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Numerical Solution For Non-Instantaneous Impulsive Mittag Leffler with Riemann-Liouville Fractional Stochastic Delay Integro-Differential Equation

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ABSTRACT

Over the last three decades, fractional differential equations have become increasingly important. Later, in applied mathematics, fractional differential equations' theory became a popular subject. In various fields, such as control, electrochemistry, polymer rheology, physics, chemistry, electromagnetic, viscoelasticity, porous media, and other domains, many mathematical models can be built using it. The research work on the monographs is referred to the theory and application of fractional differential equations. Furthermore, when studying fractional differential equations, the stochastic perturbation is inescapable, and it is critical to take the stochastic effect. In recent years, it has become clear that stochastic integrodifferential systems can better simulate real phenomena by developing computational approaches. Recently, for solving different issues, researchers have considered the spline approach as an ideal numerical method. Consequently, in this research, the Fractional Stochastic Integro-Differential (FSI-D) equations are numerically solved by proposing a Linear Cardinal B-spline (LCB-S) function. Naturally, all physical systems which evolve with respect to time are suffered from small abrupt changes in the form of impulses.Consequently, the fractional stochastic differential equation suffered by non-instantaneous impulses is carried by Riemann-Liouville Derivative (RLD),which is driven by fractional Brownian motion (fBm) and poison jumps. Following these ideas, the research work motivates to present fractional SDEs with time delay. Accordingly, the

Chelyshkov wavelet along with the Galerkin approach has been implemented to solve Stochastic Fractional Delay Differential Equations (SFDDEs).In order to achieve approximate controllability solutions for the uniqueness and existence of solutions is the fundamental prerequisite in the study of the relevant conditions, the fractional and stochastic calculus governed for solving nonlinear dynamic systems due to the lack of generic approaches. Consequently, with nonlocal conditions of the order $1<\alpha<2$ for Riemann-Liouville fractional stochastic evolution equations' approximate controllability is considered in this work. The Lebesgue dominated convergence theorem is used to obtain approximate controllability results. In this article, the test problems are conducted using Matlab software. The new method's efficiency and potential are provided by the numerical experiments. In the end, descriptive examples are included to establish the validity and effectiveness of the presented approach. The proposed method is more accurate and efficient, which is portrayed by the obtained numerical results.

Keywords: Fractional Stochastic Integro-Differential Equation, Linear Cardinal B-spline, Brownian motion, poison jumps, Riemann-Liouville, Fractional Delay Differential Equation.

1. INTRODUCTION

Derivatives and fractional integrals have a wide range of applications in the domains of nonlinear seismic oscillation, fluid-dynamic traffic, control theory, continuum and statistical mechanics, and signal processing [1]. Accordingly, to estimate the solutions of fractional differential and integral equations, many scholars have been interested in using numerical approaches [2]. Many challenges in recent years have used random functional or stochastic equations. Stochastic integral equations, for example, appear in the theory of automatic systems resulting in delay-differential equations, in the study of biological population development, in the stochastic formulation of issues in reactor dynamics, and many other problems in engineering, biology, and physics [3]. In many circumstances, accurate solutions to stochastic functional equations are impossible to find. Consequently, it's critical to use numerical approaches to determine their approximate solutions [4].

Stochastic Differential Equations

Stochastic Differential Equations (SDEs) are crucial applications in many development fields of engineering and science [5].SIEs have received more attention in recent years as a result of increased needs in finance and engineering, biological, as well as chemical, and physical sciences. However, solving the SIEs analytically is challenging, hence to overcome this obstacle, numerical solutions are required [6]. The Taylor expansion approach and the spectral collocation method are important. Bernstein polynomials, Chebychev wavelets, Bernoulli polynomials, and block pulse functions are some of the orthogonal and polynomials basic functions that have been utilised to approximate the solution for SIEs [7]. Furthermore, the numerical solution of SFIDE is discussed by a few studies in the literature. In order to solve SFIDE, based on shifted Legendre polynomials, Taheri [9] proposed a spectral collocation approach. Based on Bernstein's polynomials, Mirzaee [8] devised an effective approach for solving SFIDE. Furthermore, to solve SFIDE, based on radial basis functions, present a meshless discrete collocation approach. Because of their low cost, ease of implementation, and minimal computing effort, spline basis functions draw attention to these functions [9]. Recently, various spline basis functions have been developed by authors to solve differential equations, including modified cubic B splines and cubic trigonometric B-splines. For instance, this method has been applied for the approximate solution of the Fredholm and Volterra integral equation, a system of fractional differential equations, differential equations, integro-differential equations,etc. [10]

Impulse Fractional Differential Equations

Social macrosystems and biological are prone to impulsive disturbances, when these processes involve hereditary phenomena, some modelling is done using impulsive fractional differential equations [11].Impulsive fractional differential equations (IFDEs) are an effective mathematical tool to model in both the socialand physical sciences. Especially in the domain of IFDEs with fixed moments, there has been a lot of progress in impulsive theory. Although, all physical systems which evolve over time are suffered from small abrupt changes in the form of impulses [12]. These impulses can be specified into two cases: (i) Instantaneous Impulsive Differential Equations (IIDEs). (ii) Non-instantaneous Impulsive Differential Equations (NIIDEs). IIDEs: i.e., in the system, impulse occurs for a short period which is negligible compared with the overall period is instantaneous impulse [13]. The second type NIIDEs i.e., an impulsive disturbance which starts at a time and remains active for a finite period is non-instantaneous impulsive [14]. The statement of the impulsive condition and the lower limit of the RL fractional derivative is presented in different ways. Many researchers express the existence results by the familiar definitions of fractional derivatives defined by Caputo and RL sense [15].

Correspondingly, the research work introduced a stochastic fractional integro-differential equation using non-instantaneous impulsive with Riemann-Liouville (RL) derivative function. The remainder of the article is organized as follows. Section 2 explains the literature review of the proposed work, section 3 states the problem definition and motivation. Section 4 defines the proposed methodology and section 5 implements the test problems with results and discussion. Finally, section 5 ends with the conclusion.

2. LITERATURE SURVEY

The Rosenblatt process is researched to drive the non-instantaneous impulsive conformable fractional stochastic delay integro-differential system introduced by

Hamdy M. Ahmed *et al* [16]. For the problem under consideration, sufficient requirements for null controllability and approximation controllability are stated. Finally, an example is given to explain the results achieved. The Rosenblatt process governs the optimal mild solutions of the time-fractional stochastic Navier-Stokes equation regulated by established by K. Anukiruthika *et al* [17]. A stochastic term is added to the fundamental deterministic nonlinear time-fractional partial differential system. Fixed point theorem, stochastic analysis, fractional calculus of condensing maps, and the establishment of an adequate measure of non-compactness are used to explore solvability. In addition, recognized an optimal mild solution.

Rajesh Dhayal *et al* [18] study fractional stochastic differential equations (FSDE) with non-instantaneous impulses and Poisson jumps driven by the Rosenblatt process. The existence of solutions for the suggested stochastic system is determined using fractional calculus, fixed-point theorem, stochastic analysis, and the sectorial operator. Also, discuss the suggested control system's controllability. The key conclusions are well supported by an example. The Rosenblatt process drove the null controllability and approximate controllability for fractional neutral stochastic partial differential equations with delay were studied by Hamdy M. Ahmed *et al* [19]. The Rosenblatt process has driven the Hilfer fractional neutral stochastic partial differential equations to discuss the requirements required for null and approximate controllability. Finally, to verify the obtained results, two examples are provided.

The fractional Brownian motion (fBm) driven by the presence of mild solutions of non-instantaneous impulsive Hilfer fractional stochastic differential equations (NIHFSDEs) was investigated by S. Saravanakumar *et al* [20]. With the use of fractional calculus, semigroup theory, stochastic theory, and fixed point theorem, sufficient conditions for a class of NIHFSDEs of order $0<\beta<1$ and type $0\leq\alpha\leq1$ driven by fBmare derived. The existence of the solution is demonstrated using the Monch fixed point theorem (FPT). In order to support the theoretical result, a numerical example is also provided.

Xinjie Dai *et al* [21] consider the initial value problem of general nonlinear stochastic fractional integro-differential equations with weakly singular kernels. The effort is devoted to establishing some fine estimates to include all the cases of Abel-type singular kernels. Firstly, the continuous dependence, existence, and uniqueness of the initial value of the true solution under the local Lipschitz condition and linear growth condition are derived in detail. Secondly, the Euler-Maruyama method is introduced for solving numerically the equation, and then its strong convergence is proven under the same conditions as the well-posedness. Additionally, obtain the accurate convergence rate of this method under the linear growth condition and global Lipschitz condition. Finally, several numerical tests are reported for verification of the theoretical findings.

