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Constructions on Cyclic M-Fuzzy Group Family

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Abstract

In Crisp environment, the notion of cyclic group on a set is well known. We study an extension of this classical notion to the M- fuzzy sets to define the concept of M- cyclic fuzzy groups. By using these cyclic ,M-fuzzy groups, we then define cyclic M-fuzzy group family and investigate its structure properties with applications.

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Section – 1 Introduction

The original concept of fuzzy sets was firstly introduced in the pioneering paper [12] of Zadeh as an extension of crisp (usual) sets, by enlarging the truth value set of " grade of membership" from the two sets {0,1} to the unit interval [0,1] of real numbers. There has been tremendous interest in the fuzzy set theory due to its many applications ranging from engineering and computer science to social behavior studies. More details and historical background of fuzzy set theory can be found in , for examples , [2] , [3] , [13].

There is a quite substantial literature on fuzzy group theory. The study of fuzzy groups was started firstly by Rosenfeld [5] . He used the min operating to define his fuzzy groups and showed how some basic notions of fuzzy group theory should be extended in an elementary manner to develop the theory of fuzzy groups. It was extended by Anthony and Sherwood [1] . They used the t- norm operating instead of the min to define the t- fuzzy groups. Roventa and Spircu [6] introduced the fuzzy group operating on fuzzy sets. Sidkey and Misherf [8] defined t- cyclic fuzzy groups by using t- level sets in the crisp environment. Ray [4] defined a cyclic fuzzy group of a given fuzzy group family simply by restriction. [9],[11] results are listed. In this paper, we give a sufficient condition for a M- fuzzy subset to be a cyclicM- fuzzy group. By using this cyclic M-fuzzy group, we then define a cyclic M-fuzzy group family and investigate its structure properties with applications.

Section -2 Preliminaries

In this section, we give the preliminary definitions and results that will be required in this paper. Most contents of this section are contained in the literature that will be indicated if necessary.:

Definition2.1: Let A and B be fuzzy sets. Then A is a subset of B if $\mu_A(x) \leq \mu_B(x)$ for every $x \in U$ and it is denoted by $A \subseteq B$ or $B \supseteq A$.

Definition2.2: Two fuzzy sets A and B are called equal if $\mu_A(x) = \mu_B(x)$ for every $x \in U$ and it is denoted by A= B

Definition2.3 : Let A and B be fuzzy sets. Then the algebraic product of two fuzzy sets A and B is defined by $A \bullet B = \{(x, \mu_{\tilde{A}}(x)) / x \in U, \mu_{A \bullet B} = \mu_A \bullet \mu_B\}$

Definition2.4: Let A and B be fuzzy sets. Then the Union $A \cup B$ and Intersection $A \cap B$ are respectively defined by the equations.

$A \cup B = \{(x, \mu_{A \cup B}(x)) / x \in U, \mu_{A \cup B}(x) = \max\{(\mu_A(x), \mu_B(x)\}$ and

$A \cap B = \{(x, \mu_{A \cap B}(x)) / x \in U, \mu_{A \cap B}(x) = \min\{(\mu_A(x), \mu_B(x)\}$

Remark2.5: These definitions can be generated for countable number of fuzzy sets. If $\tilde{A}_1, \tilde{A}_2, \ldots$,are fuzzy sets with membership functions $\mu_{\tilde{A}1}(x), \mu_{\tilde{A}2}(x), \ldots$,then the membership functions of $X = U\tilde{A}_i$ and $Y = \cap \tilde{A}_i$ are defined as

$\mu_X(x) = \max\{\mu_{\tilde{A}1(x)}, \mu_{\tilde{A}2(x)} \ldots\}, x \in U$ and $\mu_Y(x) = \min\{\mu_{\tilde{A}1}(x), \mu_{\tilde{A}2}(x), \ldots\}, x \in U$ respectively.

Definition2.6: A mapping $\mu : X \rightarrow [0,1]$, where X is an arbitrary non-empty set and is called a fuzzy set in X.

