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A Note On Fuzzy Baire Spaces

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Abstract

In continuation of earlier work [8] we further investigate several properties and Characterizations of fuzzy Baire spaces.

KEY WORDS: Fuzzy dense, fuzzy nowhere dense, fuzzy first category, and fuzzy regular closed, fuzzy open hereitarily irresolvable space, fuzzy Bairespace.

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1. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. ZADEH in his classical paper [11] in the year 1965. Thereafter the paper of C. L. CHANG [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of General Topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. In recent years, fuzzy topology has been found to be very useful in solving many practical problems. SHIHONG Du et. al [4] are currently working to fuzzify the very successful 9-intersection Egenhofer model for depicting topological relations in Geographic Information Systems (GIS) query.

TANG [7] has used a slightly changed version of Chang's fuzzy topological spaces to model spatial objects for GIS databases and Structured Query Language (SQL) for GIS. The concepts of Baire spaces have been studied extensively in classical topology in [5], [6], [12], [13] and [14] . The concept of Baire spaces in fuzzy setting was introduced and studied by the authors in [8]. In this paper we discuss several characterizations of fuzzy Baire spaces.

2. PRELIMINARIES

Now we introduce some basic notions and results used in the sequel.

In this work by (X, T) or simply by X, we willdenote a fuzzy topological space due to CHANG.

DEFINITION 2. 1: Let $\lambda$and μ be any two fuzzy sets in (X, T). Then we define $\lambda \vee \mu$: X $\to$ [0, 1] as follows: $(\lambda \vee \mu) (x) = \text{Max} \{ \lambda (x), \mu (x)\}$. Also we define $\lambda \wedge \mu$: X $\to$ [0, 1] as follows: $(\lambda \wedge \mu) (x) = \text{Min} \{ \lambda (x), \mu (x)\}$.

DEFINITION 2. 2: Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define $\text{int}(\lambda) = \vee\{ \mu / \mu \leq \lambda, \mu\in T \}$ and $\text{cl} (\lambda) = \wedge \{ \mu / \lambda \leq \mu, 1- \mu \in T \}$.

For any fuzzy set in a fuzzy topological space (X, T), it is easy to see that $1 - \text{cl} (\lambda) = \text{int} (1 - \lambda)$ and $1 - \text{int} (\lambda) = \text{cl} (1 - \lambda)$ [1].

DEFINITION 2. 3 [10]: A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

DEFINITION 2. 4 [8]: A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that μ □ cl (λ). That is, int cl (λ) = 0.

DEFINITION 2. 5 [9]: A fuzzy topological space(X, T) is called a fuzzy open hereditarily irresolvable space if int cl (λ) $\neq 0$, then int (λ) $\neq 0$ for any non-zero fuzzy set λin (X, T).

DEFINITION 2. 6 [2]: A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_{σ}-set in (X, T) if $\boldsymbol{\lambda} = \vee_{i=1}^{\infty} \boldsymbol{\lambda_i}$ where $1 - \lambda_i \in T$ for $i \in I$.

DEFINITION 2. 7 [2]: A fuzzy set λ in a fuzzy topological space(X, T) is called a fuzzy G_{δ}-set in (X, T) if $\boldsymbol{\lambda} = \wedge_{i=1}^{\infty} \boldsymbol{\lambda_i}$ where $\lambda_i \in T$ for $i \in I$.

DEFINITION 2. 8 [10]: A fuzzyset λin a fuzzy topological space (X, T) is called fuzzy first categoryif $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy nowhere dense sets in (X, T).

Any other fuzzy set in (X, T) is said to be of secondcategory.

DEFINITION 2. 9 [8]: Let λ be a fuzzy first category set in(X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

DEFINITION 2. 10 [10]: A fuzzy topological space (X, T) is called fuzzy first category if $1 = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy nowhere dense sets in (X, T). A topological space which is not of fuzzy first category, is said to be of fuzzy second category.

Lemma 2. 1 [1]: For a family of $\{\lambda_\alpha\}$ of fuzzy sets of a fuzzy topological space (X, T), $\vee$ cl $(\lambda_\alpha) \leq$ cl $(\vee\lambda_\alpha)$. In case Ais a finite set, $\vee$ cl $(\lambda_\alpha) =$ cl $(\vee\lambda_\alpha)$.
Also $\vee$ int $(\lambda_\alpha) \leq$ int $(\vee\lambda_\alpha)$.

DEFINITION 2. 11 [10]: A fuzzy topological space (X, T) is called a fuzzy resolvable space if there exists a fuzzy dense set λ in (X, T) such that cl $(1-\lambda) = 1$. Otherwise (X, T) is called a fuzzy irresolvable space.

3. FUZZY BAIRE SPACES

DEFINITION 3. 1 [8]: Let (X, T) be a fuzzy topological space. Then(X, T) is called a fuzzyBaire Spaceif int $(\vee_{i=1}^{\infty} (\lambda_i)) = 0$, where $\lambda_i' s$ *are* fuzzynowhere densesets in (X, T).

