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Ulam-Hyers-Rassias Stability for a Coupled System of Nonlinear Volterra Integro-differential Equations with Finite delay

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Abstract

This work investigates Hyers-Ulam stability, and Hyers-Ulam-Rassias stability results for a coupled system of a nonlinear delay Volterra integro-differential equation NDVIDE of the form

$$\begin{cases} z_i'(t) = \mathcal{H}_i\Bigg(t, z_i(t), z_i(\beta_i(t)), \int_0^t g_i(t,s,z_i(s),z_i(\beta_i(s)))ds\Bigg), \quad \forall\, t \in \hat{J}, \\ z_i(t) = \upsilon_i(t), \quad for\; t \in [-\varrho, 0],\; i = 1,2. \end{cases}$$

Our approach is based on Pachpatte's inequality and Picard's operator. Besides, we extend and develop some well-known results, then give an illustrative example for our main results.

Keywords: Integro-differential equation with delay, Ulam stability, Pachpatte inequality, Picard operator

Mathematics Subject Classification (2020): 34K45, 34D20, 34A12.

1. INTRODUCTION

Hyers-Ulam (HU) and Hyers-Ulam-Rassias (HUR) stabilities have received immense consideration in recent times. To some extent, this is due to their likely application in model situations where we cannot assume to get the exact solution of the problem simply. We can suppose to get an approximate solution, which must be constant at a specific point. This has been done for an enormous number of diverse types of equations. Among those, we identify differential equations, functional equations and integral equations, see [1]. The main Ulam stability problem of functional equations has been developed into various kinds of equations. It has been noted that the Ulam stability theory is found to be useful in the study of differential equations, integral equations, difference equations, fractional differential equations, and other similar problems. Ulam-type stability problem was formulated by Ulam in 1941 [10], then later developed and improved by many researchers, see [4, 12]. When should the solutions of an equation that differ somewhat from the given one be close to the given equation's solution. A considerable number of research papers dealing with the UH and UHR stabilities of various kinds of differential and integral equations can be found in the literature. One can refer to for basic results and recent developments on Ulam stabilities of differential and integral equations in [2, 12, 13, 14, 15, 16, 17]. Besides, several research papers have been carried out on Ulam-Hyers stability for NDVIDEs and Volterra integral equations recently [18, 19]. In [20], UH and UHR stabilities have been investigated for the class of the following NVIDE

$$z'(t) = h\Big(t, z(t), \int_{t_0}^{t} g(t,s,z(s),z(\varsigma(s)))ds\Big),$$
$$z(t_0) = e, \ \ e \in (-\infty, \infty).$$

In [21], Pachpatte's inequality and Picard operator have been applied to study existence and uniqueness and Ulam type stabilities for the NDVIDE, i.e.,

$$\begin{cases} z'(t) = \mathcal{H}\Big(t, z(t), z(\varsigma(t)), \int_0^t g(t,s,z(s),z(\varsigma(s)))ds\Big), \ t \in [0,b], \\ z(t) = \theta(t), \quad for \ \ t \in [-r, 0]. \end{cases} \tag{1}$$

Motivated by above works, we consider the following NDVIDE

$$z_i'(t) = \mathcal{H}_i\Big(t, z_i(t), z_i(\beta_i(t)), \int_0^t g_i(t,s,z_i(s),z_i(\beta_i(s)))ds\Big), \ \forall\, t \in \hat{J}, \tag{2}$$
$$z_i(t) = \upsilon_i(t), \quad for \ \ t \in [-\varrho, 0], \ i = 1, 2, \tag{3}$$

where $J = [t_0, \mathcal{T}]$, $\upsilon_i \in C([-\varrho, 0], \mathbb{R})$, $\mathcal{H}_i : [t_0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_i : [t_0, \mathcal{T}] \times [t_0, \mathcal{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\beta_i : [t_0, \mathcal{T}] \to [-\varrho, 0]$ are continuous functions with $\beta_i(t) \leq t$, for all $i = 1, 2$.

Observe that the single problem (1) has been studied by Kucche and Shikhare [21], using Pickard's method. Here we will study a coupled system of NDVIDE (2) with a finite delay. Our approach is smooth and depends on Pachpatte's inequality and Picard's operator. Moreover, our analysis costs the minimum conditions sufficient to discuss the stability analysis in the sense of UH and UHR.

The structure of this article is as follows: In section 2, we mention some concepts and principles. Section 3 discusses UH and UHR stabilities for NDVIDE (2). An example to illustrative the obtained results is given in section 4. At the end, the last section deals with the conclusion.

