_1 & q_2 & q_2 & q_3 \\ q_2 & q_1 & q_3 & q_2 \\ q_2 & q_3 & q_1 & q_2 \\ q_3 & q_2 & q_2 & q_1 \end{pmatrix} \begin{pmatrix} rhs_{i,j,l} \\ rhs_{i+1,j+1,l} \\ rhs_{i+1,j,l+1} \\ rhs_{i,j+1,l+1} \end{pmatrix} \tag{10}
$$

and

$$
\begin{pmatrix} u_{i,j,l+1,m+1} \\ u_{i+1,j+1,l+1,m+1} \\ u_{i+1,j,l,m+1} \\ u_{i,j+1,l,m+1} \end{pmatrix} = A \begin{pmatrix} q_1 & q_2 & q_2 & q_3 \\ q_2 & q_1 & q_3 & q_2 \\ q_2 & q_3 & q_1 & q_2 \\ q_3 & q_2 & q_2 & q_1 \end{pmatrix} \begin{pmatrix} rhs_{i,j,l+1} \\ rhs_{i+1,j+1,l+1} \\ rhs_{i+1,j,l} \\ rhs_{i,j+1,l} \end{pmatrix} \tag{11}
$$

where

$A = 1/(\ell_1^4 - 4\ell_1^2\ell_2^2)$; $q_1 = \ell_1^3 - 2\ell_1\ell_2^2$; $q_2 = \ell_1^2\,\ell_2$; $q_3 = 2\ell_1\,\ell_2^2$

Similar to the rotated seven-point formula, the EDG scheme is constructed by dividing the grid points in solution domain into two types of points. The evaluation of Eq. (10) and Eq. (11) can be performed independently based on the types of points involved respectively. This means that the iterative evaluation of points from each group requires contribution of points only from the same group. Thus, iterations can be carried out on either one of the two types of points, which is only half of the total nodal points. Therefore, the method corresponds to the generation of iterations on one type of points until a certain convergence criteria are met. After the convergence is achieved, the solutions at the remaining of the total nodal points are evaluated directly once using the centred seven-point difference formula of Eq. (3). The process is repeated until the desired time level is achieved [24].

4. THE PROPOSED ACCELERATED EDG METHOD

The convergence rates of the EDG iterative method depend on the spectral properties of the coefficient matrices [11]. Usually the system (9) resulted from EDG method is large and sparse. By using the following preconditioner matrix

$$P=\begin{pmatrix} \ell_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ell_1 & 0 & -\ell_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ell_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ell_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\ell_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\ell_2 \\ 0 & 0 & 0 & 0 & \ell_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\ell_2 & 0 & 0 \end{pmatrix} \tag{12}$$

where ℓ_1 and ℓ_2 defined as equation (9), we will obtain new preconditioned system as the following:

$$\begin{pmatrix} \ell_1^2 & -\ell_1\ell_2 & -\ell_1\ell_2 & 0 & 0 & 0 & 0 & 0 \\ -\ell_1\ell_2 & \ell_1^2 & 0 & -\ell_1\ell_2 & 0 & 0 & 0 & 0 \\ -\ell_1\ell_2 & 0 & \ell_1^2 & -\ell_1\ell_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\ell_1\ell_2 & -\ell_1\ell_2 & \ell_1^2 \\ 0 & 0 & 0 & 0 & \ell_2^2 & 0 & -\ell_1\ell_2 & \ell_2^2 \\ 0 & 0 & 0 & 0 & 0 & \ell_2^2 & \ell_2^2 & \ell_1\ell_2 \\ 0 & 0 & 0 & 0 & \ell_1\ell_2 & -\ell_2^2 & -\ell_2^2 & 0 \\ 0 & 0 & 0 & 0 & \ell_2^2 & \ell_1\ell_2 & 0 & \ell_2^2 \end{pmatrix} \begin{pmatrix} u_{i,j,l,m+1} \\ u_{i+1,j+1,l,m+1} \\ u_{i+1,j,l+1,m+1} \\ u_{i,j+1,l+1,m+1} \\ u_{i,j,l+1,m+1} \\ u_{i+1,j+1,l+1,m+1} \\ u_{i+1,j,l,m+1} \\ u_{i,j+1,l,m+1} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \end{pmatrix} \tag{13}$$