THEOREM3. 1[8]: Let (X, T) bea fuzzy topological space. Then the followingare equivalent:

1. (X, T) is a fuzzy Baire space.
2. Int $(\lambda) = 0$ for every fuzzy first category set λ in (X, T).
3. cl $(\mu) = 1$ for every fuzzy residual set μ in (X, T).

PROPOSITION 3. 1: If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then $1-\lambda$ is a fuzzy first category set in (X, T).

PROOF: Since λ is a fuzzy G_δ-setin (X, T), $\lambda = \wedge_{i=1}^{\infty} (\lambda_i)$ where $\lambda_i \in T$ and since$\lambda$is a fuzzy dense set in (X, T), cl $(\lambda) = 1$. Then cl $(\wedge_{i=1}^{\infty} (\lambda_i)) = 1$. But cl $(\wedge_{i=1}^{\infty} (\lambda_i)) \leq \wedge_{i=1}^{\infty}$ cl (λ_i).Hence $1 \leq \wedge_{i=1}^{\infty}$ cl (λ_i).That is, $\wedge_{i=1}^{\infty}$ cl $(\lambda_i) = 1$. Then we have cl $(\lambda_i) = 1$ for each $\lambda_i \in T$ and hence cl (intλ_i) = 1 which implies that $1-$ cl int $(\lambda_i) = 0$ and hence int cl $(1-\lambda_i) = 0$. Therefore $1-\lambda_i$ is a fuzzy nowhere dense set in (X, T). Now $1-\lambda = 1 - \wedge_{i=1}^{\infty}(\lambda_i) = \vee_{i=1}^{\infty}(1-\lambda_i)$ Therefore $1-\lambda = \vee_{i=1}^{\infty}(1-\lambda_i)$ where$(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X, T). Hence$1-\lambda$is a fuzzy first categoryset in (X, T).

PROPOSITION 3. 2: If λ isa fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then λ is a fuzzy residual set in (X, T).

PROOF: Since λ is a fuzzy dense and fuzzy G_δ-set in (X, T), byproposition 3. 1, we have $1-\lambda$ is a fuzzy first category set in (X, T) and hence λ is a fuzzy residual set in (X, T).

PROPOSITION 3. 3: If afuzzy topological space(X, T) has a fuzzy dense and fuzzy G_δ set, then (X, T) is a fuzzy Baire space.

PROOF: Let λ be a fuzzy dense and fuzzy G_δ-set in (X, T). Then by proposition 3. 1, $1-\lambda$ is a first category set in (X, T) and $(1-\lambda) = \vee_{i=1}^{\infty}(1-\lambda_i)$ where $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X, T). But int $(1-\lambda) = 1-$ cl $(\lambda) = 1-1 = 0$ (since λ

isfuzzy dense, cl (λ) = 1). Then int ($\vee_{i=1}^{\infty}(1-\lambda_i)$) = int ($1-\lambda$) = 0 and hence (X, T) isa fuzzy Baire space.

PROPOSITION 3. 4: Ifthe fuzzy topological space (X, T)is afuzzy first category space, then (X, T) is not a fuzzy Baire space.

PROOF: Since (X, T) is a fuzzy firstcategory space, then $\vee_{i=1}^{\infty} \lambda_i = 1$, where$\lambda_i$'s are fuzzy nowhere dense sets in (X, T). Therefore int ($\vee_{i=1}^{\infty}$ (λ_i)) = int (1) = 1 $\neq$ 0. Hence int ($\vee_{i=1}^{\infty}$(λ_i)) $\neq$ 0, where λ_i 's are fuzzy nowhere dense sets in (X, T) and therefore (X, T) is notafuzzy Baire space.

PROPOSITION 3. 5: Ifλ is a fuzzy first category set in (X, T), then there is a fuzzyF_σ-setηin (X, T) such that $\lambda \leq \eta$.

PROOF: Letλbe a fuzzy first category set in (X, T). Then, $\lambda = \vee_{i=1}^{\infty} \lambda_i$, where$\lambda_i$ 'sarefuzzy nowhere dense setsin (X, T). Now 1 – cl (λ_i) is a fuzzy open setin(X, T). Then$\wedge_{i=1}^{\infty}[1-\text{cl}(\lambda_i)]$ is a fuzzy G_δ-set in (X, T). Let $\mu = \wedge_{i=1}^{\infty}[1-\text{cl}(\lambda_i)]$. Now $\wedge_{i=1}^{\infty}[1-\text{cl}(\lambda_i)] = 1-\vee_{i=1}^{\infty}(\text{cl}\,\lambda_i) \leq 1-\vee_{i=1}^{\infty}(\lambda_i) = 1-\lambda$, Hence $\mu \leq 1-\lambda$. Then $\lambda \leq 1-\mu$. Let $\eta = 1-\mu$. Since μ is afuzzy G_δ–set in (X, T), η is a fuzzyF_σ-setin (X, T) . Therefore, ifλ is a fuzzy first category set in (X, T), then there is a fuzzy F_σ-setηin (X, T) such that $\lambda \leq \eta$.