2. PRELIMINARIES

Here we will go to call back some principal definitions and conditions to discuss the Ulam type stabilities for NDVIDEs (2)
For every $\epsilon > 0$ and a nonnegative increasing continuous function $\Phi \in C([-\varrho, \mathcal{T}], \mathbb{R}_+)$. We consider the following inequalities:

$$|\mathfrak{D}_i'(t) - \mathcal{G}_i(t)| \le \epsilon, \ i = 1, 2, \ t \in \hat{J}, \tag{4}$$

$$|\mathfrak{D}_i'(t) - \mathcal{G}_i(t)| \le \Phi(t), \ i = 1, 2, \ t \in \hat{J}, \tag{5}$$

$$|\mathfrak{D}_i'(t) - \mathcal{G}_i(t)| \le \epsilon\Phi(t), \ i = 1, 2, \ t \in \hat{J}, \tag{6}$$

where

$$\mathcal{G}_i(t) = \mathcal{H}_i\bigg(t, \mathfrak{D}_i(t), \mathfrak{D}_i(\beta_i(t)), \int_0^t g_i(t, s, \mathfrak{D}_i(s), \mathfrak{D}_i(\beta_i(s)))ds\bigg), \ i = 1, 2.$$

Definition 2.1. *Equation* (2) *is called UH stable if there is a constant* $K > 0$ *such that for every* $\epsilon > 0$ *and for every solution* $\mathfrak{D}_i \in C'([-\varrho, \mathcal{T}], \mathbb{R})$ *of* (4) *there is a solution* $z_i \in C'([-\varrho, \mathcal{T}], \mathbb{R}), \ i = 1, 2$ *of* (2) *satisfying*

$$|\mathfrak{D}_i(t) - z_i(t)| \le K\epsilon, \ for\, t \in [-\varrho, \mathcal{T}].$$

Definition 2.2. *Equation* (2) *is called generalized UH stable if there is* $\theta_{\mathcal{H}_i} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{\mathcal{H}_i}(0) = 0$ *such that for every solution* $\mathfrak{D}_i \in C'([-\varrho, \mathcal{T}], \mathbb{R})$ *of* (4) *there is a solution* $z_i \in C'([-\varrho, \mathcal{T}], \mathbb{R}), \ i = 1, 2$ *of* (2) *satisfying*

$$|\mathfrak{D}_i(t) - z_i(t)| \le \theta_{\mathcal{H}_i}\epsilon, \ for\, t \in [-\varrho, \mathcal{T}], \ i = 1, 2.$$

Definition 2.3. *Equation* (2) *is called the UHR stable concerning continuous function* $\Phi \in C(\hat{J}, \mathbb{R}_+)$ *if there is a constant* $K_\Phi > 0$ *such that for every* $\epsilon > 0$ *and for every solution* $\mathfrak{D}_i \in C'([-\varrho, \mathcal{T}], \mathbb{R})$ *of* (6) *there is a solution* $z_i \in C'([-\varrho, \mathcal{T}], \mathbb{R}), \ i = 1, 2$ *of* (2) *satisfying*

$$|\mathfrak{D}_i(t) - z_i(t)| \le K_\Phi\epsilon\Phi(t), \ for\, t \in [-\varrho, \mathcal{T}], \ i = 1, 2,$$

Definition 2.4. *Equation* (2) *is called the generalized UHR stable concerning continuous function* $\Phi \in C(J,\mathbb{R}_+)$ *if there is a constant* $K_\Phi > 0$ *such that for every* $\epsilon > 0$ *and for every solution* $\mathfrak{D}_i \in C'([-\varrho,\mathcal{T}],\mathbb{R})$ *of* (5) *there is a solution* $z_i \in C'([-\varrho,\mathcal{T}],\mathbb{R}),\ i=1,2$ *of* (2) *satisfying*

$$|\mathfrak{D}_i(t) - z_i(t)| \leq K_\Phi \Phi(t),\ for\ t \in [-\varrho,\mathbb{R}],\ i=1,2,$$

Remark 2.1. Note that a function $\mathfrak{D}_i \in C'(\hat{J},\mathbb{R})$ is a solution of the estimate (4) if there is a $p_{\mathfrak{D}_i} \in C(\hat{J},\mathbb{R})$ (which depends on $\mathfrak{D}_i$) such that

(1) $|p_{\mathfrak{D}_i}(t)| \leq \epsilon,\ t \in \hat{J},$
(2) $\mathfrak{D}_i'(t) = \mathcal{H}_i\Big(t, \mathfrak{D}_i(t), \mathfrak{D}_i(\beta_i(t)), \int_0^t g_i(t,s,\mathfrak{D}_i(s),\mathfrak{D}_i(\beta_i(s)))ds\Big) + p_{\mathfrak{D}_i}(t),\ i = 1,2,\ t \in \hat{J}.$

Similar arguments hold for the inequalities (5) and (6).