Definition 2.7: Let M and G a set and a group respectively. A mapping $\mu : M \times G \rightarrow [0,1]$ is called M – fuzzy set in G . For any M- fuzzy set μ in G and $t \in [0,1]$ we define the set $U(\mu; t) = \{ x \in G / \mu(mx) \geq t, m \in M\}$ which is called an upper cut of 'μ' and can be used to the characterization of μ.

Definition 2.8 ; Let f be a non- fuzzy function from X to Y. The image $f(\tilde{A})$ of a M-fuzzy set $\tilde{A}$ on X is defined by means of the extension principle as

$$f(\tilde{A}) = \{ (my) , \mu_{f(\tilde{A})}(my) \ / \ y = f(x) , x \varepsilon X \} ,$$

where $\mu_{f(\tilde{A})}(my) = \begin{cases} \sup_{(mx)\varepsilon f^{-1}(my)} \mu_{\tilde{A}}(mx) \text{ if } f^{-1}(y,q) \neq \Phi \\ 0 \text{ otherwise} \end{cases}$

Definition2.9: A M- fuzzy set 'A' is called M- fuzzy group of G if (MFG1) A(mxy,q) $\geq$ min { A(x), A(y)} (MFG2) $A(mx^{-1})$ = A (x) (MFG3) A(me) = 1 for all x,y ε G and m ε M .

Definition 2.10 : Let G = < a > be a cyclic group. If $\tilde{A} = \{ (a^n, \mu (a^n)) / n \varepsilon z \}$ is a fuzzy group, then Ã is called a cyclic fuzzy group generated by (a, μ(a)) and denoted by < a, μ(a)>.

Definition 2.11: (M- Cyclic fuzzy group) Let A = < ma> be a M- cyclic group. If $\tilde{A} = \{ (ma^n) , \mu(ma^n)) / n \varepsilon z \}$ is a M- fuzzy group, then Ã is called a M- cyclic fuzzy group generated by < (ma) , μ (ma) >.

Section-3: Properties of M- fuzzy groups

Proposition3.1: Let 'A' be a M- fuzzy group of G Then
(i)A(mx) $\leq$ A(me) for all x εG and m ε M.
(ii) The subset G_A = { xε G / A(mx) = A(me)} is a M- fuzzy group of G.

Proof: Let x be any element of G, then A(mx,q) = min {A(mx,q), A(mx,q) } = min {A(x,) , $A(x^{-1})$} $\leq$ $A(xx^{-1})$ = A(x) and (i) is proved. To prove (ii) we have e ε G_A, then $G_A \neq \Phi$. Now let x,y ε G_A and m ε M.

$A(mxy^{-1}) \geq$ min {A(x) , $A(y^{-1})$} = min {A(x), A(e)}=min{A(e). A(e)} = A(e) but from (i) $A(mxy^{-1}) \leq A(e)$ for x,y ε G and m ε M. Therefore $A(mxy^{-1}) = A(e)$ which means mxy^{-1} ε G_A and G_A is M- fuzzy group of G.

Corollary 3.2: Let G be a finite group and A be a M- fuzzy group of G. Consider the subset H of G given by H = {xεG / A(mx) = A(e)} .Then H is a crisp subgroup of G.

Proof: It is obvious.

Proposition 3.3: If 'A' is M- fuzzy group of G, then the set U(A; t) is also M- fuzzy group for all m ε M, t ε Im(A).

Proof: Let t ε Im(A) ⊂ [0,1] and let x,y ε U(A; t), m ε M Then A(x,q) $\geq$ t, A(y,q) $\geq$ t. since A is M- fuzzy group of G it follows that

A(mxy) $\geq$ min { A(x) , A(y)} $\geq$ t hence m(xy) ε U(A; t). Let x ε U(A; t) and m ε M. Then $A(mx^{-1}) = A(x) \geq t$ which implies mx^{-1} ε U(A ; t) .Therefore U (A ; t) is M – fuzzy group of G.

Proposition 3.4: If 'A' is M- fuzzy set in G such that all non- empty level subset U(A; t) is M- fuzzy group of G then A is M- fuzzy group of G.