Lingyun He *et al* [22] proposed a step-by-step collocation method for solving a fractional-order system of nonlinear stochastic differential equations with a constant delay based on shifted Legendre polynomials. The fractional derivative is in the Caputo sense, and the problem is considered with appropriate initial conditions. In each phase, the problem is turned into a non-delay fractional-order system of nonlinear stochastic differential equations, and then the system is solved using a shifted Legendre collocation scheme. In each step, obtain a nonlinear system of equations by collocating the acquired residual at the shifted Legendre points. The proposed method's rate of convergence and convergence analysis are explored. Finally, to demonstrate the technique's accuracy in the presence of various noise measures, three test examples are supplied.

To numerically solve the fractional stochastic integro-differential (FSI-D) equations, anoperational matrix of the linear cardinal B-spline (LCB-S) functions are developed by Somayeh Abdi-Mazraeh *et al* [23]. The integer integral operational matrix, fractional Riemann-Liouville integral operational matrix, and the stochastic integral operational matrix are some of the LCB-S functions. By applying a suitable method, the FSI-D equation is converted into a linear system of algebraic equations that can be solved quickly is the scheme's major feature. An upper bound of error is determined and also the proposed method's error estimate and convergence analysis are examined. To demonstrate the new method's potential and efficiency, numerical experiments are offered.

Averaging concepts for neutral stochastic partial functional differential equations (NSPFDEs) with delayed impulses were investigated by Jiankang Liu *et al* [24]. Derive an average principle for a class of NSPFDEs with delayed impulses using inequality techniques, the semigroup approach, and some technical transformations. By decreasing complexity effectively, the acquired results enable one can concentrate on the averaged autonomous system without impulses in place of the original system. Finally, an example is developed to exemplify the theoretical findings.

An accurate and computationally efficient technique is provided by B.P.Moghaddam *et al* [25] for the approximation solution of a large class of fractional stochastic differential equations driven by Brownian motion with constant delay. For approximating the fractional-order integral, a piecewise integro quadratic spline interpolation technique is used. Statistical indicators of exact solutions are used to assess the computational scheme's performance. The convergence of the computation is also examined. When compared to the M-scheme, three families of stochastic excitation models demonstrate the novel approach's accuracy.

3. RESEARCH PROBLEM DEFINITION AND MOTIVATION

Deterministic integral equations have long been researched because they are essential for modelling engineering and scientific phenomena. These phenomena are frequently dependent on a noise source, such as Gaussian white noise, which is typically overlooked due to inadequate computing capabilities. By increasing computational power, some random factors are inserted into deterministic systems or phenomena and are created stochastic or random processes. Many scholars are interested in studying the behaviour of stochastic equations. This trend can be attributed to several applications in biology, physics, engineering, and economics. Many physical phenomena that appear randomly can be modelled mathematically by stochastic processes such as stochastic differential equations, stochastic integro-differential equations, stochastic integro-differential equations, and stochastic integral equations of fractional order.

For mathematical modelling, fractional stochastic differential equations (FSDEs) have become increasingly applied. In order to solve crucial real-world situations where randomness cannot be ignored SDEs are successfully employed. Drew The attention of not only mathematicians, but also engineers, economists, and medics are attracted by the characteristics and elegant formulation. The research work encourages to propose the fractional SDEs with time delay based on these notions. In numerous scientific domains, there are several practical applications of systems of differential equations with delay. Some systems in nature are related to time delay, which means that the system's future conditions are dependent on its past. Due to the lack of sophisticated computer devices, these systems are frequently reliant on "noise" perturbations that were previously neglected.Deterministic delay differential systems were used to model these phenomena in the future.

4. PROPOSED RESEARCH METHODOLOGY

A Stochastic Fractional Integro-Differential Equation (SFIDE) is a generalisation of the fractional Fokker-Plank equation, which explains the random walk of a particle and has a non-integer order of derivative. For handling different issues, researchers have recently regarded the spline collocation method as an ideal numerical solution. This research work studies the numerical solution for solving fractional integral equations using non-instantaneous impulsive Mittag-Leffler type function with the Riemann-Liouville fractional stochastic delay differential equations driven by fractional Brownian motion and poison jumps. Figure 1 illustrates the flow diagram of the research work.

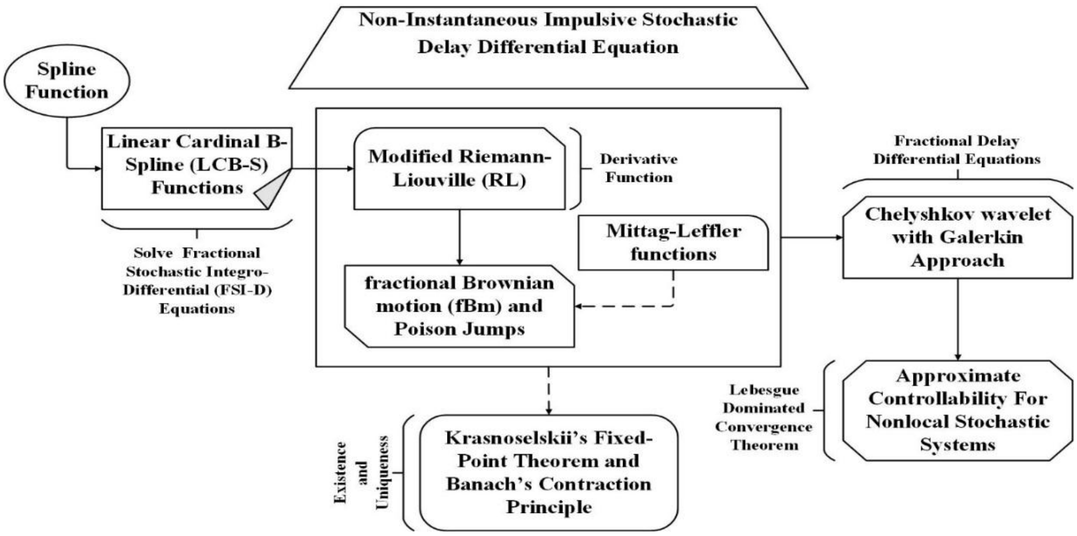


Figure 1: Flow Diagram of the Proposed Work

The article proposed a class of fractional stochastic differential equations using Spline functions. Researchers have recently regarded the spline collocation method as an ideal numerical solution for solving many problems. Subsequently, the research work proposed a Linear Cardinal B-spline (LCB-S) function to numerically solve FSI-D equations. Fractional stochastic differential equation suffered by non-instantaneous impulses with Riemann-Liouville Derivative (RLD) driven by fractional Brownian motion (fBm) and poison jumps. To obtain their fractional integration operational matrix and delay operational matrix in the Riemann-Liouville sense, the Chelyshkov wavelet basis and attributes are used. Sufficient conditions for approximate controllability for the considered problem are also established. For Riemann-Liouville fractional stochastic evolution equations with nonlocal circumstances of the order $1<\alpha<2$, consider the approximate controllability.

a. Solving Fractional Stochastic Integro Differential Equation

In order to numerically solve the FSI-D equations, the research work introduces LCB-S functions. The LCB-S functions have interpolation features, which is worth noting. Accordingly, for other fundamental functions, there is a significant benefit of the LCB-S functions, and for integration, the coefficients of each known function may be simply determined without the need. Furthermore, these functions have cardinality features, which reduces the cost of calculation.

In this article, the FSI-D problem is solved using a new approach based on the LCB-S basis functions.

$$ {}_{0}^{C}D_{\tau}^{\nu}Y(\tau)=U(\tau)+\int_0^{\tau}\lambda_1(\tau,s)Y(s)ds+\int_0^{\tau}\lambda_2(\tau,s)Y(s)dB(s) \quad (1) $$

Wherein, the Caputo fractional derivative of order $0<\nu<1$ is determined as ${}_{0}^{C}D_{\tau}^{\nu}$. Also, $B(\tau)$ is the Brownian motion process, subsequently, stochastic processes like $U(t)\in C^2([0,1]),\lambda_i(\tau,s)\in C^2([0,1]\times[0,1])$ for $i=1,2$ are defined on the probability space (Ω,F,P). There $Y(\tau)$ is an unknown function that must be determined in equation (1).