Remark 2.2. If $\mathfrak{D}_i \in C'(\hat{J},\mathbb{R})$ fulfills the estimate (4), then $\mathfrak{D}_i$ is a solution of the following integral inequality:

$$\left|\mathfrak{D}_i(t) - \mathfrak{D}_i(0) - \int_0^t \mathcal{H}_i\left(s, \mathfrak{D}_i(s), \mathfrak{D}_i(\beta_i(s)), \int_0^s g_i(s,\sigma,\mathfrak{D}_i(\sigma),\mathfrak{D}_i(\beta_i(\sigma)))d\sigma\right)ds\right| \leq \epsilon t,\ i=1,2,\ t \in \hat{J}. \tag{7}$$

Indeed, if $\mathfrak{D}_i \in C'(\hat{J},\mathbb{R})$ fulfills the estimate (4), by Remark 2.1, we would have

$$\mathfrak{D}_i'(t) = \mathcal{H}_i\left(t, \mathfrak{D}_i(t), \mathfrak{D}_i(\beta_i(t)), \int_0^t g_i(t,s,\mathfrak{D}_i(s),\mathfrak{D}_i(\beta_i(s)))ds\right) + p_{\mathfrak{D}_i}(t),\ i=1,2,\ \ t \in \hat{J}.$$

This yields that

$$\left|\mathfrak{D}_i(t) - \mathfrak{D}_i(0) - \int_0^t \mathcal{H}_i\left(s, \mathfrak{D}_i(s), \mathfrak{D}_i(\beta_i(s)), \int_0^s g_i(s,\sigma,\mathfrak{D}_i(\sigma),\mathfrak{D}_i(\beta_i(\sigma)))d\sigma\right)ds\right| \leq \int_0^t |p_{\mathfrak{D}_i}(t)| \leq \epsilon t,\ i=1,2.$$

Similar estimates can also be obtained for the inequalities (5) and (6). We use the following inequality to obtain our main results.

Theorem 2.1. (Pachpatte's inequality (see [22], p. 39)). Let $v(t)$, $h(t)$ and $p(t)$ be nonnegative continuous functions defined on $\mathbb{R}_+$, for which the inequality

$$v(t) \leq m(t) + \int_0^t h(s)\left[v(s) + \int_0^s p(\sigma)v(\sigma)d\sigma\right]ds,\ for\ t \in \mathbb{R}_+,$$

holds, where $m(t)$ is nonnegative and continuous increasing function defined on $\mathbb{R}_+$. Then

$$v(t) \leq m(t)\left[1 + \int_0^t h(s)exp\left(\int_0^s h(\sigma) + p(\sigma)d\sigma\right)ds\right],\ for\ t \in \mathbb{R}_+.$$

Now we give the definition of the Picard operator and state the abstract Gronwall lemma (see Rus [23]), which are used in our subsequent analysis.

Definition 2.5. *(Picard operator [23]). Let $(\mathcal{Y}, d)$ be a metric space. An operator $\mathfrak{B} : \mathcal{Y} \times \mathcal{Y}$ is called the Picard operator if there is $y^* \in \mathcal{Y}$ such that:*
(1) $\mathcal{H}_{\mathfrak{B}} = \{y^*\},\ wher\ \mathcal{H}_{\mathfrak{B}}\{y \in \mathcal{Y} : \mathfrak{B}(y) = y\}$ is the fixed point set of $\mathfrak{B}$,
(2) the sequence $(\mathfrak{B}^n(x_0))_{n\in\mathcal{N}}$ converges to y^* for each $y_0 \in \mathcal{Y}$.

Lemma 2.1. (Gronwall lemma [23]). Let $(\mathcal{Y}, d, \leq)$ be an ordered metric space and let $\mathfrak{B} : \mathcal{Y} \times \mathcal{Y}$ be an nondecreasing Picard operator ($\mathcal{H}_{\mathfrak{B}} = y^*_{\mathfrak{A}}$). Then for $y \in \mathcal{Y}, y \leq \mathfrak{B}(y)$ implies $y \leq y^*_{\mathfrak{B}}$, while $y \geq \mathfrak{B}(y)$ implies $y \geq y^*_{\mathfrak{B}}$.

3. ULAM STABILITY FOR NVIDE ON $\hat{J} = [T_0, T]$

First, we list the following assumptions for our convenience.
(A1) (1) The functions $\beta : [t_0, \mathcal{T}] \to [-\varrho, 0]$, is a continuous with $\beta_i(t) \leq t, forall\ i = 1, 2,$
(2) There exists real number $L_{\mathcal{H}_i}, L_{g_i} > 0$ such that

$$|\mathcal{H}_i(t, \varkappa_1, \varkappa_2, \varkappa_3) - \mathcal{H}_i(t, \hat{\varkappa}_1, \hat{\varkappa}_2, \hat{\varkappa}_3)| \leq L_{\mathcal{H}_i}(|\varkappa_1 - \hat{\varkappa}_1| + |\varkappa_2 - \hat{\varkappa}_2| + |\varkappa_3 - \hat{\varkappa}_3|),$$

$$|g_i(t, s, \varkappa_1, \varkappa_2) - g_i(t, s, \hat{\varkappa}_1, \hat{\varkappa}_2)| \leq L_{g_i}(|\varkappa_1 - \hat{\varkappa}_1| + |\varkappa_2 - \hat{\varkappa}_2|)$$

for each $(t, s) \in \hat{J} \times J$ and $\varkappa_j, \hat{\varkappa}_j, \in \mathbb{R}$ $(i = 1, 2\ and\ j = 1, 2, 3)$.

