International Journal of Pure and Applied Mathematical Sciences.
ISSN 0972-9828 Volume 15, Number 1 (2022), pp. 21-28

http://www.ripublication.com

Stability of Additive - Quadratic Functional Equations in Several Variables

Manoj Kumar[1], Sushma Devi[2], Asha Rani[3]

[1, 3]Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India.

[2]Department of Mathematics, Kanya Mahavidyalaya, Kharkhoda, Haryana, India.

Abstract

In this paper, we prove the Hyers–Ulam – Rassias stability of generalized n-variable mixed type of additive and quadratic functional equations in Banach spaces by direct method.

MSC: 39B72, 68U10, 94A08, 47H10, 39B52.

Keywords: Hyers–Ulam–Rassias stability, additive functional, quadratic functional equations.

1. INTRODUCTION

In 1940, Ulam asked the following stability problem: “Let G be a group and H be a group with metric d(., .). Given $\varepsilon > 0$, does there exists $\delta > 0$ such that if a mapping $f : G \rightarrow H$ satisfies $d(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G$, then there exists a homomorphism $a: G \rightarrow H$ with $d(f(x), a(x)) < \varepsilon$ for all $x \in G$?”

In 1941, Hyers [6] answered it. Later, it was developed as Hyers – Ulam stability by Rassias [11], Rassias [10] and Gavruta [5].

M. Ramdoss, D. Pachaiyappan, C. Park and J. R. Lee [13] introduced a new generalized n-variable mixed type functional equation of the form

$$\sum_{i=1,j=i+1}^{n-1}\left(f\left(kx_i + x_j\right)\right) + f(kx_n + x_1) - k\left[\sum_{i=1,j=i+1}^{n-1}\left(f\left(x_i + x_j\right)\right) + f(x_n + x_1)\right]$$

$$= \frac{(1-k)^2}{2}\sum_{i=1}^{n}(f(x_i) + f(-x_i)) - \frac{1-k}{k^2-k}\sum_{i=1}^{n}(k^2 f(x_i) - f(kx_i)) \quad (1.1)$$

for positive integer n, k ≥ 2 and proved its Hyers – Ulam stability in Fuzzy Modular spaces.

The functional equations

$$f(x+y) = f(x) + f(y) \text{ and } f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

are called the additive and quadratic functional equations respectively. Every solution of additive and quadratic functional equations is said to be additive mapping and quadratic mapping respectively. In this paper, we will obtain Hyers–Ulam – Rassias stability of generalized n-variable mixed type of additive and quadratic functional equations in Banach spaces.

2. MAIN RESULTS

Lemma 2.1[13]. Let a mapping f : U → V satisfy functional equation (1.1). If f is an even mapping, then f is quadratic and if f is an odd mapping, then f is additive.

Denote by

$$S(f(x_1, x_2, \ldots, x_n)) = \sum_{i=1,j=i+1}^{n-1}\left(f\left(kx_i + x_j\right)\right) + f(kx_n + x_1) - k\left[\sum_{i=1,j=i+1}^{n-1}\left(f\left(x_i + x_j\right)\right) + f(x_n + x_1)\right] - \frac{(1-k)^2}{2}\sum_{i=1}^{n}(f(x_i) + f(-x_i)) - \frac{1-k}{k^2-k}\sum_{i=1}^{n}(k^2 f(x_i) - f(kx_i)), \quad (2.1)$$

for all $n \in N$, $k \geq 2$.

Theorem 2.2. Let X be a vector space and Y be a Banach space and ϕ: $X^n \rightarrow \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty}\frac{1}{k^{2i}}\phi\left(k^i x, 0, 0, \ldots, 0\right) \text{ converges} \quad (2.2)$$

and $\lim_{m\to\infty} \frac{1}{k^{2m}} \phi(k^m x_1, k^m x_{2,\ldots} k^m x_n) = 0,$ (2.3)

for all $x_i \in$ X, i = 1, 2, . . ., n.

Suppose that f is an even mapping and satisfies

$\| \mathrm{S}f(x_1, x_2, \ldots, x_n) \| \leq \phi(x_1, x_2, \ldots, x_n)$ (2.4)

for all $x_i \in$ X, i = 1, 2, . . ., n.