where

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \end{pmatrix} = \begin{pmatrix} r\ell_1 hs_{i,j,l} \\ r\ell_1 h(s_{i+1,j+1,l} - s_{i,j+1,l+1}) \\ r\ell_1 hs_{i+1,j,l} \\ r\ell_1 hs_{i,j+1,l} \\ -r\ell_2 hs_{i+1,j,l} \\ -r\ell_2 hs_{i,j+1,l} \\ r\ell_2 hs_{i,j,l+1} \\ -r\ell_2 hs_{i+1,j+1,l+1} \end{pmatrix}$$

The process of obtaining the new preconditioned system depend on the structure of the coefficient matrix of the target system involves multiplying this preconditioner matrix P by the original system of the mentioned iterative methods to produce coefficients matrix with a spectral radius less than the spectral radius of the coefficients matrix of the original system. The resulted preconditioned Eight-Point EDG has the same solution of original Eight-Point EDG system (9), but that has more favorable spectral properties. The stability of the proposed preconditioned method will be discussed in the following section 5 and the superiority of the proposed preconditioned method in terms of number of iterations and execution time will be introduced in section 6 through numerical experiments.

5. STABILITY

The stability of a finite difference scheme must be ensured so that the errors incurred at each time level do not grow as the computation proceed [25].

Theorem 5.1 The explicit decouped group schemes (10) and (11) are unconditionally stable when $|\mu| < 1$ which satisfy conditions $(p+q+r) > 0;\ (p-r) > 0; (p-q+r) > 0,$ where the conditions are obtained from stability polynomial.

Proof.

From equation (9), the resulting system can be written as

$$A\,u_{m+1} = B\,u_m + Cu_{m-1} + b_m \tag{14}$$

where

$$A = \left(\begin{array}{ccccccccccccccccc}
R_1 & R_2 & & & & \vdots & R_4 & & & & & \vdots & & & & & 0 \\
R_3 & R_1 & R_2 & & & \vdots & & R_4 & & & & \vdots & & & & & \\
 & \ddots & \ddots & \ddots & & \vdots & & & \ddots & & & \vdots & & & & & \\
 & & R_3 & R_1 & R_2 & \vdots & & & & R_4 & & \vdots & & & & & \\
 & & & R_3 & R_1 & \vdots & & & & & R_4 & \vdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
R_5 & & & & & \vdots & R_1 & R_2 & & & & \vdots & R_4 & & & & \\
 & R_5 & & & & \vdots & R_3 & R_1 & R_2 & & & \vdots & & R_4 & & & \\
 & & \ddots & & & \vdots & & \ddots & \ddots & \ddots & & \vdots & & & \ddots & & \\
 & & & R_5 & & \vdots & & & R_3 & R_1 & R_2 & \vdots & & & & R_4 & \\
 & & & & R_5 & \vdots & & & & R_3 & R_1 & \vdots & & & & & R_4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & \vdots & R_5 & & & & & \vdots & R_1 & R_2 & & & \\
 & & & & & \vdots & & R_5 & & & & \vdots & R_3 & R_1 & R_2 & & \\
 & & & & & \vdots & & & \ddots & & & \vdots & & \ddots & \ddots & \ddots & \\
 & & & & & \vdots & & & & R_5 & & \vdots & & & R_3 & R_1 & R_2 \\
0 & & & & & \vdots & & & & & R_5 & \vdots & & & & R_3 & R_1
\end{array}\right);$$