Proof: Assume that the non-empty level set U(A; t) is M- fuzzy group of G. If t_0 =min { A(mx) , A(my)} and for x,y ε G, m ε M, then x,y ε U(A; t_0) so A(mxy) $\geq t_0$ = min {A(x) , A(y)} which implies that the condition (MFG1) is valid. For x^{-1} ε G and m ε M, then x^{-1} ε U(A ; t_0) Thus $A(mx^{-1}) = t_0 = A(mx)$ which implies that the condition (MFG2) is valid and therefore A is M- fuzzy group of G.

Proposition 3.5: A set of necessary and sufficient conditions for a M- fuzzy set of a group G to be a M- fuzzy group of G is that

$A(mxy^{-1}) \geq$ min (A(x) , A(y)} for all x,y in G and m in M.

Proof: Let A be a M- fuzzy group of G. Then

$A(mxy^{-1}) \geq \min\{ A(x) , A(y^{-1})\} = \min \{ A(x) , A(y)\}$ for x,y ε G and m ε M.

For the converse part suppose that A be a M- fuzzy set of the group G of which e is the identity element.

Now $A(myy^{-1}) \geq \min \{A(y) , A(y)\}$

or $A(e) \geq A(y)$

Now $A(mey^{-1}) \geq \min \{ A(e) , A(y)\}$

or $A(my^{-1}) \geq A(y)$ also $A(mxy) \geq \min \{A(x) , A(y^{-1})\} \geq \min\{A(x) , A(y)\}$.

Proposition 3.6: If G is a group, then prove that $\tilde{A}^m = \{ (ma^n) , \mu_{\tilde{A}}(ma^n)^m / n \in z \}$ is also a M –cyclic fuzzy group.

Proof: Let us show that $\tilde{A}^m$ satisfies three conditions (MFG1-MFG3) in definition 2.9. we can consider only its membership function because the m^{th} power of Ã effects just only the membership function of $\tilde{A}^m$.

(MFG1) Since Ã is a M- fuzzy group and $\mu_{\tilde{A}}(ma)$ ε [0,1], we have

$\mu_{\tilde{A}}((ma^{n1}), (ma^{n2}))^m \geq \min \{ \mu_{\tilde{A}}(ma^{n1}), \mu_{\tilde{A}}(ma^{n2})\}^m$

$= \min \{ \mu_{\tilde{A}}(ma^{n1})^m , \mu(ma^{n2})^m\}$

(MFG2) Now $\mu(ma^n) = \mu_{\tilde{A}}(ma^{-n})$ since Ã is a M- fuzzy group. Accordingly, we get

$(\mu_{\tilde{A}}(ma^n)^m = (\mu_{\tilde{A}}(ma^{-n}))^{m.}$

(MFG3) $\mu_{\tilde{A}}(me,q) = 1$ since Ã is M- fuzzy group, then $(\mu_{\tilde{A}}(me,q))^m = 1$.

Now an example of M- cyclic fuzzy group.

Example 3.7 : Let A be a M- cyclic group with 12 elements and generated by (ma). Let Ã be a M- fuzzy set of the group A defined as follows. $\mu_{\tilde{A}}(ma^0) = 1$, $\mu_{\tilde{A}}(ma^4) = \mu_{\tilde{A}}(ma^8) = t_1$, $\mu_{\tilde{A}}(ma^2) = \mu_{\tilde{A}}(ma^6) = \mu_{\tilde{A}}(ma^{16}) = t_2$ and $\mu_{\tilde{A}}(mx) = t_3$ for all other elements x in A, where t_1,t_2,t_3 ε [0,1] with $t_1 > t_2 > t_3$. It is clear that Ã is a M- fuzzy group of A, thus $\tilde{A} = \{ (ma^k) , \mu_{\tilde{A}}(ma^k) / k \, \varepsilon \, z \}$ is a M- cyclic fuzzy group generated by ((ma) , $\mu_{\tilde{A}}$(ma)).

