International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 17, Number 1 (2022), pp. 113-120

https://www.ripublication.com/ijde.htm

# Some Formulae and their Generalized Formulae

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**Abstract**

The discovery of a hypergeometric function has provided an intrinsic stimulation in the world of mathematics. It has also motivated the development of several domains such as complex functions, Riemann surfaces, differential equations, difference equations, arithmetic theory and so forth. The global structure of the Gauss hypergeometric function as a complex function, i.e., the properties of its monodromy and the analytic continuation, has been extensively studied by Riemann. His method is based on complex integrals. Modified Bessel Function and Hypergeometric functions are very important functions in the history of Special functions. These functions has many applications in Engineering and Sciences. Many researchers are involved in this area of classical Mathematics. They have developed so many formulae by using these functions. We have also involved in this area of classical Mathematics. We have also established so many formulae by using these formulae. In this we have developed some formulae and their Generalized formulae. These formulae are associated to Hypergeometric function and Modified Bessel function.

**Key Words :** Bessel Function,Hypergeometric Function, Boolean Algebra.

**2010 MSC NO:** 33B50,33C05, 33C10, 33D50, 33D60,33D67

## 1. Introduction

Yurry A. Brychkov[ Brychkov p.211(4.9.5)] has derived the following formulae

$$\int_0^1 x^0 cos^{-1}x\, I_0(ax)dx = \frac{1}{a}\, Shi(a). (1.1)$$

$$\int_0^1 xcos^{-1}x\, I_0(ax)dx = \frac{\pi}{2a}\, I_0(\frac{a}{2})I_1(\frac{a}{2}). (1.2)$$

$$\int_0^1 x^2 cos^{-1}x\, I_0(ax)dx = \frac{1}{a^3}[a\, cosh\, a - 2sinh\, a + Shi(a)]. (1.3)$$

The first kind of modified Bessel function is defined as

$$I_\alpha(x) = \iota^{-\alpha} J_\alpha(\iota x) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha} \quad (1.4)$$

A generalized hypergeometric function ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\ldots(k+a_p)}{(k+b_1)(k+b_2)\ldots(k+b_q)(k+1)}\, z. \quad (1.5)$$

Where $k+1$ in the denominator is present for historical reasons of notation[ Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \quad (1.6)$$

where the parameters $b_1, b_2, \cdots, b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \le q$, converges for $|z| < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$[Luke p.156(3)].

The function ${}_2F_1(a,b;c;z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, Pochhammer symbol (sometimes called the descending factorial,falling sequential product, or lower factorial) is defined as the polynomial[ Steffensen p.8]

$$(x)_\beta = x(x-1)(x-2)\ldots\ldots(x-\beta+1) = \prod_{k=1}^{\beta}(x-k+1) = \prod_{k=0}^{\beta-1}(x-k) \quad (1.7)$$

The basic operations of Boolean algebra are as follows:

AND (conjunction), denoted $\varsigma \wedge \upsilon$, satisfies $\varsigma \wedge \upsilon = 1$ if $\varsigma = \upsilon = 1$, and $\varsigma \wedge \upsilon = 0$ otherwise.

OR (disjunction), denoted $\varsigma \vee \upsilon$, satisfies $\varsigma \vee \upsilon = 0$ if $\varsigma = \upsilon = 0$, and $\varsigma \vee \upsilon = 1$ otherwise.

NOT (negation), denoted $\neg\varsigma$ , satisfies $\neg\varsigma = 0$ if $\varsigma = 1$ and $\neg\varsigma = 1$ if $\varsigma = 0$.