Let $N_1(\tau)=X_{[0,1)}(\tau)$ denote the first-order CB-S function, where the interval $[0,1)$, $X_{[0,1)}(\tau)$ is a characteristic function. In addition, the CB-S order function is r defined as

$$ N_r(\tau)=(N_{r-1}*N_1)(\tau)=\int_{-\infty}^{\infty}N_{r-1(\tau-s)}N_1(s)ds=\int_0^1 N_{r-1}(\tau-s)ds \quad (2) $$

recursively, and $\sup[N_r(\tau)]=[0,r]$. The LCB-S function of 2nd order is expressed explicitly as

$$ N_2(\tau)=\begin{cases}\tau, & \tau\in[0,1]\\ 2-\tau, & \tau\in[1,2]\\ 0, & elsewhere\end{cases} \quad (3) $$

Let $N_{r,q}(\tau)=N_2(2^r\tau-q)$, r , $q\in Z$. It is simple to demonstrate that

$$\sup[N_{r,q}(\tau)]=[2^{-r}q,2^{-r}(2+q)],\quad r,q\in Z$$

Define $S_r=\{q:[2^{-r}q,2^{-r}(2+q)]\cap(0,1)\neq\phi\},\quad r\in Z$. It is possible to derive that $\max\{S_r\}=2^r-1$ and $\min\{S_r\}=-1$, $r\in Z$. Because it is necessary that the assistance of $N_{r,q}(\tau)$ be limited to $[0,1]$,

$$\psi_q(\tau)=\psi_q^r(\tau)=N_{r,q}(\tau)X_{[1,0)}(\tau),\qquad r\in Z,\ \ q\in S_r \tag{4}$$

Consequently, it is possible to write

$$\psi_q(\tau)=\begin{cases}\begin{cases}1-2^r\tau,\ \tau\in[0,2^{-r}) & \\ 0, \quad Otherwise\end{cases} & for\ q=-1\\ \begin{cases}2^r\tau-q,\ \ \tau\in[2^{-r}q,2^{-r}(q+1)]\\ 2-(2^r\tau-q),\ \ \tau\in[2^{-r}(q+1),2^{-r}(q+2)], \\ 0,\quad Otherwise\end{cases} & for\ q=0,2^r-2\\ \begin{cases}2^r\tau-2^r+1,\quad \tau\in[1-2^{-r},1)\\ 0,\qquad Otherwise\end{cases} & for\ q=2^r-1\end{cases} \tag{5}$$

Suppose $$\Psi(\tau)=[\psi_{-1}(\tau),\psi_0(\tau),\ldots,\psi_0(\tau),\ldots,\psi_{2^r-1}(\tau)]^T \tag{6}$$

Where, $r\in Z$ is a fixed number. The LCB-S functions approximated a function $U(\tau)\in L^2[0,1]$:

$$U(\tau)\approx\sum_{q=-1}^{2^r-1}u_q\psi_q(\tau)=V^T\psi(\tau) \tag{7}$$

Where, $V=[u-1,u_0,\ldots,u_{2^r-1}]^T$, $u_q=U(2^{-r}(q+1)),q=-1,\ldots,2^r-1$. Let e_q be the $(q+2)th$ column of the order 2^r-1 unit matrix, therefore it's simple to see that

$$\psi(\xi_q)=e_q,\ \xi_q=2^{-r}(q+1),\ q=-1,\ldots,2^r-1 \tag{8}$$

Also, a function $\lambda(\tau,s)\in L^2([0,1]\times[0,1])$ may be approximated by LCB-S functions as

$$\lambda(\tau,s)\approx\sum_{q=-1}^{2^r-1}\sum_{p=-1}^{2^r-1}u_{q,p}\psi_q(\tau)\psi_p(s)=\Psi(\tau)^T\Lambda\Psi(S) \tag{9}$$

Where, Λ is the $(2^r+1)\times(2^r+1)$ coefficient matrix with the following entries $\Lambda_{p,q}$

$$\Lambda_{q,p}=U(2^{-r}(q+1),2^{-r}(p+1)),\quad q,p=-1,\ldots,2^r-1 \tag{10}$$

Suppose $P = 2^{-r}\begin{bmatrix} 1/6 & 1/12 & & & \\ 1/12 & 1/3 & 1/12 & & \\ \ddots & \ddots & \ddots & & \\ & & 1/12 & 1/3 & 1/12 \\ & & & 1/12 & 1/6 \end{bmatrix}$, $E = \begin{bmatrix} -1/2 & -1/2 & & & \\ 1/2 & 0 & -1/2 & & \\ \ddots & \ddots & \ddots & & \\ & & 1/2 & 0 & -1/2 \\ & & & 1/2 & 1/2 \end{bmatrix}$ (11)

The differentiation of $\Psi(\tau)$ in (6) is calculated as

$$\Psi'(\tau) = D\Psi(\tau),\ D = E\left(P^{-1}\right) \tag{12}$$

For the LB-S functions on $[0,1]$, D recalls $\left(2^r+1\right)\times\left(2^r+1\right)$ operational derivative matrix.

i. Operational matrix of the Stochastic Integral

The Operational matrix of the stochastic integral of equation is given by

$$I^s\psi_q(\tau) = \int_0^\tau \psi_q(s)dB(s) \tag{13}$$

Two instances were examined to calculate the above integral, which is detailed in the next part.

Case 1: $q = -1$

In this case, if $\tau \in \left[0, 2^r\right]$ then Eq. (13) is calculated as

$$I^s\psi_{-1}(\tau) = \int_0^\tau \left(1 - 2^r s\right)dB(s) = B(t) - 2^r \int_0^\tau s dB(s) \tag{14}$$

Using integration by parts, the given integral can be solved. Accordingly, it can be written

$$I^s\psi_{-1}(\tau) = B(\tau) - 2^r\left(\tau B(\tau) - \int_0^\tau B(s)ds\right) \tag{15}$$

Simpson’s rule can be used to approximate the integral in equation (15). Consequently, it can be determined that

$$\begin{aligned} I^s\psi_{-1}(\tau) &= \int_0^{2^{-r}} \left(1 - 2^r s\right)dB(s) = B\left(2^{-r}\right) - 2^r \int_0^{2^{-r}} s dB(s) \\ &= B\left(2^{-r}\right) - 2^r\left(sB(s)\Big|_0^{2^{-r}} - \int_0^{2^{-r}} B(s)ds \right) \end{aligned} \tag{16}$$

When Simpson’s rule is applied to the above integral, one can deduce that

$$I^s\psi - 1(\tau) = \frac{4}{3}B\left(2^{-r-1}\right) + \frac{1}{3}B\left(2^{-r}\right) \tag{17}$$

Case 2: $q = 2^r - 1$

In this case, if $\tau \in \left[0, 2^{-r}q\right]$ then it can be derived $I^s\psi_{2^r-1}(\tau) = 0$. If $\tau \in \left[1 - 2^{-r}, 1\right]$ then

$$I^s\psi_{2^r-1}(\tau)=\int_{1-2^{-r}}^{\tau}\left(2^r s-2^r+1\right)dB(s)$$

$$=\frac{2^r\tau-q}{3}\left[2B(\tau)-B\left(q2^{-r}\right)-4B\left(\frac{\tau+q2^{-r}}{2}\right)\right] \tag{18}$$

If $\tau>1$ then

$$I^s\psi 2^r-1(\tau)=\int_{1-2^{-r}}^{1}\left(2^r s-2^r+1\right)dB(s)$$

$$=\frac{1}{3}\left[2B\left(2^{-r}(q+1)\right)-B\left(2^{-r}q\right)-4B\left(2^{-r-1}(2q+1)\right)\right] \tag{19}$$

Using Eqs. (14)–(19), one can write

$$I^s\psi_q(\tau)=\begin{cases}\begin{cases}B(\tau)\left(1-\frac{2^{r+1}\tau}{3}\right)+\frac{2^{r+2}\tau}{3}B\left(\frac{\tau}{2}\right), & \tau\in\left[0,\frac{1}{2^r}\right]\\ \frac{4}{3}B\left(2^{-r-1}\right)+\frac{1}{3}B\left(2^{-r}\right), & \tau\geq\frac{1}{2^r} \quad \textit{for } q=-1\\ 0, & \textit{Otherwise}\end{cases}\\ \begin{cases}\frac{2^r\tau-q}{3}\left[2B(\tau)-B\left(\frac{q}{2^r}\right)-4B\left(\frac{2^r\tau+q}{2^{r+1}}\right)\right], & \tau\in\left[\frac{q}{2^r},\frac{q+1}{2^r}\right]\\ \frac{5+2q-2^{r+1}\tau}{3}B(\tau)-\frac{1}{3}B\left(q2^{-r}\right)+\frac{-2+2^r\tau-q}{3}B\left((q+1)2^{-r}\right)- & \\ \quad \frac{4}{3}B\left((2q+1)2^{-r-1}\right)+\frac{2^{r+2}}{3}\left[\tau-2^{-r}(q+1)\right] & \\ \quad \times B\left(\frac{\tau+2^{-r}(q+1)}{2}\right), & \tau\in\left[\frac{q+1}{2^r},\frac{q+2}{2^r}\right] \quad \textit{for } q=0,2^r-2\\ \frac{-1}{3}B\left(q2^{-r}\right)+\frac{1}{3}B\left((q+2)2^{-r}\right)-\frac{4}{3}B\left((2q+1)2^{-r-1}\right) & \\ \quad +\frac{4}{3}B\left((2q+3)2^{-r-1}\right), & \tau\geq\frac{q+2}{2^r}\\ 0 & \textit{Otherwise}\end{cases}\\ \begin{cases}\frac{2^r\tau-2^r+1}{3}\left[2B(\tau)-B\left(1-2^{-r}\right)-4B\left(\frac{\tau+1-2^{-r}}{2}\right)\right], & \tau\in\left[1-\frac{1}{2^r},1\right]\\ \frac{1}{3}\left[2B(1)-B\left(1-2^{-r}\right)-4B\left(1-2^{-r-1}\right)\right], & \tau\geq 1 \quad \textit{for } q=0,2^r-2\\ 0 & \textit{Otherwise}\end{cases}\end{cases}$$

