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A New Modular Relation of Ratio's of Ramanujan Quantity of degree 5

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Abstract

In this paper, We establish a new modular equations of ratios of Ramanujan Quantities $R(1,2,4;q) := \frac{\prod_{n=0}^{\infty}(1-qq^8)(1-q^3q^8)}{\prod_{n=0}^{\infty}(1-q^2q^8)(1-q^2q^8)}$ (established by Nikos Bagis) of degree 5 for $n = 2, 3, 5$ and 7 and their explicit evaluations.

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1. INTRODUCTION

In Chapter 16 of his second notebook [1], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \mid ab \mid < 1, \tag{1.1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_\infty \tag{1.4}$$

and

$$\chi(q) := (-q; q^2)_\infty. \tag{1.5}$$

In [6],[7] Nikos Bagis define Ramanujan Quantities $R(a, b, p; q)$ as

$$R(a, b, p; q) = q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{\prod_{n=0}^{\infty}(1 - q^a q^{np})(1 - q^{p-a} q^{np})}{\prod_{n=0}^{\infty}(1 - q^b q^{np})(1 - q^{p-b} q^{np})}, \tag{1.6}$$

where a, b,and p are positive rationales such that $a + b < p$. General Theorem such

$$\frac{q^{B-A}}{1 - a_1 b_1 +} \frac{(a_1 - b_1 q_1)(b_1 - a_1 q_1)}{(1 - a_1 b_1)(q_1^2 + 1) +} \frac{(a_1 - b_1 q_1^3)(b_1 - a_1 q_1^3)}{(1 - a_1 b_1)(q_1^4 + 1) +} \cdots$$
$$= \frac{\prod_{n=0}^{\infty}(1 - q^a q^{np})(1 - q^{p-a} q^{np})}{\prod_{n=0}^{\infty}(1 - q^b q^{np})(1 - q^{p-b} q^{np})} \tag{1.7}$$

where $a_1 = q^A$, $b_1 = q^B$, $q_1 = q^{A+B}$, $a = 2A + 3p/4$, $2B + p/4$, and $p = 4(A + B)$, $|q| < 1$, are proved.

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a, b; c; x)$ is defined by

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)$ for any positive integer n, and $| x |< 1$.

Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{1.8}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{1.9}$$

where $0 < x < 1$.

Let r denote a fixed natural number and assume that the following relation holds:

$$r \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.10}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.10). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

2. PRELIMINARY RESULTS

Definition 2.1. *[7]*

$$[a,p;q] = (q^{p-a};q^p)_\infty (q^a;q^p)_\infty \tag{2.1}$$

where $q = e^{-\pi\sqrt{r}}$ *and* $a, p, r > 0$.

Definition 2.2. *[7]*

$$R(a,b,p;q) := q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{[a,p;q]}{[b,p;q]} \tag{2.2}$$

Lemma 2.1. *[1, Ch. 16, Entry 25, p.40]*

$$\psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2) \tag{2.3}$$

Lemma 2.2. *[1, Ch. 17, Entry 10-11, p.122-123]*

$$\varphi(-q^2) = \sqrt{z}(1-\alpha)^{1/8} \tag{2.4}$$

$$\psi(-q) = \sqrt{\frac{1}{2}z}\{\alpha(1-\alpha)q^{-1}\}^{1/8} \tag{2.5}$$

where $q = e^{-y}$

Lemma 2.3. *[4] If* $P := \dfrac{\psi(q)}{q^{1/2}\psi(q^5)}$ *and* $Q := \dfrac{\psi(q^2)}{q\psi(q^{10})}$, *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 4 = P^2 + \frac{5}{P^2}. \tag{2.6}$$

Lemma 2.4. *[2, Ch. 25, Entry 66, p.233] If* $P = \dfrac{\psi(q)}{q^{1/2}\psi(q^5)}$ *and* $Q = \dfrac{\psi(q^3)}{q^{3/2}\psi(q^{15})}$, *then*

$$PQ + \frac{5}{PQ} = -\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 3\left(\frac{P}{Q} + \frac{Q}{P}\right). \tag{2.7}$$

Lemma 2.5. *[5] If* $P := \dfrac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}$ *and* $Q := \dfrac{\psi(q^4)}{q^2\psi(q^{20})}$, *then*

$$\begin{aligned} &\frac{P^4}{Q^4} + \frac{Q^4}{P^4} + 24\left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right) + 8\left(P^2Q^2 + \frac{25}{P^2Q^2}\right) - 20\left(Q^2 + \frac{5}{Q^2}\right) + 120 \\ &+ 3\left(P^4 + \frac{25}{P^4}\right) - 32\left(P^2 + \frac{5}{P^2}\right) = P^4\left(Q^2 + \frac{3}{Q^2}\right) + \frac{5}{P^4}\left(3Q^2 + \frac{25}{Q^2}\right). \end{aligned} \tag{2.8}$$