$$B=\left(\begin{array}{ccccccccccccccccc}
S_1 & S_2 & & & & \vdots & S_4 & & & & & \vdots & & & & & 0 \\
S_3 & S_1 & S_2 & & & \vdots & & S_4 & & & & \vdots & & & & & \\
& \ddots & \ddots & \ddots & & \vdots & & & \ddots & & & \vdots & & & & & \\
& & S_3 & S_1 & S_2 & \vdots & & & & S_4 & & \vdots & & & & & \\
& & & S_3 & S_1 & \vdots & & & & & S_4 & \vdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
S_5 & & & & & \vdots & S_1 & S_2 & & & & \vdots & S_4 & & & & \\
& S_5 & & & & \vdots & S_3 & S_1 & S_2 & & & \vdots & & S_4 & & & \\
& & \ddots & & & \vdots & & \ddots & \ddots & \ddots & & \vdots & & & \ddots & & \\
& & & S_5 & & \vdots & & & S_3 & S_1 & S_2 & \vdots & & & & S_4 & \\
& & & & S_5 & \vdots & & & & S_3 & S_1 & \vdots & & & & & S_4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & \vdots & S_5 & & & & & \vdots & S_1 & S_2 & & & \\
& & & & & \vdots & & S_5 & & & & \vdots & S_3 & S_1 & S_2 & & \\
& & & & & \vdots & & & \ddots & & & \vdots & & \ddots & \ddots & \ddots & \\
& & & & & \vdots & & & & S_5 & & \vdots & & & S_3 & S_1 & S_2 \\
0 & & & & & \vdots & & & & & S_5 & \vdots & & & & S_3 & S_1
\end{array}\right);$$

$$C=\left(\begin{array}{ccccccccccccccccc}
T_1 & & & & & \vdots & & & & & & \vdots & & & & & 0 \\
& T_1 & & & & \vdots & & & & & & \vdots & & & & & \\
& \ddots & \ddots & & & \vdots & & & & & & \vdots & & & & & \\
& & & T_1 & & \vdots & & & & & & \vdots & & & & & \\
& & & & T_1 & \vdots & & & & & & \vdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & \vdots & T_1 & & & & & \vdots & & & & & \\
& & & & & \vdots & & T_1 & & & & \vdots & & & & & \\
& & & & & \vdots & & & \ddots & & & \vdots & & & & & \\
& & & & & \vdots & & & & T_1 & & \vdots & & & & & \\
& & & & & \vdots & & & & & T_1 & \vdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & \vdots & & & & & & \vdots & T_1 & & & & \\
& & & & & \vdots & & & & & & \vdots & & T_1 & & & \\
& & & & & \vdots & & & & & & \vdots & & & \ddots & & \\
& & & & & \vdots & & & & & & \vdots & & & & T_1 & \\
0 & & & & & \vdots & & & & & & \vdots & & & & & T_1
\end{array}\right), \quad b=\begin{pmatrix} w_1 \\ w_1 \\ \\ w_1 \\ w_1 \\ \\ w_1 \\ w_1 \\ \cdots \\ \vdots \\ \cdots \\ w_1 \\ w_1 \\ \cdots \\ \vdots \\ \cdots \\ \\ w_1 \\ w_1 \end{pmatrix};$$

$$R_1=\begin{pmatrix} G_1 & G_2 & & & \\ G_3 & G_1 & G_2 & & \\ & \ddots & \ddots & \ddots & \\ & & G_3 & G_1 & G_2 \\ & & & G_3 & G_1 \end{pmatrix};\ R_2=\begin{pmatrix} G_4 & G_6^T & & & \\ G_5 & G_4 & G_6^T & & \\ & \ddots & \ddots & \ddots & \\ & & G_5 & G_4 & G_6^T \\ & & & G_5 & G_4 \end{pmatrix};\ R_3=\begin{pmatrix} G_7 & G_6 & & & \\ G_5^T & G_7 & G_6 & & \\ & \ddots & \ddots & \ddots & \\ & & G_5^T & G_7 & G_6 \\ & & & G_5^T & G_7 \end{pmatrix}$$