(20)

Applying Eqs. (6)–(7) on (20), it can be achieved as

$$I^s\Psi(\tau)=I_s\Psi(\tau) \tag{21}$$

Wherein

$$(I_s)_{p,q}=I^s\psi_q\xi_p\,,\ \xi_p=\frac{p+1}{2^r}\,,\ p,q=-1,\ldots,2^r-1. \tag{22}$$

It can be shown that I_s is a $(2^r+1)\times(2^r+1)$ matrix in the following way:

$$I_s=\begin{bmatrix}0 & \mu & \mu & \mu & \cdots & \mu\\ & \kappa_0 & \eta_0 & \eta_0 & \cdots & \eta_0\\ & & \kappa_1 & \eta_1 & \cdots & \eta_1\\ & & & \ddots & \ddots & \vdots\\ & & & & \kappa_{2^r-2} & \eta_{2^r-2}\\ & & & & & \gamma\end{bmatrix} \tag{23}$$

Where, $\mu=\frac{1}{3}\left[4B\left(\frac{1}{2^{r+1}}\right)+B\left(\frac{1}{2^r}\right)\right]$,

$\kappa_q=\frac{1}{3}\left[2B\left(\frac{q+1}{2^r}\right)-B\left(\frac{q}{2^r}\right)-4B\frac{2q+1}{2^{r+1}}\right],\ q=0,\ldots,2^{r-2}$,

$\eta_q=\frac{1}{3}\left[B\left(\frac{q+2}{2^r}\right)-B\left(\frac{q}{2^r}\right)-4B\left(\frac{2q+1}{2^{r+1}}\right)+4B\left(\frac{2q+3}{2^{r+1}}\right)\right],q=0,\ldots,2^r-2$,

$\gamma=\frac{1}{3}\left[2B(1)-B\left(\frac{2^r-1}{2^r}\right)-4B\left(\frac{2^{r+1}-1}{2^{r+1}}\right)\right]$

ii. Numerical Solution of FSI-D equation

A numerical technique for solving the FSI-D equation is described in this section. Therefore equation (5) is written as follows

$${}_{\tau_0}I_{\tau}^{1-v}Y'(\tau)=U(\tau)+\int_0^{\tau}\lambda_1(\tau,s)Y(s)ds+\int_0^{\tau}\lambda_2(\tau,s)Y(s)dB(s),\qquad 0<v<1 \tag{24}$$

Now, using LCB-S functions, approximated the functions $Y(\tau)$, $U(\tau)$, $\lambda_1(\tau,s)$ and $\lambda_2(\tau,s)$ in the following way:

$$\begin{gathered}Y(\tau)\approx y^T\Psi(\tau)=\Psi^T(\tau)y\\ U(\tau)\approx u^T\Psi(\tau)=\Psi^T(\tau)u\\ \lambda_1(\tau,s)\approx\Psi^T(\tau)\lambda_1\Psi(s)=\Psi^T(s)\lambda_1^T\Psi(\tau)\\ \lambda_1(\tau,s)\approx\Psi^T(\tau)\lambda_2\Psi(s)=\Psi^T(s)\lambda_2^T\Psi(\tau)\end{gathered} \tag{25}$$

Wherein vector u and matrices λ_1, λ_2 from equation (10) can be obtained.Furthermore, y is an unknown vector that must be discovered. Equation (26) can be written as follows using equation (24) and the operational matrices described in the preceding section.

$$y^T D I_{1-v} \Psi(\tau) = u^T \Psi(\tau) + \int_0^\tau \Psi^T(\tau) \lambda_1 \underbrace{\Psi(s)\Psi^T(s) y}_{\tilde{y}^T \Psi(s)} ds + \int_0^\tau \Psi^T(\tau) \lambda_2 \underbrace{\Psi(s)\Psi^T(s) y}_{\tilde{y}^T \Psi(s)} dB(s)$$

$$= u^T \Psi(\tau) + \Psi^T(\tau) \lambda_1 \tilde{y}^T \mathrm{I} \Psi(\tau) + \Psi^T(\tau) \lambda_2 \tilde{y}^T \mathrm{I}_s \Psi(\tau) \tag{26}$$

Also, using the problem's initial condition (1.1), one can write

$$y^T \Psi(0) = 0 \tag{27}$$

Collocating Eq. (26) at the point $\xi_j = (j+1)2^{-r}$, $j = 0, \ldots, 2^r - 1$ and using Eq. (8) it can be obtained as

$$y^T D I_{1-v} e_j = u^T e_j + e_j^T \lambda_1 \tilde{y}^T \mathrm{I} e_j + e_j^T \lambda_2 \tilde{y}^T \mathrm{I}_s e_j \tag{28}$$

Furthermore, it can be deduced from equation (27)

$$y^T e_{-1} = 0 \tag{29}$$

A linear system of equations is now obtained from equation (28) and equation (29), which is calculated to get the unknown function $Y(\tau)$ in equation (25).

b. Non-Instantaneous Impulses Stochastic Differential Equations

Fractional stochastic differential equation suffered by non-instantaneous impulses with Riemann-Liouville Derivative (RLD) has been investigated. Consistently, utilized the fractional derivatives according to the Modified Riemann-Liouville (RL) derivative for constructing some results related to Mittag-Leffler functions driven by fractional Brownian motion (fBm) and poison jumps. The form of a non-instantaneous impulsive differential equation

$$\begin{cases} w'(t) = Aw(t) + Q(t, w(t)), \quad t \in (s_l, t_{l+1}), l = 0, 1, \ldots, K \\ w(t) = D_l(t, w(t)), t \in (s_l, t_{l+1}), l = 1, 2, \ldots K \\ w(0) = w_0 \end{cases} \tag{30}$$

Banach's fixed point theorem gives a unique $X_l \in C([t_l, s_l], W)$ so that $X = D_l(t, X)$ if and only if $X = X_l(t)$. Consequently, Equation (30) is equivalent to

$$w(t) = X_l(t), \qquad t \in (t_l, s_l), \; l = 1, 2, \ldots, K \tag{31}$$

Which does not depend on $w(\cdot)$. Consequently, the model recommends that the conditions be considered, and the non-instantaneous impulsive conditions (30) must be modified.

$$w(t)=D_l\left(t,w\left(t_l^-\right)\right),\quad t\in\left(t_l,s_l\right),\quad l=1,2,\ldots,K \tag{32}$$

then, $w\left(t_l^+\right)=D_l\left(t_l,w\left(t_l^-\right)\right),\ l=1,2,\ldots,K$. The symbols $w\left(t_l^+\right)=\lim_{\delta\to 0^+}w\left(t_l+\delta\right)$ and $w\left(t_l^-\right)=\lim_{\delta\to 0^-}w\left(t_l+\delta\right)$ represent the right and left limits of $w(t)$ at point $t=t_l$, respectively.

Consider the following fractional stochastic differential equation with fBm and Poisson jumps driving non-instantaneous impulses.

$$\begin{cases} {}^cD_t^n z(t)=Az(t)+Cv(t)+K_1(t,z(t))+\int_u K_2(t,z(t),\mu)+K_3(t)\dfrac{dB^H(t)}{dt} \\ t\in\bigcup_{j=0}^{M}\left(p_j,t_{j+1}\right), \\ z(t)=W_j\left(t,z\left(t_j^-\right)\right),\ t\in\bigcup_{j=1}^{M}\left(t_j,p_j\right) \\ z(0)=z_0 \end{cases} \tag{33}$$

Where $A:D(A)\subset \mathcal{Z}\to \mathcal{Z}$ is the infinitesimal generator of a η-resolventfamily $S_\eta(t)=t\geq 0$, ${}^cD_t^n$ is the Caputo fractional derivative of order $\eta\in(1/2,1)$, and the state $z(\cdot)$ takes values in a separable Hilbert space $\mathcal{Z}, 0=t_0=p_0<t_1<p_1<\ldots<p_M<t_{M+1}=T<\infty\ J_1=[0,T]$. $B^H=\left\{B^H(t):t\geq 0\right\}$ is a Q-fBm with the Hurst parameter $1/2<H<1$ that reflects non-instantaneous impulses during the intervals $\left[t_j,p_j\right),\ j=0,1,2,\ldots,M$ and the limit $Z\left(t_j^-\right)=\lim_{s\to 0^-}z\left(t_j+\delta\right)$ represents the left limit of $z(t)$ at $t=t_j$. The $\mathcal{Z}$ valued random variable that is unaffected by B^H and the starting state z_0 is F_0 quantifiable.

i. Modified Riemann-Liouville (RL) Derivative

The modified Riemann Liouville derivative of order α is defined by the expression

$$D^\alpha f(x)=\begin{cases} \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_a^x (x-\eta)^{-\alpha-1}f(\eta)d\eta & \alpha<0 \\ \dfrac{1}{\Gamma(-\alpha)}\dfrac{d}{dx}\displaystyle\int_a^x (x-\eta)^{-\alpha}\left[f(\eta)-f(a)\right]d\eta & 0<\alpha<1 \\ \left(f^{(\alpha-m)}(x)\right)^m & m\leq\alpha<m+1 \end{cases} \tag{34}$$

