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Exponential function-its Applications

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Exponential functions are best applied in real world problems especially when the causation factor itself propagates against another factor. Its omnipresence in pure and applied mathematics has led the famous mathematician W. Rudin to say that the exponential function is "the most important function in mathematics". In applied settings, exponential functions model a relationship in which a constant change in the independent variable gives the same proportional change (i.e. percentage increase or decrease) in the dependent variable. Such a situation occurs widely in both natural and social sciences; thus, the exponential function also appears in various of contexts within physics, chemistry, engineering, mathematical biology, and economics.

By definition, a function of the form $f(x)=b^x$ in which the input variable *x* occurs as an exponent is called an Exponential function. Another form of the function is $f(x)=b^{x+c}$, where is c is a constant, and can be rewritten as $f(x)=ab^x$, where $a=b^c$.

Unique properties of exponential functions are what make it special. These functions are characterized by the fact that the growth rate of such a function (i.e., its derivative) is directly proportional to the value of the function. The constant of proportionality of this relationship is the natural logarithm of the base b:

$$\frac{d}{dx}(b^x)=b^x \log_e b$$

The constant $e \approx 2.71828...$, a transcendental number is the unique base for which the constant of proportionality is 1, so that the function's derivative is itself:

$$\frac{d}{dx}(e^x)=e^x \log_e e=e^x$$

The exponential function satisfies the fundamental multiplicative identity

$e^{x+y}=e^x.e^y$, for all $x, y \in \Re$. ($\Re$, the set of real numbers)

In fact, this identity extends to complex valued exponents also. It can be shown that complete set of continuous, non-zero solutions of the functional equation $f(x+y)=f(x).f(y)$ are the exponential functions, $f:\Re\to\Re,\ x\to b^x$, with $b>0$.

The argument of the exponential function can be any real or complex number or even an entirely different kind of mathematical object (*e.g.*, a matrix).

Geometrical visualization of the graph provides greater insights upon the exponential functions. The curve of $y=e^x$ is upward-sloping, and increases faster as x increases. The graph always lies above the x - axis but can get arbitrarily close to it for negative x; thus, the x-axis is a horizontal asymptote. As the rate of increase of the function at x is equal to the value of the function at x, the slope of the graph at any point is the height of the function at that point.

In strict analytical sense, the exponential function $\exp:\mathbb{C}\to\mathbb{C}$,($\mathbb{C}$, the set of complex numbers)can be defined by the following power series:

$$\exp:(e^x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^8}{24}+..........$$

Since the radius of convergence of this power series is infinite, this definition is applicable to all complex numbers. The constant *e* is then defined as,

$\exp(e^1)=\sum_{k=0}^{\infty}\frac{1^k}{k!}$ or it can be defined as the solution y to the equation $x=\int_1^y\frac{dt}{t}$. It can also be defined as $e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

As it satisfies the basic exponential identity , exp(x+y)= exp(x) .exp(y) ,we write it as e^x.

In Complex plane, as in the real case, the exponential function can be defined in several equivalent forms. The most common definition of the complex exponential function parallels the power series definition for real, where the real variable is replaced by a complex variable z, i.e. $\exp(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}$. Termwise multiplication of two copies of these power series in the Cauchy sense, permitted by Merten's theorem, shows that the defining multiplicative property of exponential functions continues to hold for all complex arguments: for all The definition of the complex exponential function in turn leads to the appropriate definitions extending the trigonometric functions to complex arguments. In particular, when $z=it$ (t real), from the series definition

$$\exp(it)=1+it+\frac{it}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+\frac{(it)^5}{5!}+\frac{(it)^6}{6!}+\frac{(it)^7}{7!}+.............$$

$$=1+it-\frac{t}{2!}-\frac{it^3}{3!}+\frac{t^4}{4!}+\frac{it^5}{5!}-\frac{t^6}{6!}-\frac{it^7}{7!}+.............$$

$$=1+it-\frac{t^2}{2!}-\frac{it^3}{3!}+\frac{t^4}{4!}+\frac{it^5}{5!}-\frac{t^6}{6!}-\frac{it^7}{7!}+.............$$

$$=\left(1-\frac{t}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+.............\right)+i\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+.............\right)$$

In this expansion, the rearrangement of the terms into real and imaginary parts is justified by the absolute convergence of the series. The real and imaginary parts of the above expression in fact correspond to the series expansions of $\cos t$ and $\sin t$ respectively. This correspondence provides motivation for defining cosine and sine functions for all complex arguments in terms of $\exp(\pm it)$ and the equivalent power series.

$$\cos t=\frac{1}{2}(e^{it}+e^{-it})=\sum_{k=0}^{\infty}(-)^k\frac{t^{2k}}{(2k)!} \quad \text{and} \quad \sin t=\frac{1}{2}(e^{it}-e^{-it})=\sum_{k=0}^{\infty}(-)^k\frac{t^{2k+1}}{(2k+1)!}, \quad \forall t\in\mathbb{C}$$

The functions exp, cos, and sin so defined have infinite radii of convergence by the ratio test and are therefore entire functions (*i.e.*, holomorphic on $\mathbb{C}$). The range of the exponential function is $\mathbb{C}-\{0\}$, while the ranges of the complex sine and cosine functions are both $\mathbb{C}$ in its entirety, in accord with Picard's theorem, which asserts that the range of a non-constant entire function is either all of $\mathbb{C}$, or $\mathbb{C}$ excluding one lacunary value.