$$R_4=\begin{pmatrix} & G_6 & & & \\ G_5 & & G_6 & & \\ & \ddots & \ddots & \ddots & \\ & & G_5 & & G_6 \\ & & & G_5 & \end{pmatrix};\ R_5=\begin{pmatrix} & G_6^T & & & \\ G_5^T & & G_6^T & & \\ & \ddots & \ddots & \ddots & \\ & & G_5^T & & G_6^T \\ & & & G_5^T & \end{pmatrix};S_1=\begin{pmatrix} H_1 & H_2 & & & \\ H_3 & H_1 & H_2 & & \\ & \ddots & \ddots & \ddots & \\ & & H_3 & H_1 & H_2 \\ & & & H_3 & H_1 \end{pmatrix}$$

$$S_2=\begin{pmatrix} H_4 & H_6^T & & & \\ H_5 & H_4 & H_6^T & & \\ & \ddots & \ddots & \ddots & \\ & & H_5 & H_4 & H_6^T \\ & & & H_5 & H_4 \end{pmatrix};S_3=\begin{pmatrix} H_7 & H_6 & & & \\ H_5^T & H_7 & H_6 & & \\ & \ddots & \ddots & \ddots & \\ & & H_5^T & H_7 & H_6 \\ & & & H_5^T & H_7 \end{pmatrix};S_4=\begin{pmatrix} & H_6 & & & \\ H_5 & & H_6 & & \\ & \ddots & \ddots & \ddots & \\ & & H_5 & & H_6 \\ & & & H_5 & \end{pmatrix}$$

$$S_5=\begin{pmatrix} & H_6^T & & & \\ H_5^T & & H_6^T & & \\ & \ddots & \ddots & \ddots & \\ & & H_5^T & & H_6^T \\ & & & H_5^T & \end{pmatrix};\quad T_1=\begin{pmatrix} M_1 & & & & \\ & M_1 & & & \\ & & \ddots & & \\ & & & M_1 & \\ & & & & M_1 \end{pmatrix};\quad w_1=\begin{pmatrix} L_1 \\ L_1 \\ \vdots \\ L_1 \\ L_1 \end{pmatrix}$$

with

$$G_1=\begin{pmatrix} k_1 & -k_2 & -k_2 & 0 \\ -k_2 & k_1 & 0 & -k_2 \\ -k_2 & 0 & k_1 & -k_2 \\ 0 & -k_2 & -k_2 & k_1 \end{pmatrix};\ G_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};\ G_3=\begin{pmatrix} -k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_2 \end{pmatrix}$$

$$G_4=\begin{pmatrix} 0 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -k_2 & 0 \end{pmatrix};\quad G_5=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \end{pmatrix};\ G_6=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_7=\begin{pmatrix} 0 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_2 \\ 0 & 0 & 0 & 0 \end{pmatrix};\ H_1=\begin{pmatrix} k_3 & k_2 & k_2 & 0 \\ k_2 & k_3 & 0 & k_2 \\ k_2 & 0 & k_3 & k_2 \\ 0 & k_2 & k_2 & k_3 \end{pmatrix};\ H_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$H_3=\begin{pmatrix}k_2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&k_2\end{pmatrix};\ H_4=\begin{pmatrix}0&0&0&0\\k_2&0&0&0\\0&0&0&0\\0&k_2&0&0\end{pmatrix};\ H_5=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\k_2&0&0&0\end{pmatrix};$$

$$H_6=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&k_2&0&0\\0&0&0&0\end{pmatrix};\ H_7=\begin{pmatrix}0&k_2&0&0\\0&0&0&0\\0&0&0&k_2\\0&0&0&0\end{pmatrix};\ M_1=\begin{pmatrix}k_4&0&0&0\\0&k_4&0&0\\0&0&k_4&0\\0&0&0&k_4\end{pmatrix}$$

$$L_1=\Delta t^2=\begin{pmatrix}f_{i,j,l}\\f_{i+1,j+1,l}\\f_{i+1,j,l+1}\\f_{i,j+1,l+1}\end{pmatrix};\ k_1=1+3r/2+a+b/2;\ k_2=r/4;\ k_3=2-3r/2-b/2;\ k_4=a-1.$$