Lemma 2.6. *[5] If* $P := \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ *and* $Q := \dfrac{\psi(-q^5)}{q^{5/2}\psi(-q^{25})}$ *then*

$$\begin{aligned}&\frac{Q^3}{P^3} - \frac{5Q^2}{P^2} - \frac{15Q}{P} - 5\left(PQ + \frac{5}{PQ}\right) - 5\left(Q^2 + \frac{5}{P^2}\right)\\ &- P^2Q^2 + \frac{5^2}{P^2Q^2} - 15 = 0.\end{aligned} \tag{2.9}$$

Lemma 2.7. *[5] If* $P := \dfrac{\psi(-q)\psi(-q^7)}{q^4\psi(-q^5)\psi(-q^{35})}$ *and* $Q := \dfrac{\psi(-q)\psi(-q^{35})}{q^{-3}\psi(-q^5)\psi(-q^7)}$, *then*

$$\begin{aligned}&Q^4 - \frac{1}{Q^4} + 14\left[\left(Q^3 + \frac{1}{Q^3}\right) + \left(Q^2 - \frac{1}{Q^2}\right) + 10\left(Q + \frac{1}{Q}\right)\right] + P^3 + \frac{5^3}{P^3}\\ &+ 7\left\{\left(P^2 + \frac{5^2}{P^2}\right)\left(Q + \frac{1}{Q}\right) + \left(P + \frac{5}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) + 9\right]\right\} = 0.\end{aligned} \tag{2.10}$$

3. NEW MODULAR RELATIONS OF RATION OF RAMANUJAN QUANTITIES OF $\dfrac{R(1,2,4;Q)}{R(1,2,4;Q^5)}$

In this section, we obtain certain modular relations between $U := \dfrac{R(1,2,4;q)}{R(1,2,4;q^5)}$ and $V := \dfrac{R(1,2,4;q^n)}{R(1,2,4;q^{5n})}$ for $n = 2, 3, 5$ and 7.

Theorem 3.1. *If* $U = \frac{R(q)}{R(q^5)}$ *and* $V = \frac{R(q^2)}{R(q^{10})}$, *then*

$$2\left\{U^2 + \frac{1}{U^2}\right\} + \left\{U^2 - \frac{1}{U^2}\right\}\left\{V + \frac{1}{V}\right\} = \left\{V + \frac{1}{V}\right\}^2. \tag{3.1}$$

Proof. Employing the definition (2.1) and (2.2) with $a = 1, b = 2$ *and* $p = 4$, we get

$$R(q) := R(1,2,4;q) = q^{1/8}\frac{(q;q^4)_\infty(q^3;q^4)_\infty}{(q^2;q^4)_\infty(q^2;q^4)_\infty} \tag{3.2}$$

Using the equations (1.1), (1.2) and (1.3), then the above equation can be written as

$$R(q) = q^{1/8}\frac{f(-q,-q^3)}{f(-q^2,-q^2)} = q^{1/8}\frac{\psi(-q)}{\varphi(-q^2)} \tag{3.3}$$

by using lemma(2.1), the equation (3.3) can be expressed as,

$$R(q) = q^{1/8}\frac{\psi(q^2)}{\psi(q)} \tag{3.4}$$

by substituting the above equation (3.4) in the lemma (2.3), we get

$$\begin{aligned}&(U^4V - V - U^2 + 2V^2 - V^4U^2 + 2U^4V^2 - 2V^2U^2 - V^3 + V^3U^4)\\&(U^4V - V + U^2 - 2V^2 + V^4U^2 - 2U^4V^2 + 2V^2U^2 - V^3 + V^3U^4) = 0\end{aligned} \tag{3.5}$$

On observing the behavior of the above factor near $q = 0$ and we can find a neighborhood about the origin in which the first factor is zero and the other is non-zero. And the first factor vanishes identically by the Identity theorem. This proves the theorem. □

Theorem 3.2. *If* $U = \frac{R(q)}{R(q^5)}$ *and* $V = \frac{R(q^3)}{R(q^{15})}$, *then*

$$\left\{\frac{U}{V} + \frac{V}{U}\right\}^2 + 4 = 3\left\{\frac{U}{V} + \frac{V}{U}\right\} + \left\{UV + \frac{1}{UV}\right\}. \tag{3.6}$$