These definitions for the exponential and trigonometric functions lead to Euler's formula:

$$e^{iz}=\cos z+i\sin z, \quad z\in\mathbb{C}.$$

Alternatively, if $z=x+iy$, where x and y are real, then

$$\exp(z)=\exp(x+iy)=\exp(x)\exp(iy)=\exp(x)(\cos y+i\sin y)$$

where exp, cos, and sin on the right- hand side are to be interpreted as functions of a real variable. For $t\in\mathbb{R}$, the relationship $\overline{\exp(it)}=\exp(-it)$ holds, so that $|\exp(it)|=1$, for real t and $t\to\exp(it)$, maps the real line (mod 2π) to the unit circle. Based on the relationship between $\exp(it)$ and the unit circle, it is easy to see that, restricted to real arguments, the definitions of sine and cosine given above coincide with their more elementary definitions based on geometric notions. More over the complex exponential function is periodic with period $2\pi i$ and $\exp(z+2\pi ik)=\exp(z)$ holds for all $z\in\mathbb{C}$, $k\in\mathbb{Z}$ ($\mathbb{Z}$, the set of integers).

Exponential functions have already found its applications in various fields of scientific explorations. It gave a mathematical form to the hitherto unsolvable problems, thus helping us arrive at the solutions. Few scenarios are mentioned below categorically against respective fields.

Biology

The number of micro organisms in a culture will increase exponentially until an essential nutrient is exhausted. Typically the first organism splits into two daughter organisms, who then each split to form four, who split to form eight, and so on. Because exponential growth indicates constant growth rate, it is frequently assumed that exponentially growing cells are at a steady state.

Physics

Avalanche breakdown in a dielectric material - A free electron becomes sufficiently accelerated by an externally applied electrical field that it frees up additional electrons as it collides with atoms or molecules of the dielectric media. These secondary electrons also are accelerated, creating larger numbers of free electrons. The resulting exponential growth of electrons and ions may rapidly lead to complete dielectric breakdown of the material.

Nuclear chain reaction - Each uranium nucleus that undergoes fission produces multiple neutrons, each of which can be absorbed by adjacent uranium atoms, causing them to fission in turn. If the probability of neutron absorption exceeds the probability of neutron escape (a function of the shape and mass of the uranium), $k > 0$ and so the production rate of neutrons and induced uranium fissions increases exponentially, in an uncontrolled reaction. "Due to the exponential rate of increase, at any point in the chain reaction 99% of the energy will have been released in the last 4.6 generations. It is a reasonable approximation to think of the first 53 generations as a latency period leading up to the actual explosion, which only takes 3–4 generations.

Finance
Compound interest at a constant interest rate provides exponential growth of the capital.

Pyramid schemes or Ponzi schemes also show this type of growth resulting in high profits for a
few initial investors and losses among great numbers of investors.

Computer technology

Processing power of computers - In computational complexity theory, computer algorithms of exponential complexity require an exponentially increasing amount of resources (e.g. time, computer memory) for only a constant increase in problem size.

Similarly, following Few analogies will give us a perspective upon how problems of exponents are profounded from early times.

Rice on a chessboard:
According to an old legend, vizier Sissa Ben Dahir presented an Indian King Sharim with a beautiful, handmade chessboard. The king asked what he would like in return for his gift and the courtier surprised the king by asking for one grain of rice on the first square, two grains on the second, four grains on the third etc. The king readily agreed and asked for the rice to be brought. All went well at first, but the requirement for $2\ n - 1$ grains on the nth square demanded over a million grains on the 21st square, more than a million million (aka trillion) on the 41st and there simply was not enough rice in the whole world for the final squares. The second half of the chessboard is the time when an exponentially growing influence is having a significant economic impact on an organization's overall business strategy.

Water lily:
French children are told a story in which they imagine having a pond with water lily leaves floating on the surface. The lily population doubles in size every day and if left unchecked will smother the pond in 30 days, killing all the other living things in the water. Day after day the plant seems small and so it is decided to leave it to grow until it half covers the pond, before cutting it back. They are then asked on what day half coverage will occur. This is revealed to be the 29th day, and then there will be just one day to save the pond.