**2. MAIN FORMULAE OF THE INTEGRATION**

$$\int_0^1 \frac{cos^{-1}x\, I_0(ax)}{x^n}dx = \pi\ 2^{n-2}\ \Gamma(1-n)\ {}_2\tilde{F}_3(\frac{1-n}{2},\frac{2-n}{2};1,\frac{3-n}{2},\frac{3-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.1)

$$\int_0^1 \frac{cos^{-1}x\, I_1(ax)}{x^n}dx = \pi\ a\ 2^{n-4}\ \Gamma(2-n)\ {}_2\tilde{F}_3(\frac{2-n}{2},\frac{3-n}{2};2,\frac{4-n}{2},\frac{4-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.2)

$$\int_0^1 \frac{cos^{-1}x\, I_2(ax)}{x^n}dx = \pi\ a^2\ 2^{n-6}\ \Gamma(3-n)\ {}_2\tilde{F}_3(\frac{3-n}{2},\frac{4-n}{2};3,\frac{5-n}{2},\frac{5-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.3)

$$\int_0^1 \frac{cos^{-1}x\, I_3(ax)}{x^n}dx = \pi\ a^3\ 2^{n-8}\ \Gamma(4-n)\ {}_2\tilde{F}_3(\frac{4-n}{2},\frac{5-n}{2};4,\frac{6-n}{2},\frac{6-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.4)

$$\int_0^1 \frac{cos^{-1}x\, I_{10}(ax)}{x^n}dx = \pi\ a^{10}\ 2^{n-22}\ \Gamma(11-n)\ {}_2\tilde{F}_3(\frac{11-n}{2},\frac{12-n}{2};11,\frac{13-n}{2},\frac{13-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.5)

$$\int_0^1 \frac{cos^{-1}x\, I_{19}(ax)}{x^n}dx = \pi\ a^{19}\ 2^{n-40}\ \Gamma(20-n)\ {}_2\tilde{F}_3(\frac{20-n}{2},\frac{21-n}{2};20,\frac{22-n}{2},\frac{22-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.6)

$$\int_0^1 \frac{cos^{-1}x\, I_{29}(ax)}{x^n}dx = \pi\ a^{29}\ 2^{n-60}\ \Gamma(30-n)\ {}_2\tilde{F}_3(\frac{30-n}{2},\frac{31-n}{2};30,\frac{32-n}{2},\frac{32-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.7)

$$\int_0^1 \frac{cos^{-1}x\, I_{100}(ax)}{x^n}dx = \pi\, a^{100}\, 2^{n-202}\, \Gamma(101-n)\, {}_2\tilde{F}_3(\frac{101-n}{2},\frac{102-n}{2};101,\frac{103-n}{2},\frac{103-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.8)

$$\int_0^1 \frac{cos^{-1}x\, I_{151}(ax)}{x^n}dx = \pi\, a^{151}\, 2^{n-304}\, \Gamma(152-n)\, {}_2\tilde{F}_3(\frac{152-n}{2},\frac{153-n}{2};152,\frac{154-n}{2},\frac{154-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.9)

$$\int_0^1 \frac{cos^{-1}x\, I_{251}(ax)}{x^n}dx = \pi\, a^{251}\, 2^{n-504}\, \Gamma(252-n)\, {}_2\tilde{F}_3(\frac{252-n}{2},\frac{253-n}{2};252,\frac{254-n}{2},\frac{254-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.10)

$$\int_0^1 \frac{cos^{-1}x\, I_{1251}(ax)}{x^n}dx =$$
$$\pi\, a^{1251}\, 2^{n-2504}\, \Gamma(1252-n)\, {}_2\tilde{F}_3(\frac{1252-n}{2},\frac{1253-n}{2};252,\frac{1254-n}{2},\frac{1254-n}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0)$. (2.11)

$$\int_0^1 x^n\, cos^{-1}x\, I_1(ax)dx = \pi\, a\, 2^{-n-4}\, \Gamma(n+2)\, {}_2\tilde{F}_3(\frac{n+2}{2},\frac{n+3}{2};2,\frac{n+4}{2},\frac{n+4}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1$. (2.12)

$$\int_0^1 x^n\, cos^{-1}x\, I_2(ax)dx = \pi\, a^2\, 2^{-n-6}\, \Gamma(n+3)\, {}_2\tilde{F}_3(\frac{n+3}{2},\frac{n+4}{2};3,\frac{n+5}{2},\frac{n+5}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1$. (2.13)

$$\int_0^1 x^n\, cos^{-1}x\, I_3(ax)dx = \pi\, a^3\, 2^{-n-8}\, \Gamma(n+4)\, {}_2\tilde{F}_3(\frac{n+4}{2},\frac{n+5}{2};4,\frac{n+6}{2},\frac{n+6}{2};\frac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1$. (2.14)