Equation (14) can be written as

$$[A_{\frac{(n-2)^3}{2}\times\frac{(n-2)^3}{2}\times\frac{(n-2)^3}{2}}]u_{m+1}=[B_{\frac{(n-2)^3}{2}\times\frac{(n-2)^3}{2}\times\frac{(n-2)^3}{2}}]u_m+[C_{\frac{(n-2)^3}{2}\times\frac{(n-2)^3}{2}\times\frac{(n-2)^3}{2}}]u_{m-1}+b_m$$

The system can also be rewritten as

$$u_{m+1}=A^{-1}B\,u_m+A^{-1}Cu_{m-1}+A^{-1}b_m \tag{15}$$

If we let $v_{m+1}=(u_m,u_{m-1})^T$, then equation (15) can be written in the following partitioned matrix form

$$\begin{pmatrix}u_{m+1}\\u_m\end{pmatrix}=\begin{pmatrix}A^{-1}B&A^{-1}C\\I&0\end{pmatrix}\begin{pmatrix}u_m\\u_{m-1}\end{pmatrix}+\begin{pmatrix}A^{-1}b_m\\0\end{pmatrix}\quad\Rightarrow v_{m+1}=Pv_m+d_m$$

where d_m is a column vector of known values. We can observe that this technique has reduced a three-level difference equation to a two-level difference equation. The matrices *A*, *B* and *C* have the same system of linearly independent eigenvectors. The eigenvalues μ of *P* are given by

$$\begin{vmatrix}a_k^{-1}b_k-\mu&a_k^{-1}c_k\\1&-\mu\end{vmatrix}=0,\quad k=1,2,...,(n-1)$$

where a_k,b_k and c_k are the eigenvalues of *A*, *B* and *C* respectively. The matrices *A*, *B* and *C* from Eq. (14) can be written as

$$A=G_1I+(G_2+G_3)E+G_4I+(G_5+G_6^T)E+G_7I+(G_5^T+G_6)E+(G_5+G_6)E+(G_5^T+G_6^T)E+L_1F,$$

$$B=H_1I+(H_2+H_3)E+H_4I+(H_5+H_6^T)E+H_7I+(H_5^T+H_6)E+(H_5+H_6)E+(H_5^T+H_6^T)E+L_1F,$$

$$C=(a-1)I \tag{16}$$

where *E* is the matrix with unity values along each diagonal just above and below the main diagonal and zeroes elsewhere, *F* is a column vector of known values. From (16), we can write the eigenvalues of the matrices *A*, *B* and *C* as the following:

$$a_k = 1 + a + b/2 + r(1 + (5/2)\sin^2(i\pi/2m)),$$
$$b_k = 2 - b/2 - r(1 + (5/2)\sin^2(i\pi/2m)), \qquad (17)$$
$$c_k = a - 1$$

Using Eq. (17), we will get

$$1 + a + b/2 + r(1 + (5/2)\sin^2(i\pi/2m))\mu^2 + (b/2 - 2 + r(1 + (5/2)\sin^2(i\pi/2m))\mu + (1 - a) = 0,$$
$$p\mu^2 + q\mu + r = 0$$

Under the transformation $\mu = (1+z)/(1-z)$, we may write the stability polynomial $P(\mu)$ as

$$P(\mu) = P(\frac{1+z}{1-z}) = p(\frac{1+z}{1-z})^2 + q(\frac{1+z}{1-z}) + r \Rightarrow (p - q + r)z^2 + 2(p - r)z + (p + q + r) = 0$$

Again, the necessary and sufficient conditions for $|\mu| < 1$ are that $(p + q + r) > 0; (p - r) > 0$ and $(p - q + r) > 0$. From the first condition, $(p + q + r) > 0$, we can see,

$$1 + a + b/2 + r(1 + (5/2)\sin^2(i\pi/2m)) + (b/2 - 2 + r(1 + (5/2)\sin^2(i\pi/2m)) + (1 - a) > 0,$$
$$\Rightarrow \; b + r(2 + 5\sin^2(i\pi/2m) > 0$$