Then there exists a unique quadratic mapping Q : X → Y satisfying (1.1) and the inequality

$\|f(x) - Q(x)\| \leq \frac{1}{k^2-k} \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \phi\left(k^i x, 0, 0, \ldots, 0\right)$ (2.5)

and function Q is given by

$Q(x) = \lim_{m\to\infty} k^{-2m} f(k^m x)$ for all $x \in$ X. (2.6)

Proof: Letting$(x_1, x_2, \ldots, x_n)$ by $(x, 0, \ldots, 0)$ in (2.4).

$\Rightarrow \left\|f(kx) + f(x) - k[2f(x)] - \frac{(1-k)^2}{2}\left(2f(x)\right) - \frac{1-k}{k^2-k}[k^2 f(x) - f(kx)]\right\| \leq \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|f(kx) + f(x) - 2kf(x) - (1-k)^2\left(f(x)\right) - \frac{1-k}{k^2-k} k^2 f(x) + \frac{1-k}{k^2-k} f(kx)\right\|$

$\leq \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\| (1 + \frac{1-k}{k^2-k}) f(kx) + (1 - 2k - (1-k)^2 - \frac{1-k}{k^2-k} k^2) f(x)\right\| \leq \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|(\frac{k^2-2k+1}{k^2-k}) f(kx) + \frac{1}{k^2-k}(k^2 - k - 2k(k^2 - k) - (k^2 - k)(1-k)^2 - (1-k)k^2) f(x)\right\|$

$\leq \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|(\frac{k^2-2k+1}{k^2-k}) f(kx) + (\frac{2k^3-k^2-k^4}{k^2-k}) f(x)\right\| \leq \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|(\frac{k^2-2k+1}{k^2-k})[f(kx) - k^2 f(x)]\right\| \leq \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|(\frac{(k-1)^2}{k(k-1)})[f(kx) - k^2 f(x)]\right\| \leq \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|f(kx) - k^2 f(x)\right\| \leq \frac{k}{k-1} \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|f(x) - k^{-2} f(kx)\right\| \leq \left(\frac{k}{k-1}\right) \frac{1}{k^2} \phi(x, 0, 0, \ldots, 0)$

$\Rightarrow \left\|f(x) - \frac{1}{k^2} f(kx)\right\| \leq \left(\frac{k}{k-1}\right) \frac{1}{k^2} \phi(x, 0, 0, \ldots, 0)$ (2.7)

Replacing x by kx, we obtain

$\Rightarrow \left\| f(kx) - \frac{1}{k^2} f(k^2 x) \right\| \leq \left(\frac{k}{k-1}\right) \frac{1}{k^2} \phi(kx, 0, 0, \ldots, 0)$

$\Rightarrow \left\| \frac{1}{k^2} f(kx) - \frac{1}{k^4} f(k^2 x) \right\| \leq \left(\frac{k}{k-1}\right) \frac{1}{k^{2+2}} \phi(kx, 0, 0, \ldots, 0)$

Again replacing x with k^2x in (2.7), we obtain

$\Rightarrow \left\| f(k^2 x) - \frac{1}{k^2} f(k^3 x) \right\| \leq \left(\frac{k}{k-1}\right) \frac{1}{k^2} \phi(k^2 x, 0, 0, \ldots, 0)$

$\Rightarrow \left\| \frac{1}{k^4} f(k^2 x) - \frac{1}{k^6} f(k^3 x) \right\| \leq \left(\frac{k}{k-1}\right) \frac{1}{k^{2+4}} \phi(k^2 x, 0, 0, \ldots, 0)$

Generalizing the above inequality, we obtain

$$\left\| \frac{1}{k^{2l}} f(k^l x) - \frac{1}{k^{2m}} f(k^m x) \right\| \leq \sum_{j=l}^{m} \left(\frac{k}{k-1}\right) \frac{1}{k^{2j+2}} \phi\left(k^j x, 0, 0, \ldots, 0\right), \tag{2.8}$$

for all non-negative integers m and l with $m > l$ and for all $x \in X$. It follows from (2.8) that the sequence $\{\frac{f(k^m x)}{k^{2m}}\}$ is a Cauchy sequence for all x ∈ X. Since Y is complete, the sequence $\{\frac{f(k^m x)}{k^{2m}}\}$ converges.