(A2) Let $\Phi \in ([-\varrho, \mathcal{T}], \mathbb{R}_+)$ is nonnegative, increasing and continuous and there is a constant $\eta > 0$ such that

$$\int_0^t \Phi(s)ds \leq \eta\Phi(t),\ for\ t \in \hat{J}.$$

Theorem 3.1. Let $\mathcal{H}_i$ and g_i in (2) satisfy the hypothesis (A1) and suppose that (A2) holds. If $TL_{\mathcal{H}_i}[2 + L_{g_i}T] < 1, i = 1, 2$, then the following assertions hold.

(1) Equation (3) with initial value (2) has a unique solution $z_i \in C([-\varrho, \mathcal{T}], \mathbb{R}) \cap C'(\hat{J}, \mathbb{R})$,

(2) Equation (2) is UHR stable concerning the function Φ.
Proof. (1) Observe first that in view of assumption (A1)(1), the equation (3) with initial value (2) is equivalent to the following integral equations:

$$z_i(t) = \upsilon_i(0) + \int_0^t \mathcal{H}_i\bigg(t, z_i(t), z_i(\beta_i(t)), \int_0^s g_i(s,\sigma, z_i(\sigma), z_i(\beta_i(\sigma)))d\sigma\bigg)ds, \quad \forall\, t \in \hat{J},$$
$$z_i(t) = \upsilon_i(t),\ t \in [-\varrho, 0],\ i = 1, 2.$$

Let the Banach space $\mathcal{Y} = C([-\varrho, \mathcal{T}], \mathbb{R})$ with Chebyshev norm $\|.\|$, and define the operator $B_{\mathcal{H}_i} : \mathcal{Y} \to \mathcal{Y}$ by

$$B_{\mathcal{H}_i}(z_i)(t) = \upsilon_i(0) + \int_0^t \mathcal{H}_i\bigg(t, z_i(t), z_i(\beta_i(t)), \int_0^s g_i(s,\sigma, z_i(\sigma), z_i(\beta_i(\sigma)))d\sigma\bigg)ds, \quad \forall\, t \in \hat{J},$$
$$B_{\mathcal{H}_i}(z_i)(t) = \upsilon_i(t),\ t \in [-\varrho, 0],\ i = 1, 2.$$

We can now show that $B_{\mathcal{H}_i}$ has a fixed point using the contraction principle. Observe that

$$|B_{\mathcal{H}_i}(z_i)(t) - B_{\mathcal{H}_i}(\mathfrak{D}_i)(t)| = 0,\ z_i, \mathfrak{D}_i \in C([-\varrho, \mathcal{T}], \mathbb{R}),\ t \in [-\varrho, \mathcal{T}], i = 1, 2. \tag{8}$$

Next, for any $t \in \hat{J}$, we can write

$$\begin{aligned}
&|B_{\mathcal{H}_i}(z_i)(t) - B_{\mathcal{H}_i}(\mathfrak{D}_i)(t)|\\
&\le \int_0^t L_{\mathcal{H}_i}\bigg\{|z_i(s) - \mathfrak{D}_i(s)| + |z_i(\beta_i(s)) - \mathfrak{D}_i((\beta_i(s))|\\
&\quad + \int_0^s \big[L_{g_i}\big\{|z_i(\sigma) - \mathfrak{D}_i(\sigma)| + |z_i(\beta_i(\sigma)) - \mathfrak{D}_i((\beta_i(\sigma))\big]d\sigma\bigg\}ds\\
&\le \int_0^t L_{\mathcal{H}_i}\bigg\{\max_{0\le\tau_1\le\sigma}|z_i(\tau_1) - \mathfrak{D}_i(\tau_1)| + \max_{0\le\tau_1\le\sigma}|z_i(\beta_i(\tau_1)) - \mathfrak{D}_i((\beta_i(\tau_1))|\\
&\quad + \int_0^s \bigg[\max_{0\le\tau_2\le\sigma} L_{g_i}\big\{|z_i(\tau_2) - \mathfrak{D}_i(\tau_2)| + \max_{0\le\tau_2\le\sigma}|z_i(\beta_i(\tau_2)) - \mathfrak{D}_i((\beta_i(\tau_2))\bigg]d\sigma\bigg\}ds\\
&\le \int_0^t L_{\mathcal{H}_i}\bigg\{\max_{-\varrho\le\tau_1\le\mathcal{T}}|z_i(\tau_1) - \mathfrak{D}_i(\tau_1)| + \max_{-\varrho\le\sigma_1\le\mathcal{T}}|z_i(\sigma_1) - \mathfrak{D}_i(\sigma_1)|\\
&\quad + \int_0^s \bigg[\max_{-\varrho\le\tau_2\le\mathcal{T}} L_{g_i}\big\{|z_i(\tau_2) - \mathfrak{D}_i(\tau_2)| + \max_{0\le\sigma_2\le\mathcal{T}}|z_i(\sigma_2) - \mathfrak{D}_i(\sigma_2)|\bigg]d\sigma\bigg\}ds\\
&\le \int_0^t 2L_{\mathcal{H}_i}\bigg\{\|z_i - \mathfrak{D}_i\|_C + 2\int_0^s L_{g_i}\|z_i - \mathfrak{D}_i\|_C d\sigma\bigg\}ds \qquad (9)\\
&\le T L_{\mathcal{H}_i}(2 + L_{g_i}T)\|z_i - \mathfrak{D}_i\|_C.
\end{aligned}$$