This condition is satisfied for $\beta(x, y, z, t) \geq 0$ and all variables angle π. The second condition $(p - r) > 0$ gives

$$1 + a + b/2 + r(1 + (5/2)\sin^2(i\pi/2m)) - (1 - a) > 0 \Rightarrow 2a + b/2 + r(1 + (5/2)\sin^2(i\pi/2m)) > 0.$$

This condition is satisfied for $\alpha(x, y, z, t) \geq 0$ and $\beta(x, y, z, t) \geq 0$ and all variables angle π . The third condition $(p - q + r) > 0$ gives

$$1 + a + b/2 + r(1 + (5/2)\sin^2(i\pi/2m)) - (b/2 - 2 + r(1 + (5/2)\sin^2(i\pi/2m)) + (1 - a) > 0 \Rightarrow 4 > 0.$$

which is always true for all variables. Therefore, the explicit decoupled group iterative scheme (10) and (11) are unconditionally stable for $0 < r < \infty$, that is for all choices of $h, k > 0$. □

Remark 5.2 Since the proposed Preconditioned Eight-Point EDG (PEP EDG) scheme (13) has the same solution as the original EP EDG scheme (9) and the coefficient matrix has the smaller spectral radius less than that of the coefficient matrix of the original method. Therefore, by using the same manner of theorem 5.1, we can easily prove that the Preconditioned Eight-Point EDG iterative scheme (13) is also unconditionally stable for $0 < r < \infty$, that is for all choices of $h, k > 0$.

6. NUMERICAL RESULTS

In this section, two illustrative examples will introduce to confirm and justify our results. Several numerical experiments will be carried out on several mesh sizes of 26, 42, 84, 168, 248 and 318, with the values of relaxation factor (Gauss Seidel relaxation scheme) for the various mesh sizes set equal to 1.0. In this experimental work, the convergence criteria used throughout the experiments was the l_∞ norm with the error tolerance set equal to $\varepsilon = 10^{-11}$. The computer processing unit was Intel(R) Core(TM) i7-7500U CPU with a memory of 8 Gb. The RMS errors are tabulated at T=2 for a fixed $\lambda = k/h$ =3.2. Preconditioned method was deemed efficient through investigations which revealed their superiority in the context of execution time (measured in seconds), number of iterations (k) and RMS error.

Example 6.1 Consider $f(x,y,z,t) = (\beta^2 - 2\alpha - 2).\exp(t).\sinh(x).\sinh(y).\sinh(z)$, with initial condition $u(x,y,z,0) = \sinh(x).\sinh(y).\sinh(z),\quad u_t(x,y,z,0) = -[\sinh(x).\sinh(y).\sinh(z)]$ and boundary condition

$u(0,y,z,t) = 0,\ u(x,0,z,t) = 0, u(x,y,0,t) = 0,\ u(1,y,z,t) = \exp(-t).\sinh(1).\sinh(y).\sinh(z),$

$u(x,1,z,t) = \exp(-t).\sinh(x).\sinh(1).\sinh(z),$

$u(x,y,1,t) = \exp(-t).\sinh(x).\sinh(y).\sinh(1).$

The exact solution is $u(x,y,z,t) = \exp(-t).\sinh(x).\sinh(y).\sinh(z).$

Throughout the computation, we will put the values of α=10.0, β=5.0. From table 1, it can be observed that the proposed preconditioned Eight-Point EDG require lesser computing times than the original Eight-Point EDG method while maintaining the same degree of accuracies. In this example the execution timings of (PEP EDG) is only about 60%–80% of the original Eight-Point EDG method. Furthermore, the proposed preconditioned method reduced the number of iterations by about 55%–65% as shown in table 1.