The one-parameter Mittag-Leffler function is denoted by $E_\alpha(x)$ and defined by

$$E_\alpha(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)},\quad \alpha\in C,\alpha>0 \tag{35}$$

This function plays a crucial role in classical calculus for $\alpha = 1$it becomes the exponential function, that is $e^x = E_1(x)$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} \tag{36}$$

The other important function which is a generalization of series is represented by:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in C, \alpha > 0 \tag{37}$$

The functions (35) and (37) play important role in fractional calculus, also note that when $\beta = 1$in (37), then (35) is obtained which mean that

$$E_{\alpha,1}(x) = E_\alpha(x) \tag{38}$$

ii. Fractional Brownian Motion and Poisson Jumps

Let $L^2(F_T, Ƶ)$be the Banach space of all $F_T$measurable, square-integrable random variables with values in the Hilbert space $Ƶ$. Let $\bar{q} = \bar{q}(t), t \in D_{\bar{q}}$be a stationary F_t-Poisson point process with a characteristic measure λ.Let $N_1(dt, d\mu)$be the Poisson counting measure associated with $\bar{q}$, that is, $N_1(t,U) = \sum_{s \in D_q, s \le t} I_U(\bar{q}(s))$with the measurable set $U \in B(X/\{0\})$,which denotes the Borelσ-field of $X/\{0\}$.Let $N(dt, d\mu) = N(dt, d\mu) - dt\lambda(d\mu)$be the compensated Poisson measure that is independent of fBm. The fBm$W^H$with $1/2 < H < 1$admits the following integral representation

$$W^H(t) = \int_0^t K_H(t, p) dW(p) \tag{39}$$

Where the kernel $|K_H(t, p)|$ is defined as follows and the Wiener process is defined as W

$$K_H(t, p) = P_H p^{1/2-H} \int_p^t (\tau - p)^{H-3/2} \tau^{H-1/2} d\tau \quad for\ t > p \tag{40}$$

Assume, $K_H(t, p) = 0$if $t \le p$. Notice that $\frac{\partial K_H}{\partial t}(t, p) = P_H \left(t/p \right)^{H-1/2} (t-p)^{H-3/2}$, Where, $P_H = [H(2H-1)/\bar{\beta}(2-2H, H-1/2)]^{1/2}$ and $\bar{\beta}(\cdot,\cdot)$is a Beta function. For $\Psi \in L^2([0,T])$, that the integral of the function Ψ with respect to fBmW^H is defined by

$$\int_0^T \Psi(p) dW^H(p) = \int_0^T K_H * \Psi(p) W(p) \tag{41}$$

Where, $K_H^* \Psi(p) = \int_0^T \phi(t) \frac{\partial K_H}{\partial t}(t, p) dt$. Let $Qe_k = \lambda_k e_k$ with finite trace

$Tr(Q)=\sum_{k=1}^{\infty}\lambda_k<\infty$ define anoperator $Q\in L(X,X)$, where $\{e_k:k=1,2,\cdots\}$ is a complete orthonormal basis in X and $\{\lambda_k\geq 0:k=1,2,\ldots\}$ are real numbers. With covariance Q, $B^H(t)$ be X valued is determined as follows.

$$B_H(t)=\sum_{k=1}^{\infty}\sqrt{\lambda_k e_k}W_k^H(t) \tag{42}$$

Where, $W_k^H(k=1,2,\ldots)$are real independent fBm.

Lemma 1:(Krasnoselskii's fixed point theorem) Let $\hat{N}$ be a bounded closed and convex subset of N, and let N be a Banach space, and let F_1, F_2 be maps of $\hat{N}$ into such that $F_1(x)+F_2(x\in\hat{N})$ for every pair $x,y\in\hat{N}$. The equation $F_{1x}x+F_2x=$ has a solution on $\hat{N}$ if F_1 is a contraction and F_2 is entirely continuous.

iii. Existence and Uniqueness of the Solution

Prove the existence of system solutions in this section (33). The following assumptions are required to prove our primary findings. There are two positive constants M_f and M_h, and the functions $f,h:J\times C_v\rightarrow H$ are continuous, that the function must satisfy.

$$E\|f(t,x)-f(t,y)\|^2\leq M_f\|x-y\|_{C_v}^2, \qquad E\|f(t,x)\|^2\leq M_f\left(1+\|x\|_{C_v}^2\right)$$

and

$$E\|h(t,x)-f(t,y)\|^2\leq M_h\|x-y\|_{C_v}^2, \qquad E\|h(t,x)\|^2\leq M_h\left(1+\|x\|_{C_v}^2\right)$$

for every $x,y\in C_v, t\in J$

Theorem 1: Assume that the assumptions $\lambda>0$,the system (33) has a mild solution on $[0,T]$, provided that

$$\left[8l^2M^2M_\mu+4l^2M_h+4l^2M_f\left(\frac{MT^\alpha}{\Gamma(1+\alpha)}\right)+4l^2M_g\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)+4l^2nM^2\sum_{k=1}^{n}\beta_k\right]$$
$$\times\left[6+\frac{48T^{2\alpha}}{\lambda^2\alpha^2}\left(\frac{\alpha MM_B}{\Gamma(1+\alpha)}\right)\right]\leq 1$$

and

$$L=3l^2\left[M_h+M_f\frac{T^{2\alpha}}{\alpha^2}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)+M_g\frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2\right]<1$$

Proof: For any $\lambda > 0$, define the operator $\Phi : \mathrm{B}_T \to \mathrm{B}_T$ by

$$(\Phi x)(t) = \begin{cases} \varphi(t), \ t \in (-\infty, 0); \\ \Im(t)[\varphi(0) - \mu(x) - h(0,\varphi)] + h(t, x_t) + \\ \quad \int_0^t (t-s)^{\alpha-1} S(t-s) B u^{\lambda}(s) ds + \\ \quad \int_0^t (t-s)^{\alpha-1} S(t-s) f(s, x_s) ds + \\ \quad \int_0^t (t-s)^{\alpha-1} S(t-s) g(s, x_s) dW(s) + \\ \quad \sum_{0<\tau_k<t} \Im(t-\tau_k) I_k\left(x\left(\overline{\tau_k}\right)\right), \quad t \in J \end{cases}$$

In space B_T, this demonstrates that the operator Φ has a fixed position, which is the mild solution of (33). Let $x(t) = z(t) + \hat{\varphi}(t), -\infty < t \le T$, where $\hat{\varphi}(t)$ is defined by

$$\hat{\varphi}(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0) \\ \Im(t)\varphi(0), & t \in J \end{cases}$$

Then $\hat{\varphi}(t) \in B_T$, and it is clear that x satisfies (21) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \Im(t)[-\mu(z+\hat{\varphi}) - h(0,\varphi)] + h(t, z_t + \hat{\varphi}_t) + \int_0^t (t-s)^{\alpha-1} S(t-s) B u^{\lambda}(s) ds +$$
$$\int_0^t (t-s)^{\alpha-1} S(t-s) f(s, z_s + \hat{\varphi}_s) ds + \int_0^t (t-s)^{\alpha-1} S(t-s) g(s, z_s + \hat{\varphi}_s) dW(s) +$$
$$\sum_{0<\tau_k<t} \Im(t-\tau_k) I_k\left(z(\tau_{\bar{k}}) + \hat{\varphi}(\tau_{\bar{k}})\right), \quad t \in J$$

Set $B_T^0 = \{z \in B_T, z_0 = 0 \in C_v\}$, and for any $z \in B_T^0$, the system defines

$$\|z\|_{B_T^0} = \|z_0\|_{C_h} + \sup_{s\in[0,T]} \left(E\|z(s)\|^2\right)^{\frac{1}{2}}$$
$$= \sup_{s\in[0,T]} \left(E\|z(s)\|^2\right)^{\frac{1}{2}}, \qquad x \in B_T$$

Thus, $\left(B_T^0, \|\cdot\|_{B_T^0}\right)$ is a Banachspace.prove that Φ_1 is a contraction on B_r. Let $t \in J$ and $z_1, z_2 \in B_r$, have

$$E\|\Phi_1 z_1(t) - \Phi_1 z_2(t)\|^2 \le 3E\|h(t, z_{1,t} + \hat{\varphi}_t) - h(t, z_{2,t} + \hat{\varphi}_t)\|^2 +$$
$$3E\left\|\int_0^t (T-s)^{\alpha-1} S(T-s)[f(s, z_{1,s} + \hat{\varphi}_s) - f(s, z_{2,s} + \hat{\varphi}_s)] ds\right\|^2 +$$
$$3E\left\|\int_0^t (T-s)^{\alpha-1} S(T-s)[g(s, z_{1,s} + \hat{\varphi}_s) - g(s, z_{2,s} + \hat{\varphi}_s)] dW(s)\right\|^2 +$$
$$\le 3M_h \|z_{1,t} - z_{2,t}\|_{C_v}^2 + 3M_f \frac{T^{\alpha}}{\alpha}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2 \int_0^t (T-s)^{2(\alpha-1)} \|z_1(s) - z_2(s)\|^2 \le L \sup_{s\in J} \|z_1(s) - z_2(s)\|^2$$

Where, $L = 3l^2\left[M_h + M_f \frac{T^{2\alpha}}{\alpha^2}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right) + M_g \frac{T^{2\alpha-1}}{2\alpha-1}\left(\frac{\alpha M}{\Gamma(1+\alpha)}\right)^2\right] < 1$ hence Φ_1 is a contraction.Consequently, Lemma 1, Φ has a fixed point, which is a mild solution of (33).

c. Fractional Delay Differential Equations

In order to obtain their fractional integration operational matrix and delay operational matrix in the Riemann-Liouville sense, the Chelyshkov wavelet basis and attributes are used. To solve Stochastic Fractional Delay Differential Equations (SFDDEs), these operational matrices were used in conjunction with the Galerkin technique. Consequently, some nonlinear and linear SFDDEs are used to verify the efficiency and accuracy of the proposed Chelyshkov wavelet technique.