$$\int_0^1 x^n \cos^{-1}x\, I_4(ax)dx = \pi\, a^4\, 2^{-n-10}\, \Gamma(n+5)\ {}_2\tilde{F}_3(\tfrac{n+5}{2},\tfrac{n+6}{2};5,\tfrac{n+7}{2},\tfrac{n+7}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1.$ (2.15)

$$\int_0^1 x^n \cos^{-1}x\, I_5(ax)dx = \pi\, a^4\, 2^{-n-10}\, \Gamma(n+5)\ {}_2\tilde{F}_3(\tfrac{n+5}{2},\tfrac{n+6}{2};6,\tfrac{n+7}{2},\tfrac{n+7}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1.$ (2.16)

$$\int_0^1 x^n \cos^{-1}x\, I_6(ax)dx = \pi\, a^6\, 2^{-n-14}\, \Gamma(n+7)\ {}_2\tilde{F}_3(\tfrac{n+7}{2},\tfrac{n+8}{2};7,\tfrac{n+9}{2},\tfrac{n+9}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1.$ (2.17)

$$\int_0^1 x^n \cos^{-1}x\, I_7(ax)dx = \pi\, a^7\, 2^{-n-16}\, \Gamma(n+8)\ {}_2\tilde{F}_3(\tfrac{n+8}{2},\tfrac{n+9}{2};8,\tfrac{n+10}{2},\tfrac{n+10}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1.$ (2.18)

$$\int_0^1 x^n \cos^{-1}x\, I_{17}(ax)dx = \pi\, a^{17}\, 2^{-n-36}\, \Gamma(n+18)\ {}_2\tilde{F}_3(\tfrac{n+18}{2},\tfrac{n+19}{2};18,\tfrac{n+20}{2},\tfrac{n+20}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1.$ (2.19)

$$\int_0^1 x^n \cos^{-1}x\, I_{107}(ax)dx =$$
$$\pi\, a^{107}\, 2^{-n-216}\, \Gamma(n+108)\ {}_2\tilde{F}_3(\tfrac{n+108}{2},\tfrac{n+109}{2};108,\tfrac{n+110}{2},\tfrac{n+110}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -1.$ (2.20)

$$\int_0^1 x^n \cos^{-1}x\, I_0(\tfrac{x}{a})dx = \pi\, 2^{-n-2}\, \Gamma(n+1)\ {}_2\tilde{F}_3(\tfrac{n+1}{2},\tfrac{n+2}{2};1,\tfrac{n+3}{2},\tfrac{n+3}{2};\tfrac{1}{4a^2})$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) > -1.$ (2.21)

$$\int_0^1 x^n \cos^{-1}x\, I_1\left(\frac{x}{a}\right)dx = \frac{\pi\, 2^{-n-4}\, \Gamma(n+2)\, {}_2\tilde{F}_3\left(\frac{n+2}{2},\frac{n+3}{2};2,\frac{n+4}{2},\frac{n+4}{2};\frac{1}{4a^2}\right)}{a}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) > -1.$ (2.22)

$$\int_0^1 x^n \cos^{-1}x\, I_2\left(\frac{x}{a}\right)dx = \frac{\pi\, 2^{-n-6}\, \Gamma(n+3)\, {}_2\tilde{F}_3\left(\frac{n+3}{2},\frac{n+4}{2};3,\frac{n+5}{2},\frac{n+5}{2};\frac{1}{4a^2}\right)}{a^2}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) > -1.$ (2.23)

$$\int_0^1 x^n \cos^{-1}x\, I_3\left(\frac{x}{a}\right)dx = \frac{\pi\, 2^{-n-8}\, \Gamma(n+4)\, {}_2\tilde{F}_3\left(\frac{n+4}{2},\frac{n+5}{2};4,\frac{n+6}{2},\frac{n+6}{2};\frac{1}{4a^2}\right)}{a^3}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) > -1.$ (2.24)