Proof. Employing the above equation (3.4) in the lemma (2.4), we get (3.6). □

Theorem 3.3. *If* $U = \frac{R(q)}{R(q^5)}$ *and* $V = \frac{R(q^4)}{R(q^{20})}$, *then*

$$\begin{aligned}&+20\left\{U^4 + \frac{1}{U^4}\right\} - 32\left\{U^2 + \frac{1}{U^2}\right\} + \left\{V^4 + \frac{1}{V^4}\right\}\\&+\left\{V^3 + \frac{1}{V^3}\right\}\left[\left\{U^4 - \frac{1}{U^4}\right\} + 4\left\{U^2 - \frac{1}{U^2}\right\}\right]\\&+2\left\{V^2 + \frac{1}{V^2}\right\}\left[3\left\{V^4 + \frac{1}{V^4}\right\} - 8\right] + 5\left\{V + \frac{1}{V}\right\}\\&\times\left[3\left\{U^4 - \frac{1}{U^4}\right\} - 4\left\{U^2 - \frac{1}{U^2}\right\}\right] = 0.\end{aligned} \tag{3.7}$$

Proof. Employing the above equation (3.4) in the lemma (2.5), we get

$$\begin{aligned}&(-V - 15V^5 - V^7 - 15V^3 + 6V^6 + U^4 + 20V^4 + 6V^2 + VU^8 + V^7U^8 + V^8U^4\\&- 4V^7U^2 + 20V^3U^2 - 4VU^2 + 4VU^6 + 15V^3U^8 - 20V^3U^6 + 4V^7U^6 - 20V^5U^6\\&+ 15V^5U^8 - 16V^6U^4 - 16U^4V^2 + 30V^4U^4 + 20U^8V^4 - 32U^6V^4 - 32V^4U^2\\&+ 6U^8V^6 + 6U^8V^2 + 20V^5U^2)(-V - 15V^5 - V^7 - 15V^3 - 6V^6 - U^4 - 20V^4\\&- V^8U^4 + 20V^5U^2 - 4V^7U^2 + 20V^3U^2 - 4VU^2 + 4VU^6 + 15V^3U^8 - 20V^3U^6\\&+ 4V^7U^6 - 20V^5U^6 + 15V^5U^8 + 16V^6U^4 + 16U^4V^2 - 30V^4U^4 - 20U^8V^4\\&+ 32U^6V^4 + 32V^4U^2 - 6U^8V^6 - 6V^2 - 6U^8V^2 + VU^8 + V^7U^8) = 0\end{aligned} \tag{3.8}$$

On observing the behavior of the above factor near $q = 0$ and we can find a neighborhood about the origin in which the first factor is zero and the other is non-zero. And the first factor vanishes identically by the Identity theorem. This proves the theorem. □

Theorem 3.4. *If* $U = \frac{R(q)}{R(q^{25})}$ *and* $V = \frac{R(q)R(q^{25})}{R^2(q^5)}$, *then*

$$\begin{aligned}&\left\{V^4 + \frac{1}{V^4}\right\} - 14\left\{V^3 + \frac{1}{V^3}\right\} - 4\left\{V^2 + \frac{1}{V^2}\right\} + \left\{V^+\frac{1}{V}\right\} \\ &+ 5\left\{U + \frac{1}{U}\right\}\left[\left\{V^3 + \frac{1}{V^3}\right\} + 2\left\{V^2 + \frac{1}{V^2}\right\} - 2\left\{V^+\frac{1}{V} + 3\right\}\right] \\ &+ 28\left\{U^2 + \frac{1}{U^2}\right\}\left[\left\{V^2 + \frac{1}{V^2}\right\} + \left\{V^+\frac{1}{V}\right\} + 1\right] + 42 = 0.\end{aligned} \tag{3.9}$$

Proof. Employing the above equation (3.4) in the lemma (2.6), we get (3.9). □

Theorem 3.5. *If* $U = \frac{R(q)R(q^7)}{R(q^5)R(q^{35})}$ *and* $V = \frac{R(q)R(q^{35})}{R(q^5)R(q^7)}$, *then*

$$\begin{aligned}&\left\{U^4 + \frac{1}{U^4}\right\} + 28\left\{U^2 + \frac{1}{U^2}\right\} - 42\left\{U + \frac{1}{U}\right\} \\ &- \left\{V^3 + \frac{1}{V^3}\right\} + 28\left\{V^2 + \frac{1}{V^2}\right\} - 7\left\{V + \frac{1}{V}\right\} \\ &= 28\left[\left\{UV + \frac{1}{UV}\right\} + \left\{\frac{U}{V} + \frac{V}{U}\right\}\right] + 7\left\{U + \frac{1}{U}\right\}\left\{V^2 + \frac{1}{V^2}\right\}.\end{aligned} \tag{3.10}$$

Proof. Employing the above equation (3.4) in the lemma (2.7), we get (3.10). □

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