So define a function $Q : X \to Y$ by

$$Q(x) = \lim_{m\to\infty} \left\{\frac{f(k^m x)}{k^{2m}}\right\}, \tag{2.9}$$

and for all $x \in X$.

$$\begin{aligned}\left\| SQ(x_1, x_2, \ldots, x_n) \right\| &= \Big\| \sum_{i=1,j=i+1}^{n-1} \left(Q\left(kx_i + x_j\right)\right) + Q(kx_n + x_1) \\ &\quad -k\left[\sum_{i=1,j=i+1}^{n-1} \left(Q\left(x_i + x_j\right)\right) + Q(x_n + x_1)\right] \\ &\quad - \frac{(1-k)^2}{2} \sum_{i=1}^{n} (Q(x_i) + Q(-x_i)) - \frac{1-k}{k^2-k} \sum_{i=1}^{n} [k^2 Q(x_i) - Q(kx_i)] \Big\| \\ &= \Big\| \lim_{m\to\infty} [\sum_{i=1,j=i+1}^{n-1} \frac{1}{k^{2m}} f(k^m (kx_i + x_j) + \frac{1}{k^{2m}} f(k^m (kx_n + x_1)] \\ &\quad - k\Big[\sum_{i=1,j=i+1}^{n-1} \frac{1}{k^{2m}} f(k^m (x_i + x_j\Big) + \frac{1}{k^{2m}} f(k^m (kx_n + x_1))] \\ &\quad - \frac{(1-k)^2}{2} [\sum_{i=1}^{n-1} \frac{2}{k^{2m}} f(k^m (x_i)] \frac{1}{k^{2m}} (\frac{1-k}{k^2-k}) [\sum_{i=1}^{n-1} (f(k^m x_i) - f(k^m x_1))] \\ &= \lim_{m\to\infty} \frac{1}{k^{2m}} \left\| Sf(k^m x_1, k^m x_{2,\ldots,} k^m x_n) \right\| \\ &\leq \lim_{m\to\infty} \frac{1}{k^{2m}} \left\| \phi(k^m x_1, k^m x_{2,\ldots,} k^m x_n) \right\| = 0,\end{aligned}$$

for all $x_i \in X$, $i = 1, 2, \ldots, n$.

So $SQ(x_1, x_2, \ldots, x_n) = 0$.

Hence the mapping $Q : X \to Y$ is quadratic.

Moreover, letting $l = 0$ and limit as $m \to \infty$ in (2.8), we have (2.5).

Now, let $T : X \to Y$ another quadratic mapping satisfying (2.4) and (1.1).

$$\|Q(x) - T(x)\| = \frac{1}{k^{2m}} \|Q(k^m x) - T(k^m x)\|.$$

$$\Rightarrow \quad \|Q(x) - T(x)\| \leq \frac{1}{k^{2m}} [\|Q(k^m x) - f(k^m x)\| + \|T(k^m x) - f(k^m x)\|],$$

which tends to zero as $m \to \infty$ for all $x \in X$. This proves uniqueness of Q. So there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.5).

Corollary 2.3. Let $p < 2$ and θ be positive real numbers and let $f : X \to Y$ be a mapping such that

$$\|Sf(x_1, x_2, \dots, x_n)\| \leq \theta (\|x_1\|^p + \|x_2\|^p + \cdots \|x_n\|^p)$$

for all $x_i \in X$, $i = 1, 2, \dots, n$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \left(\frac{k}{(k-1)(k^2 - k^p)}\right) \theta (\|x\|^p)$$

Theorem 2.4. Let X be a vector space and Y be a Banach space and $\phi : X^n \to \mathbb{R}^+$ be a function such that

$\sum_{i=0}^{\infty} \frac{1}{k^i} \phi(k^i x, 0, 0, \dots, 0)$ converges (2.10)

and $\lim_{m\to\infty} \frac{1}{k^{2m}} \phi(k^m x_1, k^m x_2, \dots, k^m x_n) = 0$, (2.11)

for all $x_i \in X$, $i = 1, 2, \dots, n$. Suppose that f is an odd function and satisfies

$\|Sf(x_1, x_2, \dots, x_n)\| \leq \phi(x_1, x_2, \dots, x_n)$, (2.12)

for all $x_i \in X$, i = 1, 2, . . ., n. Then there exists a unique additive mapping $Q : X \to Y$ satisfying

$$\sum_{i=1, j=i+1}^{n-1} \left(f(kx_i + x_j)\right) + f(kx_n + x_1) - k\left[\sum_{i=1, j=i+1}^{n-1} \left(f(x_i + x_j)\right) + f(x_n + x_1)\right] = \frac{1-k}{k^2 - k} \sum_{i=1}^{n} k^2 f(x_i) - f(kx_i), \tag{2.13}$$

for $n \in N$ and the inequality

$\|f(x) - Q(x)\| \leq \frac{k}{k-1} \sum_{i=0}^{\infty} \frac{1}{k^i} \phi(k^i x, 0, 0, \dots, 0)$. (2.14)

The function Q is given by

$Q(x) = \lim_{m\to\infty} k^{-m} f(k^m x)$ for all $x \in X$.

