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Role of Automorphism Groups in Geometric Function Theory

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Abstract

In this note we explain the role of Automorphism group of a domain in the Complex plane to understand its Geometric nature. We survey some of the important papers and explore the ideas to understand the inter play between Automorphism groups of domains and their Geometry.

Keywords: Automorphism groups, conformal self maps.

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1. INTRODUCTION

Felix Klein's idea was to understand the geometry of a domain by way of mappings which preserve the geometry of the domain. In complex function theory conformal self maps of a domain serve this purpose.

Let D be a domain in $\mathbb{C}$. If f is a one-to-one, onto holomorphic function from D to D (f^{-1} exists and is holomorphic) then it is called an *automorphism* of D. The set of automorphisms of D forms a group under the composition of mappings and is denoted by Aut(D).

The automorphism group of a domain can be made in to a topological group which has a manifold structure. So, we can further talk about its dimension.

In this article we explain how to explore the properties of automorphism groups to understand the geometry of domains. We also demonstrate some of the topological properties of $Aut(D)$ for further investigation.

2. TOPOLOGICAL STRUCTURE ON $Aut(D)$

A bounded domain (i.e, non-empty connected open set) in $\mathbb{C}$ is called n - connected ($1 \leq n < \infty$ an integer) if $\mathbb{C} \setminus D$ has n connected components, and exactly one of them is unbounded. For $n = 1$ it is called simply connected, for $n = 2$ it is called doubly connected. To avoid the set-topological complications caused by unbounded components, we like to use the following equivalent criterion for n - connectedness : D is n - connected if and only if its complement in the Riemann sphere $\mathbb{C} \cup \{\infty\}$ has n connected components. In the sequel, we may consider D be a smooth bounded domain in the complex plane (some of the results mentioned here may not require the smoothness of the boundary).

Example 1. *Unit disc:* $|z| < 1$ *is an example of simply connected domain and the Annulus:* $5 < |z| < 10$ *is an example of doubly connected.*

The following theorem is useful and proved by H. Cartan [5]

Theorem 2. ***(Cartan))*** *Suppose* D *is a bounded domain in* $\mathbb{C}$ *and* $p \in D$. *Let* $f \in Aut(D)$. *If* $f(p) = p$ *and* $f'(p) = 1$ *then* $f \equiv I$ *where* I *denotes the identity.*

Theorem 3. *Suppose* D *is a domain in* $\mathbb{C}$ *and* $p \in D$. *The mapping*

$$F : Aut(D) \to \mathbb{C} \times \mathbb{C} \tag{1}$$

defined by

$$\varphi \longmapsto (\varphi(p), \varphi'(p))$$

is one-to-one.

Proof. Suppose $\varphi_1, \varphi_2 \in Aut(D)$. Suppose $F(\varphi_1) = F(\varphi_2)$, i.e,

$$(\varphi_1(p), \varphi_1'(p)) = (\varphi_2(p), \varphi_2'(p)),$$

which implies $\varphi_1(p) = \varphi_2(p)$ and $\varphi_1'(p) = \varphi_2'(p)$.

The mapping

$$\psi = \varphi_1 \circ \varphi_2^{-1}$$

satisfies the hypothesis of the Theorem 2, from which it follows that $\psi \equiv I$. Then we have

$$\varphi_1(z) \equiv \varphi_2(z).$$

Therefore, F is one-to-one. □

We have shown that $Aut(D)$ can be identified with a subset of the Euclidean space $\mathbb{C}^2$. In fact, the main result in this topic is due to Cartan See [4], that the $Aut(D)$ has a real Lie group structure in the compact open topology.

The compact open topology is used on function spaces. Since the mapping of the co domain here is a metric space, this is the same as the topology of uniformly convergence on compact subsets.

Finally, it turns out that, $Aut(D)$ is a topological group and has a manifold structure. Therefore we can talk about its dimension, and its topological properties such as compactness, non-compactness etc.

3. THE DIMENSION OF $Aut(D)$

The automorphism groups of the unit disc: $\mathbb{D}$, the Complex plane: $\mathbb{C}$, the Riemann sphere: $\hat{\mathbb{C}}$ and Annulus: $\mathbb{A}$ are well known and can be found in any standard text book in complex analysis. It is known that $Aut(D)$ is a finite group if D is a finitely connected domain with connectivity > 2. See [2], [3].

Even though the automorphism groups are same we can not guarantee the domains are conformally equivalent, because the annuli $1 < |z| < 2$ and $1 < |z| < 5$ have the same automorphism group but the two annuli are not conformally equivalent.

If D is a finitely connected domain in Riemann sphere, its dimension (real) is ≤ 6. One may obtain the dimension of the automorphism group by analyzing the automorphisms of each domain See[6]. In the following table we mention the dimension of the Automorphism groups of some standard domains in $\mathbb{C}$ (and Riemann sphere $\mathbb{C} \cup \{\infty\}$).

Domain	Notation for D	Dimension of $Aut(D)$
Unit disc	$\mathbb{D}$	3
Complex plane	$\mathbb{C}$	4
Annulus	$\mathbb{A}$	1
Riemann Sphere	$\hat{\mathbb{C}}$	6
Punctured plane	$\mathbb{C}^*$	2
Finite connectivity $n > 2$	D with n holes	0

Table 1: Dimensions of Aut(D)

4. CONCLUDING REMARKS

From the Table 1 we see that we can compare the dimensions of the automorphism groups to conclude whether the given domains are conformally equivalent or not. For example $\mathbb{D}$ and $\mathbb{A}$ are not conformally equivalent.

We can also compare the topological properties of the Automorphism groups. For instance, The automorphism group of the unit disc, $Aut(\mathbb{D})$ is non-compact but the Automorphism group of an Annulus, $Aut(\mathbb{A})$ is compact. Therefore we can conclude that these two domains are not conformally equivalent.

There are also other ways to conclude the given domains are not conformally equivalent. For example $\mathbb{D}$ and $\mathbb{C}$ are not conformally equivalent, by simply using Livouville's theorem ! Similarly $\mathbb{C}$ and $\hat{\mathbb{C}}$ are not conformally equivalent since $\mathbb{C}$ is non-compact where as $\hat{\mathbb{C}}$ is compact.

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