From (8) and (9), it follows that

$$\|B_{\mathcal{H}_i}(z_i)(t)-B_{\mathcal{H}_i}(\mathfrak{D}_i)(t)\|_C \leq \mathcal{T}L_{\mathcal{H}_i}(2+L_{g_i}\mathcal{T})\|z_i-\mathfrak{D}_i\|_C,\ z_i,\mathfrak{D}_i \in C([-\varrho,\mathcal{T}],\mathbb{R}), i=1,2.$$

Since $\mathcal{T}L_{\mathcal{H}_i}(2+L_{g_i}\mathcal{T})<1,\ i=1,2$. On the complete space $\mathcal{Y}$, the operator $B_{\mathcal{H}_i}$ is a contraction. Also the operator $B_{\mathcal{H}_i}$ has a fixed point $z_i^*:[-\varrho,\mathcal{T}]\to\mathbb{R},\ i=1,2,$ which provides a solution of the problem (2), (3), by Banach contraction principle

(2) Let $\mathfrak{D}_i \in C([-\varrho,\mathcal{T}],\mathbb{R})\cap C'(\hat{J},\mathbb{R})$, be a solution of the estimate (6). Denote by $z_i \in C([-\varrho,\mathcal{T}],\mathbb{R})\cap C'(\hat{J},\mathbb{R})$, the unique solution of the problem:

$$z_i'(t)=\mathcal{H}_i\bigg(t,z_i(t),z_i(\beta_i(t)),\int_0^t \mathcal{H}_i(t,s,z_i(s),z_i(\beta_i(s)))ds\bigg),\ \forall\, t\in\hat{J},$$
$$z_i(t)=\mathfrak{D}_i(t),\ for\ t\in[-\varrho,0],\ i=1,2.$$

Then assumption (A1)(i) allows to write the following (equivalent to the above problem) integral equation:

$$z_i(t)=\mathfrak{D}_i(0)+\int_0^t \mathcal{H}_i\bigg(t,z_i(t),z_i(\beta_i(t)),\int_0^s g_i(s,\sigma,z_i(\sigma),z_i(\beta_i(\sigma)))d\sigma\bigg)ds, \quad \forall\, t\in\hat{J}, \tag{10}$$

$$z_i(t)=\mathfrak{D}_i(t),\ t\in[-\varrho,0],\ i=1,2. \tag{11}$$

If $\mathfrak{D}_i \in C([-\varrho,\mathcal{T}],\mathbb{R})\cap C'(\hat{J},\mathbb{R})$, fulfills the estimate (6), then using assumption(A2) and Remarks 2.1 and (3), we obtain

$$\bigg|\mathfrak{D}_i(t)-\mathfrak{D}_i(0)-\int_0^t \mathcal{H}_i\bigg(s,\mathfrak{D}_i(s),\mathfrak{D}_i(\beta_i(s)),\int_0^s g_i(s,\sigma,\mathfrak{D}_i(\sigma),\mathfrak{D}_i(\beta_i(\sigma)))d\sigma\bigg)ds\bigg|$$
$$\leq \int_0^t |p_{\mathfrak{D}_i}(t)| \leq \int_0^t \epsilon\Phi(s)ds \leq \epsilon\eta\Phi(t),\ t\in\hat{J},\ i=1,2. \tag{12}$$

Observe that $|\mathfrak{D}_i(t)-z_i(t)|=0$ for $t\in[-\varrho,0]$. Next, using assumption (A2)(2), the equation(11) and the estimate in (12), for any $t\in\hat{J}$, we can write

$$|\mathfrak{D}_i(t)-z_i(t)| = \bigg|\mathfrak{D}_i(t)-\mathfrak{D}_i(0)-\int_0^t \mathcal{H}_i\bigg(s,z_i(s),z_i(\beta_i(s)),\int_0^s g_i(s,\sigma,z_i(\sigma),z_i(\beta_i(\sigma)))d\sigma\bigg)ds\bigg|$$
$$\leq \bigg|\mathfrak{D}_i(t)-\mathfrak{D}_i(0)-\int_0^t \mathcal{H}_i\bigg(s,\mathfrak{D}_i(s),\mathfrak{D}_i(\beta_i(s)),\int_0^s g_i(s,\sigma,\mathfrak{D}_i(\sigma),\mathfrak{D}_i(\beta_i(\sigma)))d\sigma\bigg)ds\bigg|$$
$$+\int_0^t \bigg|\mathcal{H}_i\bigg(s,\mathfrak{D}_i(s),\mathfrak{D}_i(\beta_i(s)),\int_{t_0}^s g_i(s,\sigma,\mathfrak{D}_i(\sigma),\mathfrak{D}_i(\beta_i(\sigma)))d\sigma\bigg)ds$$
$$-\int_0^t \bigg|\mathcal{H}_i\bigg(s,z_i(s),z_i(\beta_i(s)),\int_0^s g_i(s,\sigma,z_i(\sigma),z_i(\beta_i(\sigma)))d\sigma\bigg)ds\bigg|$$