Definition 3.8: Let e be the identity element of the group A. we define the identity M-fuzzy group E by E = { {me) , $\mu_{\tilde{A}}$(me) / $\mu_{\tilde{A}}$(e) = 1}.

Corollory 3.9: The M- fuzzy group $\tilde{A}^n$ is a M- fuzzy subgroup of $\tilde{A}^m$, if $m \leq n$.

Proof: Clearly $\tilde{A}^n$ and $\tilde{A}^m$ are M- fuzzy groups by (2.9). For all x ε[0,1], $(mx)^m \geq (mx)^n$ implies that $\tilde{A}^n \subset \tilde{A}^m$ (since $\mu_{\tilde{A}}{}^n(mx) \leq \mu_{\tilde{A}}{}^m(mx)$ for all x ε G and m ε M).

Proposition 3.10 : If $\tilde{A}^i$ and $\tilde{A}^j$ are M- cyclic fuzzy groups, then $\tilde{A}^i \cup \tilde{A}^j$ is also cyclic M- fuzzy group if i<j

Proof : since i<j, then we have $\mu_{\tilde{A}}{}^i > \mu_{\tilde{A}}{}^j$

(MFG1) $=\max\{\mu_{\tilde{A}}{}^i (ma^n a^m), \mu_{\tilde{A}}{}^j (ma^n a^m)\}$

$=\max \{\mu_{\tilde{A}}(ma^n a^m)^i, (\mu(ma^n a^m))^j\}$

$=(\mu_{\tilde{A}}(ma^n a^m))^i$

$>\min \{\mu_{\tilde{A}}{}^i (ma^n), \mu_{\tilde{A}}{}^i(ma^m)\}$

$>\min\{\max \{\mu_{\tilde{A}}{}^i(ma^n), \mu_{\tilde{A}}{}^i(ma^m)\}, \max (\mu_{\tilde{A}}{}^i(ma^n), \mu_{\tilde{A}}{}^j(ma^m)\}$

$= \min \{\max\{\mu_{\tilde{A}}{}^i(ma^n), \mu_{\tilde{A}}{}^j(ma^n)\}, \max (\mu_{\tilde{A}}{}^i(ma^m), \mu_{\tilde{A}}{}^i(ma^m)\}\}$

$\geq \min \{\mu_{\tilde{A}i \cup \tilde{A}j} (ma^n), \mu_{\tilde{A}i \cup \tilde{A}j}(ma^m)\}$

MFGI is satisfied.

(MFG2) $\mu_{\tilde{A}i \cup \tilde{A}j} (ma^{-n})$

$= \max \{\mu_{\tilde{A}i}(ma^{-n}), \mu_{\tilde{A}i}(ma^{-n})\}$

$= \max \{\mu_{\tilde{A}}(ma^{-n})^i, \mu_{\tilde{A}}(ma^{-n})^j\}$

$= \max \{\mu_{\tilde{A}}(ma^n)^i, \mu_{\tilde{A}}(ma^n)^j\}$

$= \max \{\mu_{\tilde{A}i}(ma^n), \mu_{\tilde{A}j}(ma^n)\}$

$= \mu_{\tilde{A}i \cup \tilde{A}j} (ma^n)$

MFG2 is satisfied.

(MFG3) $\mu_{\tilde{A}i \cup \tilde{A}j}$ (me) $= \max \{\mu_{\tilde{A}i} (me) \, \mu_{\tilde{A}j} (me)\}$

= max {1,1} ($\tilde{A}$ is a M- fuzzy group.

= 1

$\tilde{A}^i \cup \tilde{A}^j$ forms M-cyclic fuzzy group.

Proposition3.11 : If $\tilde{A}_i$ and $\tilde{A}_j$ are M- cyclic fuzzy groups, then $\tilde{A}_i \cap \tilde{A}_j$ is also a cyclic M- fuzzy group.

Proof: This theorem may be proved similarly to theorem. (3.10).

Remark: Since a cyclic M- fuzzy group is an abelian group, it is clear that μ(mxy) = μ(myx) for x,y ε A, m ε M. Therefore, the cyclic M- fuzzy groups $\tilde{A}^m$, $\tilde{A}_i \cup \tilde{A}_j$ and $\tilde{A}_i \cap \tilde{A}_j$ are also normal M- fuzzy groups.