To build the Chelyshkov wavelet base is the major goal of this current section. The Chelyshkov polynomials are defined explicitly by

$$\rho_{n,M(t)}=\sum_{j=0}^{M-n} a_{j,n} t^{n+j} \tag{43}$$

$n = 0,1,\cdots,M$ in which

$$a_{j,n} = (-1)^j \binom{M-n}{j}\binom{M+n+j+1}{M-n} \tag{44}$$

With respect to the weight function $w(t) = 1$, these polynomials are orthogonal over the interval [0,1].

$$\int_0^1 \rho_{n,M(t)}\rho_{m,M(t)}\, dt = \frac{\delta_{mn}}{m+n+1} \tag{45}$$

Where, δ_{mn} is Kronecker delta. Furthermore, using Rodrigues' formula, these polynomials can be deduced as:

$$\rho_{n,M(t)}=\frac{1}{(M-n)!}\frac{1}{t^{M-n}}\frac{d^{M-n}}{dt^{M-n}}\left(t^{M+n+1}(1-t)^{M-n}\right), \qquad n=0,\cdots,M \tag{46}$$

The functions are $\rho_{n,M(t)},\ n=0,1,\cdots,M$ polynomials of degree M for a fixed integer number M based on the Chelyshkov polynomials definition. In the [0,1] interval, this could be the essential distinction between Chelyshkov polynomials and other common orthogonal polynomials, such as shifted Legendre polynomials. Chelyshkov polynomials can express any integrable function $f(t)$ on the interval $[0,1)$ as:

$$f(t) \sim \sum_{i=0}^{M} c_i \rho_{i,M}(t) = C^T \phi(t) \tag{47}$$

Where, B and $\phi(x)$ are $(M+1)$ vectors are given by $C = [c_0, c_1, \cdots, c_M]$, $\phi(t) = \left[\rho_{0,M}(t), \rho_{1,M}(t), \cdots, \rho_{M,M}(t)\right]$ and $c_i = (2i + 1)\int_0^1 \rho_{i,M}(t) f(t)$

i. Construction of Chelyshkov Wavelets

Wavelets can be derived from the translation and dilation of a mother wavelet function ψ, accordingly, it is a collection of functions. The model has a family of continuous wavelets functions since the translation and dilation parameters change over time. On the interval [0, 1], Chelyshkov wavelets $\psi_{nm}(x)$ are described by

$$\psi_{nm}(t) = \begin{cases} \sqrt{2m+12}^{\frac{k}{2}} \rho m, M\left(2^k t - n\right) & \frac{n}{2^k} \\ 0 & Otherwise \end{cases} \tag{48}$$

Where, $n = 0,1,\cdots,2^k - 1$ and $m = 0,1,\cdots,M$ and $\rho_{m,M}(t)$are the Chelyshkov polynomials of degree m defined in (48). On the interval [0,1], the set of Chelyshkov wavelets $\psi_{nm}(t), n = 0,1,\cdots 2^k - 1,\ m = 0,1,\cdots,M$ is an orthonormal set. The set of Chelyshkov wavelets $\psi_{nm}(t), n = 0,1, m = 0,1,3$ can be calculated for $M = 3$and $k = 1$.

$$\psi_{00}(t) = \begin{cases} 4\sqrt{2}(-70t^3 + 60t^2 - 15t + 1), & 0 \le t < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

$$\psi_{01}(t) = \begin{cases} 4\sqrt{2}(42t^3 + 30t^2 + 5t), & 0 \le t < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

$$\psi_{02}(t) = \begin{cases} 8\sqrt{10}(-7t^2 + 3t^2), & 0 \le t < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

$$\psi_{03}(t) = \begin{cases} 8\sqrt{14}t^3, & 0 \le t < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

$$\psi_{10}(t) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{2}(-280t^3) + 660t^2 - 510t + 129 & \frac{1}{2} \le t < 1 \end{cases}$$

$$\psi_{11}(t) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{6}168t^3 - 372t^2 + 266t - 61, & \frac{1}{2} \le t < 1 \end{cases}$$

$$\psi_{12}(t) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{10}(-56t^3 + 108t^2 - 66t + 13), & \frac{1}{2} \le t < 1 \end{cases}$$

$$\psi_{12}(t) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ \sqrt{14}(2t-1)^3 & \frac{1}{2} \le t < 1 \end{cases}$$

The wavelet $\psi_{nm}(t)$ is clearly a polynomial of degree M across the interval $\left[\frac{n}{2^k}, \frac{n+1}{2^k}\right)$ based on the Chelyshkov wavelet's definition. Furthermore, using Chelyshkov wavelets, every square-integrable function $f(t)$ over $[0, 1)$can be represented as:

$$f(t) \simeq f_{M,k}(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm}\psi_{nm}(x) = C^T\psi(t) \tag{49}$$

Where,C and $\psi(t)$ are $\hat{m} = 2^k(M+1)$vectors are given as

$$C = \left[c_{00}, \cdots c_{0(M-1)}, \cdots, c_{(2^k-1)0,\cdots,}c_{(2^k-1)(M-1)}\right]^T$$

$$\Psi(x) = \left[\psi_{00}(x), \ldots, \psi_{0M}, \ldots, \psi_{(2^k-1)0}(x), \ldots, \psi_{(2^k-1)M}(x)\right]^T$$

and

$$c_{nm} = \langle \psi_{nm}(t), f(t) \rangle = \int_0^1 \psi_{nm}(t) f(t) dt \tag{50}$$

The expansion (49) may be rewritten as:

$$f_{M,k}(x) = \sum_{i=1}^{\hat{m}} c_i \psi_i(x) = C^T \Psi(x)$$

Where, $C = [c_1, c_2, \ldots, c_{\hat{m}}]$, $\Psi(x) = [\psi_1(x), \psi_2(x), \ldots, \psi_{\hat{m}}(x)]$ and $c_i = c_{nm}$, $\psi_i(x) = \psi_{nm}(x)$,

$i = n(M+1) + m + 1$.

d. Approximate Controllability Problem of Riemann-Liouville

On the other hand, in mathematical control theory, the concept of dynamical system controllability is a fundamental concept, that is important in many fields of engineering and science. With nonlocal conditions of order $1 < \alpha < 2$, consider the approximate controllability of Riemann-Liouville fractional stochastic evolution equations in this research paper. Under the assumption that the associated linear system is approximately controllable, the approximate controllability of this nonlinear Riemann-Liouville fractional nonlocal stochastic systems of order $1 < \alpha < 2$ is investigated.

The following were investigated in this article: Fractional evolution equation of Riemann-Liouville

$$\begin{cases} {}^{r}D_t^{\alpha} x(t) = Ax(t) + Bu(t) + f\left(t, x(t), \int_0^t h(t,s,x(s))ds\right) + \sigma(t, x(t))\frac{dw(t)}{dt}, \ t \in [0,b] \\ J_t^{2-\alpha} x(0) + g(x) = x_0 \\ D_t^1\left(J_t^{2-\alpha} x\right)(0) + q(x) = x_1 \end{cases} \quad (51)$$

Where, $1 < \alpha < 2$, ${}^{r}D_t^{\alpha}$ is Riemann-Liouville's fractional derivatives of order α; $x(\cdot)$ takes values in a separable real Hilbert space H with the inner product $< \cdot, \cdot >$ and norm $\|\cdot\|$; and $x_0, x_1 \in H$. Here, A is a sectorial operator of the type (M, θ, α, μ) defined from the domain $D(A) \subset H$ into H. Let K be another separable Hilbert space. This work estimated the solution operator of Riemann-Liouville's fractional evolution equation with nonlocal conditions of the order $1 < \alpha < 2$.

Theorem 2: Assume that the assumptions $(H1)-(H4)$, $(A1), (A2)$ hold, and $\tilde{Q}_\alpha(t), T_\alpha(t)$ are compact, then the system (1) is approximately controllable on J.