REMARKS: If int (η) = 0 in the above proposition, then (X, T) is a fuzzy Baire space. For, int (λ) $\leq$ int (η) = 0 implies that int (λ) = 0 and hence (X, T) is a fuzzy Baire space.

THEOREM 3. 2 [9]: If (X, T) is a fuzzy open hereditarily irresolvable space and if int (μ)= 0, then int cl (μ) = 0, for any non-zero fuzzy set μin (X, T).

PROPOSITION 3. 6: If cl ($\wedge_{i=1}^{\infty}(\lambda_i)$) = 1, where λ_i 's are fuzzy dense setsin a fuzzy openhereditarily irresolvable space, then (X, T) is a fuzzy Baire space.

PROOF: Now cl ($\wedge_{i=1}^{\infty}(\lambda_i)$) = 1, where cl ($\lambda_i$) = 1, implies that int ($\vee_{i=1}^{\infty}$ $(1-\lambda_i)$) = 0, where int ($1-\lambda_i$) = 0. Let $\mu_i = 1-\lambda_i$. Then, int ($\vee_{i=1}^{\infty}$ (μ_i)) = 0, where int (μ_i) = 0. Since (X, T) isa fuzzy open hereditarily irresolvable space, int (μ_i)= 0 implies that int cl (μ_i) = 0. Henceμ_i is a fuzzy nowheredense set in (X, T). Hence int ($\vee_{i=1}^{\infty}$ (μ_i))= 0, where μ_i's are fuzzy nowhere dense sets in (X, T), implies that (X, T) is a fuzzy Baire space.

PROPOSITION 3. 7: If (X, T) is a fuzzy Baire irresolvable space, then cl ($\vee_{i=1}^{\infty}(\lambda_i)$)$\neq$ 1, where λ_i 's are fuzzy nowhere dense sets in (X, T).

PROOF: Let λ be a fuzzy first category set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy nowhere dense sets in (X, T). Since (X, T) is a fuzzy Baire space, int (λ) = 0. Then 1 – int (λ) =1, whichimplies that cl $(1 - \lambda) = 1$. Since (X, T) is afuzzy irresolvable space, cl $[1-(1-\lambda)] \neq 1$. Hence cl $(\lambda) \neq 1$ and therefore cl $(\vee_{i=1}^{\infty}(\lambda i\,)) \neq 1$, where λ_i 's are fuzzy nowhere dense sets in (X, T).

PROPOSITION 3. 8: If (X, T) is a fuzzy Baire space and if $\vee_{i=1}^{\infty}(\lambda_i) = 1$, where λ_i's are fuzzy regular closed sets in (X, T), then cl $(\vee_{i=1}^{\infty}(int\ (\lambda_i)) = 1$.

PROOF: Suppose that int (λ_i)= 0 for each $i \in I$. Now λ_i is a fuzzy regular closed in (X, T) implies that λ_i is a fuzzy closed in (X, T). Also int $(\lambda_i) = 0$, implies that int cl $(\lambda_i) = 0$ and hence λ_i is a fuzzy nowhere dense set in(X, T). Now$\vee_{i=1}^{\infty}(\lambda_i) = 1$impliesthat int $(\vee_{i=1}^{\infty}(\lambda_i)) = \text{int}\,(1) = 1$ …. . (1). Since(X, T) is a fuzzy Baire Space, int $(\vee_{i=1}^{\infty}(\lambda_i)) = 0$ where λ_i's are fuzzy nowhere dense sets in (X, T) . From (1) we have 0 = 1, which is a contradiction, Hencewe have int $(\lambda_i) \neq 0$, for atleast one i. Hence $(\vee_{i=1}^{\infty}\text{int}\,(\lambda_i)) \neq 0$. Now$(\vee_{i=1}^{\infty}\text{ cl}\,(\lambda_i)) \leq cl\,(\vee_{i=1}^{\infty}(\lambda_i))$ implies that $(\vee_{i=1}^{\infty}\text{ cl (int}\,(\lambda_i)) \leq cl\,(\vee_{i=1}^{\infty}\text{ (int}\,(\lambda_i))$.

That is, $cl\,(\vee_{i=1}^{\infty}\text{ int}\,(\lambda_i)) \geq (\vee_{i=1}^{\infty}\text{ cl (int}\,(\lambda_i)) = (\vee_{i=1}^{\infty}(\lambda_i)\,) = 1$, [since λ_i is fuzzy regular closed]. Then cl $(\vee_{i=1}^{\infty}\text{int}\,(\lambda_i)) \geq 1$. That is, cl $(\vee_{i=1}^{\infty}\text{int}\,(\lambda_i)) = 1$.

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