$$\leq \epsilon\eta\Phi(t) + \int_0^t L_{\mathcal{H}_i}\Big\{|\mathfrak{D}_i(s) - z_i(s)| + |\mathfrak{D}_i(\beta_i(s)) - z_i(\beta_i(s))| + \int_0^s L_{g_i}\big[|\mathfrak{D}_i(\sigma) - z_i(\sigma)| + |\mathfrak{D}_i(\beta_i(\sigma)) - z_i(\beta_i(\sigma))|\big]d\sigma\Big\}ds,\ i = 1, 2. \quad (13)$$

According to (13), let the operator $\mathfrak{B} : C([-\varrho, \mathcal{T}], \mathbb{R}_+) \to C([-\varrho, \mathcal{T}], \mathbb{R}_+)$ defined by

$$\mathfrak{B}(v_i)(t) = \mathfrak{D}_i(t),\ t \in [-\varrho, 0],\ i = 1, 2,$$

$$\mathfrak{B}(v_i)(t) = \epsilon\eta\Phi(t)+L_{\mathcal{H}_i}\int_0^t \Big\{v_i(s)-w_i(\beta_i(s))+L_{g_i}\int_0^s v_i(\sigma))-w_i(\beta_i(\sigma))\Big\}, \quad \forall\, t \in \hat{J}.$$

Next, we show that $\mathfrak{B}$ is a Picard operator (see Definition 2.5). To this end, observe first that for any $v_i, w_i \in C([-\varrho, \mathcal{T}], \mathbb{R}_+)$ we have $|\mathfrak{B}(v_i)(t) - \mathfrak{B}(w_i)(t)| = 0,\ t \in [-\varrho, 0], i = 1, 2.$

Using hypothesis (A1)(ii), for all $t \in [-\varrho, 0], i = 1, 2$, we can write

$$\begin{aligned}
&|\mathfrak{B}(v_i)(t) - \mathfrak{B}(w_i)(t)| \\
&\leq L_{\mathcal{H}_i}\int_0^t \Big\{|v_i(s) - w_i(s)| + |v_i(\beta_i(s)) - w_i(\beta_i(s))| \\
&\qquad + L_{g_i}\int_0^s \big[|v_i(\sigma) - w_i(\sigma)| + |v_i(\beta_i(\sigma)) - w_i(\beta_i(\sigma))|\big]d\sigma\Big\}ds \\
&\leq \int_0^t L_{\mathcal{H}_i}\Big\{\max_{0\leq\tau_1\leq\sigma}|v_i(\tau_1) - w_i(\tau_1)| + \max_{0\leq\tau_1\leq\sigma}|v_i(\beta_i(\tau_1)) - w_i((\beta_i(\tau_1))| \\
&\qquad + \int_0^s \Big[\max_{0\leq\tau_2\leq\sigma} L_{g_i}\big\{|v_i(\tau_2) - w_i(\tau_2)| + \max_{0\leq\tau_2\leq\sigma}|v_i(\beta_i(\tau_2)) - w_i((\beta_i(\tau_2))\Big]d\sigma\Big\}ds \\
&\leq \int_0^t L_{\mathcal{H}_i}\Big\{\max_{-\varrho\leq\tau_1\leq\mathcal{T}}|v_i(\tau_1) - w_i(\tau_1)| + \max_{-\varrho\leq\sigma_1\leq\mathcal{T}}|v_i(\sigma_1) - w_i(\sigma_1)| \\
&\qquad + \int_0^s \Big[\max_{-\varrho\leq\tau_2\leq\mathcal{T}} L_{g_i}\big\{|v_i(\tau_2) - w_i(\tau_2)| + \max_{0\leq\sigma_2\leq\mathcal{T}}|v_i((\sigma_2) - w_i((\sigma_2)\Big]d\sigma\Big\}ds \\
&\leq \int_0^t L_{\mathcal{H}_i}\Big\{2\|v_i-w_i\|_C+2\int_0^s L_{g_i}\|v_i-w_i\|_C d\sigma\Big\}ds \leq \mathcal{T}L_{\mathcal{H}_i}(2+L_{g_i}\mathcal{T})\|v_i-w_i\|_C.
\end{aligned}$$

Therefore,

$$\|\mathfrak{B}(v_i) - \mathfrak{B}(w_i)\|_C \leq \mathcal{T}L_{\mathcal{H}_i}(2 + L_{g_i}\mathcal{T})\|v_i - w_i\|_C,\ v_i, w_i \in C([-\varrho, \mathcal{T}], \mathcal{R}), i = 1, 2.$$