Proof: Replacing $(x_1, x_2, \ldots, x_n)$ by $(x, 0, \ldots, 0)$ in (2.12), we obtain

$$\| f(kx) + f(x) - k(f(x) + f(x)) - \tfrac{1-k}{k^2-k}[k^2 f(x) - f(kx)] \| \leq \phi(x,0,0,\ldots,0)$$

$$\Rightarrow \| f(kx) + f(x) - 2kf(x) - \tfrac{1-k}{k^2-k} k^2 f(x) + \tfrac{1-k}{k^2-k} f(kx) \| \leq \phi(x,0,0,\ldots,0)$$

$$\Rightarrow \left\| \left(1 + \tfrac{1-k}{k^2-k}\right) f(kx) + (1 - 2k - \tfrac{1-k}{k^2-k} k^2) f(x) \right\| \leq \phi(x,0,0,\ldots,0)$$

$$\Rightarrow \| (\tfrac{k^2-2k+1}{k^2-k}) f(kx) + \tfrac{1}{k^2-k}(k^2 - k - 2k(k^2-k) - (1-k)k^2) f(x) \|$$

$$\leq \phi(x,0,0,\ldots,0)$$

$$\Rightarrow \| (\tfrac{k^2-2k+1}{k^2-k}) f(kx) - (\tfrac{k(k^2-2k+1)}{k^2-k} f(x) \| \leq \phi(x,0,0,\ldots,0)$$

$$\Rightarrow \| (\tfrac{(k-1)^2}{k(k-1)})[f(kx) - kf(x)] \| \leq \phi(x,0,0,\ldots,0)$$

$$\Rightarrow \| f(kx) - kf(x) \| \leq \tfrac{k}{k-1} \phi(x,0,0,\ldots,0)$$

$$\Rightarrow \| f(x) - \tfrac{1}{k} f(kx) \| \leq \left(\tfrac{k}{k-1}\right) \tfrac{1}{k} \phi(x,0,0,\ldots,0)$$

Replacing x by kx, we obtain

$$\| f(kx) - \tfrac{1}{k} f(k^2 x) \| \leq \left(\tfrac{k}{k-1}\right) \tfrac{1}{k} \phi(kx,0,0,\ldots,0)$$

$$\Rightarrow \quad \| \tfrac{1}{k} f(kx) - \tfrac{1}{k^2} f(k^2 x) \| \leq \left(\tfrac{k}{k-1}\right) \tfrac{1}{k^{1+1}} \phi(kx,0,0,\ldots,0).$$

Again replacing x with $k^2 x$, we obtain

$$\| f(k^2 x) - \tfrac{1}{k} f(k^3 x) \| \leq \left(\tfrac{k}{k-1}\right) \tfrac{1}{k} \phi(k^2 x,0,0,\ldots,0)$$

$$\Rightarrow \| \tfrac{1}{k^2} f(k^2 x) - \tfrac{1}{k^3} f(k^3 x) \| \leq \left(\tfrac{k}{k-1}\right) \tfrac{1}{k^{2+1}} \phi(k^2 x,0,0,\ldots,0).$$

Generalizing the above inequality, we obtain

$$\| \tfrac{1}{k^l} f(k^l x) - \tfrac{1}{k^m} f(k^m x) \| \leq \sum_{j=l}^{m} \left(\tfrac{k}{k-1}\right) \tfrac{1}{k^{j+1}} \phi\left(k^j x,0,0,\ldots,0\right), \tag{2.15}$$

for all non-negative integers m and l with $m > l$ and for all $x \in X$. It follows (2.15) that the sequence $\{\frac{f(k^m x)}{k^m}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(k^m x)}{k^m}\}$ converges.