Proof: Let $x^{\in}(\cdot)$ be a fixpoint of Ψ in B_r. $x^{\in}(\cdot)$ is a mild solution of the system (51). By the stochastic Fubini theorem, the system gets

$$x^{\in}(b) = \tilde{x}_b - \in \left(\in I + \Gamma_0^b\right)^{-1}\left[E\tilde{x}_b + \int_0^b \tilde{\phi}(\eta)dw(\eta)\right] - Q_\alpha(b)\left(x_0 - g\left(x^{\in}\right)\right) - T_\alpha(b)\left(x_1 - q\left(x^{\in}\right)\right)$$
$$- \in \int_0^b \left(\in I + \Gamma_s^b\right)^{-1} T_\alpha(b-\eta) f\left(\eta, x^{\in}(\eta), \int_0^\eta h\left(\eta, \tau, x^{\in}(\tau)\right)d\tau\right)d\eta$$
$$- \in \int_0^b \left(\in I + \Gamma_s^b\right)^{-1} T_\alpha(b-\eta)\sigma\left(\eta, x^{\in}(\eta)\right)dw(\eta)$$

Rearranging the equation as

$$x^{\in}(b) = \tilde{x}_b - \in \left(\in I + \Gamma_0^b\right)^{-1}\left[E\tilde{x}_b + \int_0^b \tilde{\phi}(\eta)dw(\eta)\right] - Q_\alpha(b)\left(x_0 - g\left(x^{\in}\right)\right) - T_\alpha(b)\left(x_1 - q\left(x^{\in}\right)\right)$$
$$- \in \int_0^b \left(\in I + \Gamma_s^b\right)^{-1} T_\alpha(b-\eta)\sigma\left(\eta, x^{\in}(\eta)\right)dw(\eta) f\left(\eta, x^{\in}(\eta), \int_0^\eta h\left(\eta, \tau, x^{\in}(\tau)\right)d\tau\right)d\eta$$

By the assumption on f, σ, the model gets the sequences $\left\{f\left(t, x^{\in}, \int_0^t h\left(t, s, x^{\in}\right)\right)\right\}$ and $\left\{\sigma\left(t, x^{\in}\right)\right\}$ are uniformly bounded on J. Therefore, there are subsequence's still denoted by $\left\{f\left(t, x^{\in}, \int_0^t h\left(t, s, x^{\in}\right)\right)\right\}$ and $\sigma\left(t, x^{\in}\right)$ that weakly converge to say f^*, σ^* in $L_2(J \times H \times H, H)$ and $L^2\left(J \times H, L_2^0(K, H)\right)$ respectively. From the above equation, the model has

$$t^{2-\alpha} E\left\|x^{\in}(b) - \tilde{x}_b\right\|^2 \le 7b^{2-\alpha} E\left\| \in \left(\in I + \Gamma_0^b\right)^{-1}\left[E\tilde{x}_b - T_\alpha(b)\left(x_1 - q\left(x^{\in}\right)\right)\right]\right\|^2 +$$

$$7E\left\| \in \left(\in I + \Gamma_0^b\right)^{-1} \tilde{Q}_\alpha(b)\left(x_0 - g\left(x^{\in}\right)\right)\right\|^2 + 7b^{2-\alpha} E\left(\int_0^b \left\| \in \left(\in I + \Gamma_0^b\right)^{-1} \tilde{\phi}(\eta)\right\|^2 ds\right) +$$

$$7b^{2-\alpha} E\left\|\int_0^b \in \left(\in I + \Gamma_s^b\right)^{-1} T_\alpha(b-\eta)\left[f\left(\eta, x^{\in}(\eta), \int_0^\eta h\left(\eta, \tau, x^{\in}(\tau)\right)d\tau\right) - f^*\right]ds\right\|^2 +$$

$$7b^{2-\alpha} E\left\|\int_0^b \in \left(\in I + \Gamma_s^b\right)^{-1} T_\alpha(b-\eta)\sigma^* dw(s)\right\|^2$$

For all $t \in [0, b]$, the operator $\in \left(\in I + \Gamma_s^b\right)^{-1} \to 0$ strongly as $\in \to 0^+$ and $\left\| \in \left(\in I + \Gamma_s^b\right)^{-1}\right\| \le 1$. Therefore, by the Lebesgue dominated convergence theorem and compactness of $\tilde{Q}_\alpha(t), T_\alpha(t)$, it follows that $\left\|x^{\in}(b) - x_b\right\|_{C_{2-\alpha}}^2 \to 0^+$ as $\in \to 0^+$. This proves the approximate controllability of the system.

4. EXPERIMENTATION AND RESULT DISCUSSION

Initially, two numerical examples are shown in this part to demonstrate the efficiency of the suggested strategy. In a specific scenario, it's worth nothing that example 1 and 2 have the exact solution; as a result, the emphasis is on this example to demonstrate the method's efficiency and potential. Consequently, for the primary problem, the perturbation results and numerical simulations are presented.

Table 1: System Configuration

MATLAB	Version R2020a
Operating System	Windows 10 Home
Memory Capacity	6GB DDR3
Processor	Intel Core i3 @ 3.5GHz

The computations have been executed on a personal computer using MATLAB R2020a software on an Intel Core i3-9700 PC with 6GB of RAM. Table 1 illustrates the simulation machine configuration of Matlab which implements an appropriate numerical integration procedure of stochastic differential problems.The maximum absolute error values are obtained as follows in the present work:

$$\|e\|_\infty = \max \max_{t \in [0,1]} \left|Y(\tau) - Y_N(\tau)\right|$$

The numerical solution and the exact answer are represented by $Y_N(\tau)$ and $Y(\tau)$, respectively. Also, as for the proposed approach, this form N specifies the number of interpolation points utilised to estimate the solution function.

Table 2: Representation of Absolute Error for Example 1

Proposed Method Absolute Error		
N	$\|e\|_\infty$	CPU Times
5	5.8406e − 2	000.65
9	1.4517e − 2	000.76
17	2.7413e − 3	001.85
33	7.0342e − 4	009.65
65	1.4240e − 4	106.72

Table 2 illustrates the estimation of the absolute error of the proposed methodology. The results reveal that the proposed approach takes less time to run than the other methods. Consequently, to solve the fractional stochastic integro-differential equation in a realistic and efficient manner, the numerical results in this table show that the proposed approach can be utilised.

Numerical Examples

In this part, some illustrated cases have been provided to demonstrate the efficiency and applicability of the proposed wavelet technique. In both nonlinear and linear cases, the numerical examples are considered.

Example 1: Consider the FSI-D equation as

$$ {}_{0}^{c}D_{\tau}^{v}Y(\tau)=\frac{\tau^{5}e^{\tau}}{5}+\frac{6\tau^{3-v}}{\Gamma(4-v)}+\int_{0}^{\tau}e^{\tau}sY(s)ds+\sigma\int_{0}^{\tau}e^{\tau}sY(s)dB(s), \quad s,\tau\in[0,1] $$

with the initial condition $Y(0)=0$. The exact stochastic solution $Y(0)$ of this equation is unknown. In this case $\sigma=0$ and $v=0.95$, the exact solution is $Y(t)=t^{3}$. Here, the proposed cubic B-spline collocation method is used for obtaining the numerical solution of Eq. (60) for $\sigma=0$ and $v=0.95$ with $r=4$.

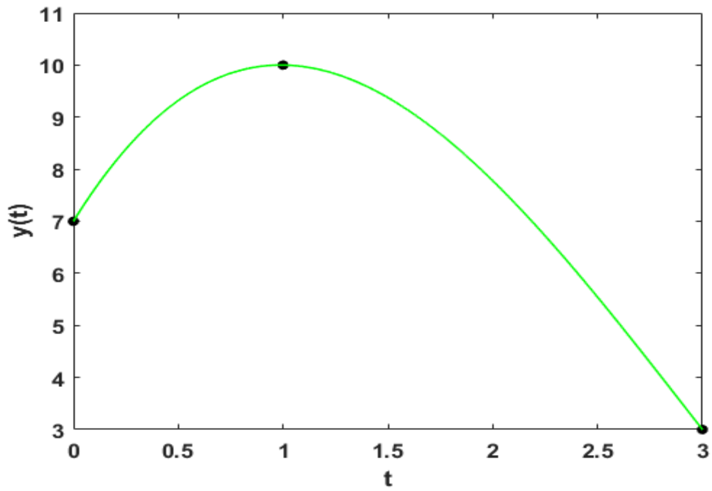


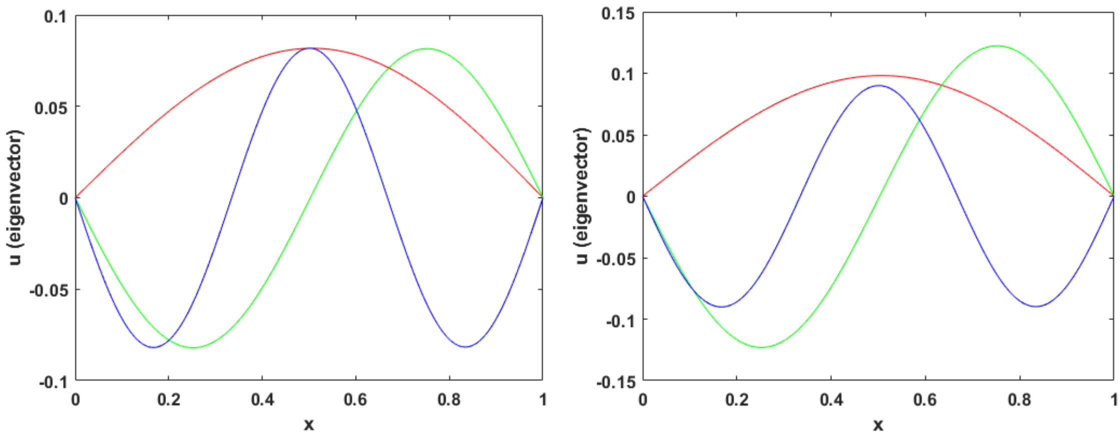
Figure 2: Numerical Solution in case of $r = 4, v = 0.85,$ and $\sigma = 0,0.35,0.70,1$ for Example 1