Since $\mathcal{T}L_{\mathcal{H}_i}(2 + L_{g_i}\mathcal{T}) < 1$, $\mathfrak{B}$ is a contraction on $C([-\varrho, \mathcal{T}], \mathbb{R}_+)$ by Banach contraction principle, we conclude that $\mathfrak{B}$ is a Picard operator and $\mathcal{H}_{\mathfrak{B}} = \{v_i^*\}$ Then, for $t \in \hat{J},\ i = 1, 2$, we have.

$$v_i^*(t) = \epsilon\eta\Phi(t) + L_{\mathcal{H}_i}\int_0^t \Big\{v_i^*(s) - v_i^*(\beta_i(s)) + L_{g_i}\int_0^s \big[v_i^*(\sigma) - v_i^*(\beta_i(\sigma))\big]d\sigma\Big\}ds.$$

Note that v_i^* is increasing and $(v_i^*)' \geq 0$ on $\hat{J}$. Therefore $v_i^*(\beta_i(t)) \leq v_i^*(t)$ for $\beta_i(t) \leq t$,
$t \in \hat{J},\ i = 1, 2$, and hence

$$v_i^*(t) \leq \epsilon\eta\Phi(t) + \int_0^t 2L_{\mathcal{H}_i}\left(u_i^*(s) + \int_0^s 2L_{g_i}\big[v_i^*(\sigma)\big]d\sigma\right)ds.$$

Next, using Pachpatte's inequality given in Theorem 2.1, we obtain

$$\begin{aligned} v_i^* &\leq \epsilon\eta\Phi(t)\left[1 + \int_0^t 2L_{\mathcal{H}_i}exp\left(\int_0^s [2L_{\mathcal{H}_i} + L_{g_i}]d\sigma\right)ds\right] \\ &\leq \epsilon\eta\Phi(t)\left\{1 + 2L_{\mathcal{H}_i}\left(\frac{exp(2L_{\mathcal{H}_i} + L_{g_i})\mathcal{T} - 1}{2L_{\mathcal{H}_i} + L_{g_i}}\right)ds\right\} \end{aligned} \tag{14}$$

Taking $K_\Phi = \eta\left\{1 + 2L_{\mathcal{H}_i}\left(\frac{exp(2L_{\mathcal{H}_i}+L_{g_i})\mathcal{T}-1}{2L_{\mathcal{H}_i}+L_{g_i}}\right)ds\right\}$ from inequality (14), we get

$$v_i^*(t) \leq K_\Phi\epsilon\Phi(t),\ t \in [-\varrho, \mathcal{T}]$$

For $v_i(t) = |\mathfrak{D}_i(t) - z_i(t)|$ the inequality (13) gives that $v_i(t) \leq \mathfrak{B}(v_i)(t)$. So, we have proved that $\mathfrak{B} : C([-\varrho, \mathcal{T}], \mathbb{R}_+) \to C([-\varrho, \mathcal{T}], \mathbb{R}_+)$ is an increasing Picard operator such that for $v_i \in C([-\varrho, \mathcal{T}], \mathbb{R}_+)$, $v_i(t) \in \mathfrak{B}v_i(t)$ and $\mathcal{H}_{\mathfrak{B}} = \{v_i^*\}$.
Hence, using the abstract Gronwall lemma (Lemma 2.1), we obtain $v_i(t) \leq v_i^*(t),\ t \in [-\varrho, \mathcal{T}]$, implying that

$$|\mathfrak{D}_i(t) - z_i(t)| \leq K_\Phi\epsilon\Phi(t),\ \forall\, t \in [-\varrho, \mathcal{T}],\ i = 1, 2. \tag{15}$$

Thus, equation (2) is UHR stable concerning the function Φ. Theorem 3.1 is proved.

Corollary 3.1. Let the functions $\mathcal{H}_i$ and g_i in (2) satisfy (A1) and assume that (A2) holds. If $\mathcal{T}L_{\mathcal{H}_i}(2 + L_{g_i}\mathcal{T}) < 1,\ i = 1, 2$, then the problem (2), (3) has a unique solution and the equation (2) is generalized UHR stable concerning the function Φ.

Proof. In the proof of Theorem 3.1, if we take $\epsilon = 1$, then, we get (cf. (15)):

$$|\mathfrak{D}_i(t) - z_i(t)| \leq K_\Phi\Phi(t),\ \forall\, t \in [-\varrho, \mathcal{T}],\ i = 1, 2,$$

showing that the equation (2) is generalized UHR stable with concerning to the function Φ. Using arguments similar to those applied in the proof of Theorem 3.1, one can prove UH stability of equation (2). Observing that for $\Phi = 1, for\ all\ t \in [-\varrho, \mathcal{T}]$, the assumption (A2) holds, we can state the following corollary of Theorem 3.1.