Definition3.12 : Let Ã be a M- cyclic fuzzy group, then the following set of M- cyclic fuzzy groups { Ã , $Ã^2$, $Ã^3$, ………….$Ã^m$,…. E } is called the cyclic M- fuzzy group family generated by Ã. It will be denoted by < Ã >.

Proposition 3.13 : Let < A > = { A , A^1 , A^2,-------------A^m-------E} then $\bigcup^{\infty} A^P = A$.

and $\bigcap_{m=1}^{\infty} A^P = E$

Proof ; The proof is immediate from propositions (3.10) and (3.11).

Proposition 3.14: Let Ã be a cyclic M- fuzzy group. Then $A \supset A^2 \supset A^3 \ldots\ldots \supset A^m \ldots\ldots\ldots \supset E$

Proof: It is known that $\mu_{Ã}(ma)\ \varepsilon\ [0,1]$. Hence
$\mu_{Ã}(ma).\supset \mu_{Ã}(ma)^2 \supset \mu_{Ã}(ma^2) \supset (\mu_{Ã}(ma^2))^{2,} \ldots\ldots\ldots\ldots\ \mu_{Ã}(ma^n) \supset (\mu_{Ã}(ma^n))^2$.

By using the definition of M- fuzzy subsets, this gives that $Ã \supset Ã^2$. By generalizing it for any natural numbers i and j with i < j, we then obtain
$(\mu_{Ã_i}(ma))^i \geq (\mu_{Ã_j}(ma))^j$, $(\mu_{Ã}(ma^2))^i \geq (\mu_{Ã}(ma^2))^j \ldots\ldots\ldots\ldots\ (\mu_{Ã_i}(ma^n) \geq (\mu_{Ã_j}(ma^n))^{j.}$
So $Ã_i \supset Ã_j$ for any natural numbers i and j with i ≤ j, which means that
$Ã \supset Ã^2 \supset Ã^3 \supset \ldots\ldots\ldots\ldots\ldots\ldots \supset Ã^m \ldots\ldots\ldots\ldots$

Finally , we get $E = \bigcap_{n=1}^{\infty} Ã^n$, which is immediate from proposition since

$$\lim \mu_{Ã}(ma)^n = \begin{cases} 1 & \text{if } ma = m \\ 0 & \text{if } ma \neq me. \end{cases}$$
we then obtain the required relations.

Corollary 3.15 : Let < Ã> = { Ã , $Ã^2$, $Ã^3$ ………… $Ã^m$ ……. E }. Then
$Ã \supset Ã^2 \supset \ldots\ldots\ldots\ldots \supset Ã^m \ldots\ldots \supset E$.

Proof: The proof is similar to that of proposition (3.14).

Proposition 3.16 : Let f be a group homomorphism’s of a cyclic M- fuzzy group Ã. Then the image of Ã under f is a cyclic M- fuzzy group.

Proof: It is well known that in the theory of classical groups, the image of any cyclic group is a cyclic group, and a homomorphic image of a fuzzy subgroup is an fuzzy subgroup[5 , Proposition 5.8] . From these results and Definition (2.8) , it is clearly seen that the image of Ã under f is a cyclic M- fuzzy group.

Proposition 3.17 : Let { $\tilde{A}^m$, $\tilde{A}^{m-1}$,………… Ã } be a finite M- cyclic fuzzy group family. Then $\tilde{A}^m \times \tilde{A}^{m-1} \times \ldots\ldots\ldots\ldots\ldots \times \tilde{A} = \tilde{A}^m$.

Proof: Using the definition of the product of M- fuzzy groups and Theorem 3.12, it is proved easily.

Conclusion

In this paper , we extend cyclic M- fuzzy groups to M- fuzzy sets to define the concept of cyclic M- fuzzy groups. We give a sufficient condition for a M- fuzzy subset to be a cyclic M-fuzzy group family and investigate its structure properties with applications.

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