$$\int_0^1 x^n \cos^{-1}x\, I_8\left(\frac{x}{a}\right)dx = \frac{\pi\, 2^{-n-18}\, \Gamma(n+9)\, {}_2\tilde{F}_3\left(\frac{n+9}{2},\frac{n+10}{2};9,\frac{n+11}{2},\frac{n+11}{2};\frac{1}{4a^2}\right)}{a^8}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) > -1.$ (2.25)

$$\int_0^1 \frac{\cos^{-1}x\, I_2\left(\frac{x}{a}\right)}{x^n}dx = \frac{\pi\, 2^{n-6}\, \Gamma(3-n)\, {}_2\tilde{F}_3\left(\frac{3-n}{2},\frac{4-n}{2};3,\frac{5-n}{2},\frac{5-n}{2};\frac{1}{4a^2}\right)}{a^2}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) < 1.$ (2.26)

$$\int_0^1 \frac{\cos^{-1}x\, I_5\left(\frac{x}{a}\right)}{x^n}dx = \frac{\pi\, 2^{n-12}\, \Gamma(6-n)\, {}_2\tilde{F}_3\left(\frac{6-n}{2},\frac{7-n}{2};6,\frac{8-n}{2},\frac{8-n}{2};\frac{1}{4a^2}\right)}{a^5}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) < 1.$ (2.27)

$$\int_0^1 \frac{\cos^{-1}x\, I_{10}\left(\frac{x}{a}\right)}{x^n}dx = \frac{\pi\, 2^{n-22}\, \Gamma(11-n)\, {}_2\tilde{F}_3\left(\frac{11-n}{2},\frac{12-n}{2};11,\frac{13-n}{2},\frac{13-n}{2};\frac{1}{4a^2}\right)}{a^{10}}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) < 1.$ (2.28)

$$\int_0^1 \frac{\cos^{-1}x\, I_{15}(\frac{x}{a})}{x^n}\,dx = \frac{\pi\, 2^{n-32}\, \Gamma(16-n)\, {}_2\tilde{F}_3(\frac{16-n}{2},\frac{17-n}{2};16,\frac{18-n}{2},\frac{18-n}{2};\frac{1}{4a^2})}{a^{15}}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) < 1.$ (2.29)

## 3. GENERALIZED FORMULA OF THE FORMULAE

$$\int_0^1 \frac{\cos^{-1}x\, I_n(ax)}{x^n}\,dx = \pi\, 2^{-n-2}\, a^n\, {}_2\tilde{F}_3(\tfrac{1}{2},1;n+1,\tfrac{3}{2},\tfrac{3}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge Re(n) < 1 \wedge (Im(a) > 0 \vee Re(a) > 0).$ (3.1)

$$\int_0^1 x^n \cos^{-1}x\, I_n(ax)dx = \sqrt{\pi}\, a^n\, 2^{-n-2}\, \Gamma(\tfrac{1}{2}+n)\, {}_1\tilde{F}_2(n+\tfrac{1}{2};n+\tfrac{3}{2},n+\tfrac{3}{2};\tfrac{a^2}{4})$$

for $Re(a) \geq 0 \wedge (Im(a) > 0 \vee Re(a) > 0) \wedge Re(n) > -\frac{1}{2}.$ (3.2)

$$\int_0^1 x^n \cos^{-1}x\, I_n(\tfrac{x}{a})dx = \sqrt{\pi}\, 2^{-n-2}\, \Gamma(n+\tfrac{1}{2})\, {}_1\tilde{F}_2(n+\tfrac{1}{2};n+\tfrac{3}{2},n+\tfrac{3}{2};\tfrac{1}{4a^2})$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) > -\frac{1}{2}.$ (3.3)

$$\int_0^1 \frac{\cos^{-1}x\, I_n(\frac{x}{a})}{x^n}\,dx = \frac{\pi\, 2^{-n-2}\, {}_2\tilde{F}_3(\frac{1}{2},1;n+1,\frac{3}{2},\frac{3}{2};\frac{1}{4a^2})}{a^n}$$

for $(Im(a) < 0 \vee (Re(a) > 0) \wedge Re(a) \geq 0 \wedge Re(n) < 1.$ (3.4)

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