Example 6.2 Consider $f(x,y,z,t) = ((\beta^2 - 4)\cos t - 2\alpha \sin t))\sinh(x).\sinh(y).\sinh(z)$, with initial condition, $u(x,y,z,t) = \cos(t).\sinh(x).\sinh(y).\sinh(z), u(x,y,z,0) = 0$ and boundary condition

$u(0,y,z,t) = 0,\ u(x,0,z,t) = 0, u(x,y,0,t) = 0,\ u(1,y,z,t) = \cos(t).\sinh(1).\sinh(y).\sinh(z),$

$u(x,1,z,t) = \cos(t).\sinh(x).\sinh(1).\sinh(z),$

$u(x,y,1,t) = \cos(t).\sinh(x).\sinh(y).\sinh(1).$

The exact solution is $u(x,y,z,t) = \cos(t).\sinh(x).\sinh(y).\sinh(z).$

Throughout the computation, we will put the values of α=10.0, β=0.0. In this example the execution timings of (PEP EDG) is only about 50%–71% of the original Eight-Point EDG method. Furthermore, the proposed preconditioned method reduced the number of iterations by about 45%–56% as shown in table 2.

Table 1. Comparison of the number of iterations, Execution time and RMS error (Example 6.1)

N	**Method**	**Elapsed Time (s)**	**No. of iterations (k)**	**RMS Error**
26	EP EDG	0.049	37	7.28E-4
	PEP EDG	0.013	13	6.03E-4
42	EP EDG	0.561	56	5.32E-4
	PEP EDG	0.207	21	3.81E-4
84	EP EDG	19.513	87	2.74E-4
	PEP EDG	8.644	33	1.47E-4
168	EP EDG	697.584	116	6.63E-5
	PEP EDG	320.731	47	4.52E-5
248	EP EDG	1046.663	234	8.92E-6
	PEP EDG	502.922	93	5.37E-6
318	EP EDG	1286.228	294	5.04E-6
	PEP EDG	584.356	115	4.61E-6

Table 2. Comparison of the number of iterations, Execution time and RMS error (Example 6.2)

N	**Method**	**Elapsed Time (s)**	**No. of iterations (k)**	**RMS Error**
26	EP EDG	0.068	45	5.43E-3
	PEP EDG	0.033	21	3.22E-3
42	EP EDG	0.605	64	8.81E-3
	PEP EDG	0.369	31	6.79E-3
84	EP EDG	22.784	93	9.88E-4
	PEP EDG	10.426	48	7.46E-4
168	EP EDG	714.334	126	9.69E-4
	PEP EDG	354.891	66	5.86E-4
248	EP EDG	1122.082	238	9.04E-5
	PEP EDG	543.678	113	8.52E-5
318	EP EDG	1334.305	298	6.86E-6
	PEP EDG	604.468	137	6.75E-6

7. Conclusion

In this article, we have formulated new preconditioned iterative scheme based on Eight-Point EDG method for solving the 3D- second order hyperbolic telegraph equation. The stability of the proposed method was analyzed and proven that its unconditional stable. From observation of all experimental results, it can be concluded that the proposed method may be a good alternative to solve this type of equations and many other numerical problems.

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References

[1] M. Dehghan, A. Mohebbi, The combination of collocation finite difference and multigrid methods for solution of the two-dimensional wave equation, Numer. Methods Partial Differ. Equ. 24 (2008) 897–910.

[2] M. Dehghan, A. Mohebbi, High order implicit collocation method for the solution of two dimensional linear hyperbolic equation, Numer. Methods Partial Differ. Equ. 25(1) (2009) 232–243.

[3] M. Dehghan, A. Ghesmati, Combination of meshless local weak and strong (MLWS) forms to solve the two dimensional hyperbolic telegraph equation, Eng. Anal. Bound. Elem. 34(4) (2010) 324–336.

[4] F. Gao, C. Chi, Unconditionally stable difference schemes for a one-space-dimensional linear hyperbolic equation, Appl. Math. Comput. 187 (2007) 1272–1276.

[5] R.K. Mohanty, An operator splitting technique for an unconditionally stable difference method for a linear three space dimensional hyperbolic equation with variable coefficients, Appl. Math. Comput. 162 (2005) 549–557.

[6] R.K. Mohanty, New unconditionally stable difference scheme for the solution of multi-dimensional telegraphic equations, Int. J. Comput. Math. 86(12) (2009) 2061–2071.