Corollary 3.2. Let the functions $\mathcal{H}_i$ and g_i in (2) satisfy the hypothesis (A1). If $TL_{\mathcal{H}_i}(2 + L_{g_i}\mathcal{T}) < 1,\ i = 1, 2,$ then the problem (2), (3) has a unique solution and the equation (2) is UH stable.

Proof. In the proof of Theorem 3.1, if we take $\Phi = 1,\ \forall\, t \in [-\varrho, \mathcal{T}]$ then, we get (cf. (15)):

$$|\mathfrak{D}_i(t) - z_i(t)| \leq K\epsilon,\ \forall\, t \in [-\varrho, \mathcal{T}],\ i = 1, 2,$$

and the result follows.

Corollary 3.3. Let $\mathcal{H}_i$ and g_i in (2) satisfy the hypothesis (A1). If $\mathcal{T}L_{\mathcal{H}_i}(2 + L_{g_i}\mathcal{T}) < 1,\ i = 1, 2,$ then the problem (2), (3) has a unique solution and the problem (2) is generalized UHR stable.

Proof. The result follows from Corollary 3.2, by taking $\mathcal{H}_i(\epsilon) = K\epsilon,\ i = 1, 2.$

3.2. Applications. In this segment, we consider some important special cases of the problem (2), (3)
Fix any $\varrho > 0$, and define $\beta_i(t) = t - \varrho,\ t \in [t_0, \mathcal{T}],\ i = 1, 2.$ Then we obtain the following special cases of the problem (2), (3):

$$z_i'(t) = \mathcal{H}_i\left(t, z_i(t), z_i(t-\varrho), \int_0^t g_i(t, s, z_i(s), z_i(s-r)ds\right), \quad \forall\, t \in \hat{J}, \tag{16}$$

$$z_i(t) = \upsilon_i(t), \quad for\ \ t \in [-\varrho, 0],\ i = 1, 2. \tag{17}$$

For an NVID difference equation, this is an initial value problem. Take the following inequality into account:

$$\left|\mathfrak{D}_i'(t) - \mathcal{H}_i\left(t, \mathfrak{D}_i(t), \mathfrak{D}_i(t-\varrho), \int_0^t g_i(t, s, \mathfrak{D}_i(s), \mathfrak{D}_i(s-\varrho)ds\right)\right| \leq \epsilon\Phi(t),\ t \in [-\varrho, 0],\ i = 1, 2,$$

where $\epsilon, \Phi\ and\ \upsilon_i\ i = 1, 2,$ correspond to the values given in Section 2. (Preliminaries). For the problem (16), (17), we have the following theorem as an application of Theorem 3.1.

Theorem 3.2. Under the assumptions (A1) - (A2) and $TL_{\mathcal{H}_i}[2 + L_{g_i}T] < 1,\ i = 1, 2.$ Then the problem (16), (17) has a unique solution $z_i \in C([-\varrho, \mathcal{T}], \mathbb{R}) \cap C^{'}(\hat{J}, \mathbb{R}),\ i = 1, 2,$ and the equation (16) is UHR stable with concerning to the function Φ.

Another special case of the problem (2), (3) we obtain by taking the delay $\beta_i(t) = t^2,\ t \in [t_0, \mathcal{T}],\ i = 1, 2$. Then we have

$$z_i'(t) = \mathcal{H}_i\Bigg(t, z_i(t), z_i(t^2), \int_0^t g_i(t, s, z_i(s), z_i(t^2)ds\Bigg), \quad \forall\, t \in \hat{J}, \tag{18}$$

$$z_i(t) = \upsilon_i(t), \quad for\ t \in [-\varrho, 0],\ i = 1, 2, \tag{19}$$

which is an initial value problem for NVIDE. Consider the following inequality:

$$\left|\mathfrak{D}_i'(t) - \mathcal{H}_i\Bigg(t, \mathfrak{D}_i(t), \mathfrak{D}_i(t^2), \int_0^t g(t, s, \mathfrak{D}_i(s), \mathfrak{D}_i(s^2)ds\Bigg)\right| \leq \epsilon\Phi(t),\ t \in [-\varrho, 0],\ i = 1, 2,$$

where ϵ, Φ *and* υ_i $i = 1, 2$, correspond to the values given in Section 2. (Preliminaries). For the problem (18), (19), we have the following theorem as an application of Theorem 3.1:

Theorem 3.3. Under the assumptions (A1) - (A3) and $TL_{\mathcal{H}}[2 + L_g T] < 1$. Then the problem (18), (19) has a unique solution $z_i \in C([-\varrho, \mathcal{T}], \mathbb{R}) \cap C'([0, T], \mathbb{R}),\ i = 1, 2$ and equation (18) is UHR stable with concerning to the function Φ.

Other Ulam type stability results for equations (16) and (18) can be obtained by using the corresponding results from Section 3.1.

4. CONCLUSION

Using Pachpatte's inequality and Picard's operator method, we examined the stability analysis in the sense of UH and UHR for NDVIDE (2).

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