The exact solution (in case $\sigma = 0$) is contrasted to the numerical solutions for the cases, $v = 0.85$ and $\sigma = 0,0.35,0.70,1$ in figure 2. In other words, moving away from $\tau = 0$ and approaching to $\tau = 1$, it significantly reduces the accuracy of these procedures. The proposed method is accurate throughout the interval $[0,1]$ unlike these two methods.

Table 2: Analytical solutions of Example 1 for $\sigma = 0$, $r = 5$ and various values of v

τ_i	$v = 0.35$	$v = 0.65$	$v = 0.95$	exact$(\sigma = 0)$
0	0.00000	0.00000	0.00000	0.00000
0.5	0.000709	0.00047	0.00041	0.00102
1	0.00812	0.00797	0.02617	0.00800
1.5	0.02715	0.02702	0.06275	0.12500
2	0.06369	0.06357	0.34237	0.34300
2.5	0.34330	0.51223	0.51139	0.34300
3	0.02715	0.0654	0.01256	0.24261

The findings of this experiment for r = 5, $v = 0.35,0.65,0.95$ are presented in table 2. Table 2 demonstrates that the proposed method has a high level of accuracy. The provided results revealed that the recommended technique can effectively solve the problem.



(a) (b)

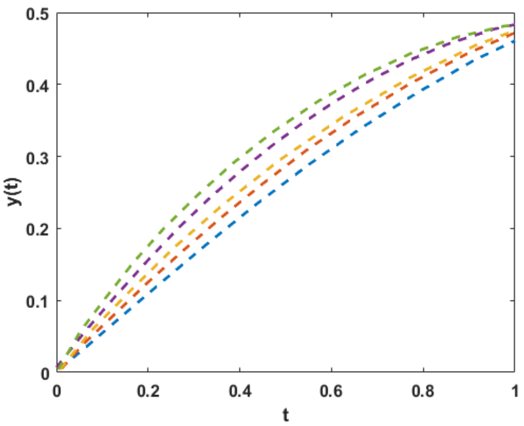
Figure 3: Graph of Orthogonal Polynomials in the Interval [0, 1].

The figure 3 depicts the orthogonal polynomials of the graph plotted in the interval [0,1]. Figure 3(a) denotes the Chelyshkov polynomials, and figure 3(b) stated the shifted Legendre polynomials.It is obvious that the functions $\rho_{n,M}(t), n = 0,1,\ldots,M$ are polynomials of degree M for a fixed integer number M. In the interval [0, 1], this may be the most important distinction between Chelyshkov polynomials and other orthogonal polynomials, such as shifted Legendre polynomials where the $k-th$ polynomial has a degree k. In terms of weight $w(t)=1$, it's worth noting that the Chelyshkov polynomials are orthogonal. Consequently, they are more efficient and trustworthy when applied to the approximate solution of functional equations.

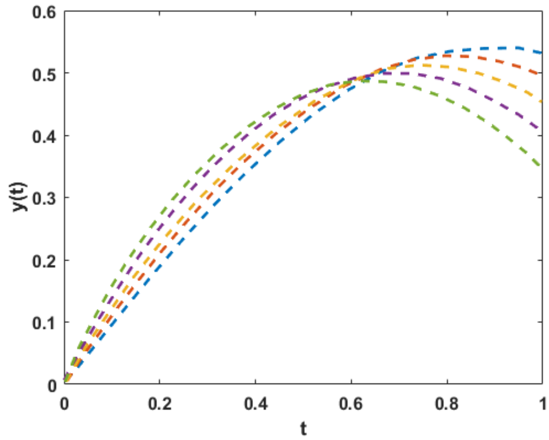
Example 2: Consider the nonlinear pantograph delay differential equations in this example.

$$\begin{cases} D^{\alpha} y_1(t) = 2y_2\left(\frac{1}{2}t\right) + y_3(t) + t\cos\left(\frac{1}{2}t\right) \\ D^{\alpha} y_2(t) = -2y_3^2\left(\frac{1}{2}t\right) - t + 1 \qquad 0 \le t \le 1 \;\; 0 < \alpha \le 1 \\ D^{\alpha} y_3(t) = y_1(t) + y_2(t) - t\cos(t) \\ y_1(0) = -1,\; y_2(0) = 0,\; y_3(0) = 0 \end{cases}$$

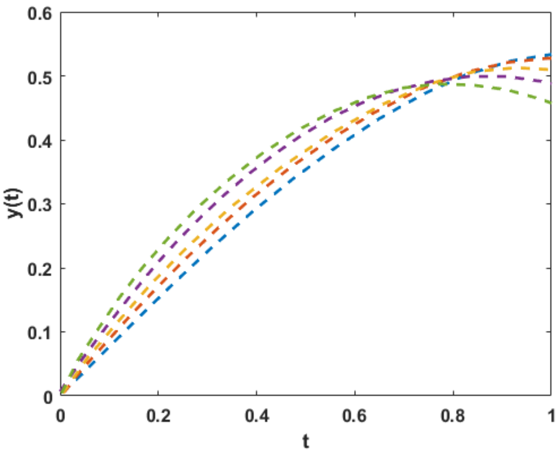
Where $y_1(t) = -\cos(t), y_2(t) = t\cos(t)$and $y_3(t) = \sin(t)$ are the actual solutions for $\alpha = 1$. For various values of M, k and α, the proposed Chelyshkov wavelet approach also solves this problem.



(a)



(b)



(c)

Figure 4: Numerical results for different values of α and $\hat{m}=18$ (Example 3)

The approximation and exact solutions for $\hat{m}=18$ various fractional orders α are shown in figure 4. Figure 4(a) stated the solutions for $y_1(t)$ and figure 4(b) plotted the Graph with $y_2(t)$. Similarly, for figure 4(c) the graph lines are denoted the $y_3(t)$ value the exact solutions are derived from the test example 2. Based on these findings, for solving nonlinear SFDDEs, and numerical solutions converge to the precise solution as fractional order α approaches 1, the Chelyshkov wavelet method is effective.

6. RESEARCH CONCLUSION

Developing computational techniques, it has become clear that Stochastic Delay Differential Systems (SDDEs) can better simulate real-world phenomena. Consequently, the article considers a fractional stochastic differential equation using Linear Cardinal B-spline (LCB-S) functions to numerically solve the Fractional Stochastic Integro-Differential (FSI-D) equations. Fractional stochastic differential equation suffered by non-instantaneous impulses with Riemann-Liouville Derivative (RLD) driven by fractional Brownian motion (fBm) and poison jumps. In order to obtain the fractional integration operational matrix and delay operational matrix in the Riemann-Liouville sense, the Chelyshkov wavelet basis and attributes are used. Sufficient conditions for approximate controllability for the considered problem are also established.Accordingly, for Riemann-Liouville fractional stochastic evolution equations with nonlocal circumstances of the order $1<\alpha<2$, the approximate controllability is considered.

The existence of a mild solution for a stochastic integrodifferential system with finite delay is described and demonstrated in this theory using fixed-point theorems of the Krasnosel'skii and Banach types. Some test issues are presented to demonstrate the important properties of the suggested algorithm, such as reliability, accuracy, and efficiency. These implementations were carried out in MATLAB software. The system of fractional-order delay differential equations is reduced into a system of algebraic equations are the important features of the proposed method. The results are formulated using stochastic analysis techniques. In the later part, investigate the stability results through the continuous dependence of solutions on initial conditions.The applicability, validity, and efficiency of the suggested method are confirmed by a comparison of obtained results using a given scheme in the case of classical stochastic differential equations. This numerical technique's convergence analysis is discussed. Finally, to demonstrate the suggested method's efficiency and efficacy, various instances are used. The family of stochastic differential equations with variable delay in the state has been considered in this research.

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