So define a function $Q : X \to Y$ by

$$Q(x) = \lim_{m\to\infty} \left\{\tfrac{f(k^m x)}{k^m}\right\}, \tag{2.16}$$

for all x ∈ X.

$$\|SQ(x_1,x_2,\ldots,x_n)\| = \| \sum_{i=1,j=i+1}^{n-1} \left(Q\left(kx_i + x_j\right)\right) + Q(kx_n + x_1)$$

$$-k\left[\sum_{i=1,j=i+1}^{n-1} \left(Q(x_i + x_1)\right) + Q(x_n + x_1)\right]$$

$$-\frac{1-k}{k^2-k}\sum_{i=1}^{n}(k^2 Q(x_i) - Q(kx_i))\|$$

$$= \| \lim_{m\to\infty} [\sum_{i=1,j=i+1}^{n-1} \frac{1}{k^m} f(k^m(kx_i + x_j) + \frac{1}{k^m} f(k^m(x_i + x_1)]$$

$$- k[\sum_{i=1,j=i+1}^{n-1} \frac{1}{k^m} f(k^m(x_i + x_j) + \frac{1}{k^m} f(k^m(kx_n + x_1))]$$

$$-\frac{1}{k^m}(\frac{1-k}{k^2-k})[\sum_{i=1}^{n-1}(k^2 f(k^m x_i) - f(k^m(kx_i))]\|$$

$$= \lim_{m\to\infty} \frac{1}{k^m} \|Sf\left(k^m x_1, k^m x_{2,\ldots.} k^m x_n\right)\|$$

$$\leq \lim_{m\to\infty} \frac{1}{k^m} \|\phi\left(k^m x_1, k^m x_{2,\ldots.} k^m x_n\right)\| = 0,$$

for all $x_i \in X$, $i = 1, 2, \ldots, n$.

So $SQ(x_1, x_2, \ldots, x_n) = 0$.

Hence the mapping $Q : X \to Y$ is additive.

Moreover, letting $l = 0$ and limit as $m \to \infty$ in (2.15), we have (2.14).

Now, let $T : X \to Y$ another additive mapping satisfying (2.14) and (2.13)

$$\|Q(x) - T(x)\| = \frac{1}{k^m} \|Q(k^m \mathrm{x}) - T(k^m \mathrm{x})\|$$

$$\|Q(x) - T(x)\| \leq \frac{1}{k^m}\left[\|\mathrm{Q}(k^m \mathrm{x}) - \mathrm{f}(k^m \mathrm{x})\| + \|\mathrm{T}(k^m \mathrm{x}) - \mathrm{f}(k^m \mathrm{x})\|\right],$$

which tends to zero as $m \to \infty$ for all $x \in X$. This proves uniqueness of Q.

So there exists a unique additive mapping $Q : X \to Y$ satisfying (2.14).

Corollary 2.5. Let $p < 1$ and θ be positive real numbers and let $f : X \to Y$ be a mapping such that

$$\|Sf(x_1,x_2,\ldots,x_n)\| \leq \theta\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right),$$

for all $x_i \in X$, $i = 1, 2, \ldots, n$.

Then there exists a unique additive mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \left(\frac{k}{(k-1)(k-k^p)}\right)\theta\left(\|x\|^p\right).$$

REFERENCES

[1] Aoki T., On the Stability of the Linear Transformation in Banach spaces, J. Math. Soc. Japan **2**(1950), 64-66.

[2] Czerwik S., On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg, **62**(1992), 59-64.

[3] Forti G. L., Hyers-Ulam stability of functional equations in several variables, Aequations Mathematicae, **50**(1995), 143-190.

[4] Gajda Z., On stability of additive mappings, *International Journal of Mathematics and Mathematical Sciences,***3**(14)(1991), 431–434.

[5] Gavruta P., A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, Journal of Mathematical Analysis and Applications, **3**(184)(1994), 431–436.

[6] Hyers, D. H., On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U. S. A., **27**(1941), 222-224.

[7] Hyers D.H., Isac G. and Rassias T.M., Stability of Functional Equations in Several Variables,Birkhäuser, Boston (34)(1998), 45-77.

[8] Lee J., Kim J., Park C.,A fixed point approach to the stability of an additive-quadratic-cubic-quartic functional equation,Fixed Point Theory Appl. (185780)(2010), 1-16.

[9] Park C., Jang S., Lee J. and Shin D., On the stability of an AQCQ-functional equation in random normed spaces, J. Inequal. Appl. **34**(2011).

[10] Rassias T.M., On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc. **2**(72) (1978), 297-300.

[11] Rassias T. M., On the stability of the functional equations in Banach spaces, Journal of Mathematical Analysis and Applications, **1**(251)(2000), 264-284.

[12] Ulam S.M., Problems in Modern Mathematics; John Wiley& Sons: New York, NY, USA, (1940), 63-69.

[13] Ramdoss M., Pachaiyappan D. and Lee J. R, Stability of generalized n-variable mixed-type functional equation in Fuzzy modular spaces, Journal of inequalities and Applications, **1**(2021), 1-14.