[7] R.K. Mohanty, M.K. Jain, An unconditionally stable alternating direction implicit scheme for the two space dimensional linear hyperbolic equation, Numer. Methods Partial Differ. Equ. 17(6) (2001) 684–688.

[8] R.K. Mohanty, M.K. Jain, U. Arora, An unconditionally stable ADI method for the linear hyperbolic equation in three space dimensions, Int. J. Comput. Math. 79 (2002) 133–142.

[9] N.H.M. Ali, L.M. Kew, New explicit group iterative methods in the solution of two dimensional hyperbolic equations, J. Comput. Phys. 231 (2012) 6953–6968.

[10] A. M. Saeed, N. H. M. Ali, " Preconditioned Modified Explicit Decoupled Group Method In The Solution Of Elliptic PDEs," Applied Mathematical Sciences. 4 (2010) 1165-1181.

[11] A. M. Saeed, N. H. M. Ali, " On the Convergence of the Preconditioned Group Rotated Iterative Methods In The Solution of Elliptic PDEs," Applied Mathematics &Information Sciences. 5(2011) 65-73.

[12] A. M. Saeed, N. AL-harbi, Group Splitting with SOR/AOR Methods for Solving Boundary Value Problems: A Computational Comparison, European Journal of

Pure And Applied Mathematics, 2021;14: 905-914.

[13] A. M. Saeed, N. AL-harbi, An Accelerated Numerical Solution of Elliptic Equations by using Nine Explicit Group Iterative Method, Advances in Dynamical Systems and Applications (ADSA). 16 (2021) 1479-1497.

[14] N. H. M. Ali , A. M. Saeed, " Preconditioned Modified Explicit Decoupled Group for the Solution of Steady State Navier-Stokes Equation," Applied Mathematics & Information Sciences. 7 (2013)1837-1844.

[15] A. M. Saeed, N. H. M. Ali, " Accelerated Solution Of Two Dimensional Diffusion Equation," World Applied Sciences Journal. 7(2014)1906-1912.

[16] A. M. Saeed, " Fast Iterative Solver For The 2-D Convection-Diffusion Equations," Journal Of Advances In Mathematics. 9(2014) 2773-2782.

[17] A. M. Saeed, " Efficient Group Iterative Method for Solving the Biharmonic Equation," British Journal of Mathematics & Computer Science. 9(2015) 237-245.

[18] A. M. Saeed, " Improved Rotated Finite Difference Method for Solving Fractional Elliptic Partial Differential Equations," American Scientific Research Journal for Engineering, Technology, and Sciences. 26(2016) 261-270.

[19] A. M. Saeed, " Solving Poisson's Equation Using Preconditioned Nine Point Group SOR Iterative Method," International Journal of Mathematics and Statistics Invention. 4(2016) 20-26.

[20] A. M. Saeed, " A numerical method based on explicit finite difference for solving fractional hyperbolic PDE's," International Journal of Applied Mathematical Research (IJAMR).5(2016) 202-205.

[21] A. M. Saeed, " A new Approach on Numerical Solutions of Burger's Equation using PMEDG Iterative Method," Journal of Asian Scientific Research. 4(2017) 263-270.

[22] A. M. Saeed, " Solving 2D Time-Fractional Diffusion Equations by Preconditioned Fractional EDG Method," Journal of Progressive Research in Mathematics. 14(2018) 2388 - 2394.

[23] Abdulkafi M. Saeed, Preconditioned Fractional MEG Iterative Method for Solving the Time Fractional Advection-Diffusion Equation, Advances in Dynamical Systems and Applications (ADSA). 16 (2021) 1183-1198.

[24] L.M. Kew, N.H.M. Ali. New explicit group iterative methods in the solution of three dimensional hyperbolic telegraph equations, J. Comput. Phys. 294 (2015) 382-404.

[25] J.C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, second ed., University of Wisconsin-Madison, Wadsworth & Brooks/Cole Advanced Books & Software Publishers, New York, 2004.