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Efficacy of Chebyshev Wavelet Collocation Method through Nonlinear Ordinary Differential Equations

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Abstract:

The differential equation plays an important role in the mathematical modeling of many physical phenomena that exists in the field of science and engineering. In the present paper, the numerical solution of nonlinear initial and boundary value problems is obtained by using Chebyshev Wavelet Collocation Method (CWCM) and Haar Wavelet Collocation Method (HWCM). The nonlinear equation is linearised using the quasilinearization technique and then the wavelet collocation methods are applied to linearised differential equations to convert it into a system of algebraic equations which can be handled easily. The solution is obtained at the specified points using Lagrange's interpolation technique. The results obtained by wavelet collocation methods are compared with the exact solutions. The tables and graphs show the efficiency of the wavelet collocation methods and we analyze that the method is accurate for a smaller number of collocation points. Thus, the comparative study highlights the advantages of wavelet collocation methods.

Keywords: Wavelets, Chebyshev wavelet, Haar wavelet, Quasilinearization, Collocation points.

1 INTRODUCTION

Many physical problems that arise in the field of science and engineering are modelled either by singular or nonlinear ordinary differential equations. Therefore, numerical methods are often required to find the solution to such problems [1]. During the 1990s, an effort has been made to solve these differential equations using wavelet methods [2]. In the field of mathematical research, wavelet analysis is a new technique that has gained the interest of most researchers and investigators [1]. Applications of these wavelets can be found in the field of signal processing and image processing like denoising, compression, audio and image enhancement and its effects, detection of discontinuities and edges, optimal control, etc.

Recently, methods based on the orthogonal functions and polynomials series, including wavelets, are being used to approximate the solution to various problems. The advantage of these wavelets is that it consists of combined properties of polynomials and compact support which help in being good at modelling localized features and hence in solving various differential and integral equations [3]. Another advantage of the method is that it converts the problem of differential equations to a system of algebraic equations and helps in obtaining the solution to the problems. Also, it doesn't require the approximation of the nonlinear term, unlike other numerical methods [3, 4].

Ersoy [3] obtained the numerical solutions of Bratu and Duffing equations by using Laguerre wavelets and these solutions are compared with the Datardar-Jafari method and Taylor matrix method. Balaji [6] solved Duffing equation using Legendre wavelets. They obtained the numerical solutions in the presence of both integral and non-integral forcing terms by considering the operational matrix method with the Gaussian quadrature formula that helps in converting Duffing equation to a system of algebraic equations which can be solved easily. The elliptic partial differential equations along with the Dirichlet boundary conditions are solved numerically by Aziz et al. [7] using Haar wavelets. They have considered the Hadamard product and Kronecker delta product in addition to the commonly used product of matrices. They have applied Broyden's method in order to solve nonlinear test problems and observed that rate of convergence is better as the number of collocation points is increased. Beylkin et al. [8] in their work computed numerical solutions of heat and Burger's equation using the algorithm such as adaptive application of operation to functions and the adaptive point-wise product of functions. Chen et al. [9] determined the numerical solution of the nonlinear fractional differential equation using Legendre wavelets and the obtained results are compared with the finite difference solution and exact solutions which shows good agreement of the results.

Yang et al. [10] obtained a numerical solution of the nonlinear ordinary differential equation using the wavelet Homotopy Analysis Method (wHAM) and the results are compared with the normal Homotopy Analysis Method (HAM). They observed that the wHAM has a faster rate of convergence and cpu time is less than that of HAM. Sumana et al. [11] obtained numerical solutions of one-dimensional and two-dimensional second-kind Fredholm integral equations using the Haar wavelet collocation method and the results are compared with the exact solutions. They observed that the method has better accuracy even for a small number of grid points. The fractional differential equations are solved using the Haar wavelet collocation method by Shiralashetti et al. [12] for different levels of resolutions. They found that the results of the comparison, are in good agreement with the exact solutions. The nonhomogeneous and the non-planar Burgers equations are solved for numerical solutions using Haar wavelets by Sumana et al. [13, 14]. They observed that the approximate solutions are in good agreement with the exact solutions and Finite difference solutions. Cubic spline interpolation and Lagrange's interpolation are respectively used in order to obtain the solution at specified points.

Singh et al. [15] have used the Haar wavelet collocation method to obtain the numerical solutions of the Lane-Emden equation in an integral form. They have done a comparison of the obtained numerical solution with the solutions found in the literature and the exact solutions. The convergence analysis of the method is also presented. Sumana et al. [16] solved two-dimensional Laplace and Poison equations using 3-scale Haar wavelets, and they observed that the error is very negligible, which leads to the better convergence of the method. The error analysis of the 3-scale Haar wavelet method proves that the solution improves with the increase in the levels of resolution of the wavelet. Jong et al. [17] determined the numerical solution of fractional differential equations using Haar wavelets and convergence of the method has been discussed. It is seen that the approximate solutions coincide with the exact solutions.

Even though the Haar wavelets have better convergence, due to their non-smooth character, the accuracy is less. Thus, to overcome this drawback, we consider the Chebyshev wavelets which have a smooth character so that we get high accuracy in the approximate solution [18]. Sripathy et al. [1] have obtained an approximation solution for linear and nonlinear differential equations by using an operational matrix of a derivative of shifted second-kind Chebyshev wavelets. They observed a good agreement with the exact solution with less computational time. Oruc et al. [2] used the Chebyshev wavelet method in order to obtain the numerical solution of one-dimensional coupled Burger's equation and the solutions are compared with the exact solution and the solutions found in the literature by the Finite element method, Haar wavelet method and Spectral methods.

Shiralashetti et al. [18] have obtained the approximate solution of linear and nonlinear ordinary differential equations by the Chebyshev wavelet collocation method and observed that Chebyshev wavelets are better than Haar wavelets. Adibi et al. [19] obtained the numerical solution of Fredholm integral equations of the first kind using Chebyshev wavelets and results were tabulated which coincides with the exact solutions. Hosseini et al. [20] applied spectral methods and the Chebyshev wavelet Galerkin method to solve ordinary differential equations in which at least one of the solutions is not analytic and noticed that Chebyshev wavelet Galerkin method produces more accurate results when compared to the spectral methods.

Celik [21, 22] used Chebyshev wavelets to determine the solution of the Bessel differential equation of order zero and the Lane-Emden equation and class of linear and nonlinear nonlocal boundary value problems of second and fourth order. They noted that the accuracy of the method increases as the number of grid points increases. Heydari et al. [23] obtained the solution of partial differential equations using the Chebyshev wavelet collocation method with a smaller number of grid points which gave the accurate solutions. Hossein et al. [24] achieved a good agreement of numerical solution with the exact solutions by solving the nonlinear system of integrodifferential equation using Chebyshev wavelets. Youssri et al. [25] have discussed the algorithm based on spectral second-kind Chebyshev wavelets in solving linear, nonlinear, singular, and Bratu-type equations. They have noticed the efficiency and the accuracy of the method for a smaller number of collocation points. In this paper, Haar and Chebyshev wavelets are applied to get the solution of linear and nonlinear differential equations, and their solutions are compared with the exact solutions.

The flow of this paper is as follows. In section 2, we describe the Chebyshev wavelet's definition and its properties. Section 3 is devoted to the Haar wavelets definition. The quasilinear technique has been explained in section 4. The method of solution through numerical examples is discussed, and numerical solutions are represented graphically in section 5. Then the final conclusion has made in section 6.

2 CHEBYSHEV WAVELETS

The family of Chebyshev wavelets $[22, 26]$ are defined on the interval $[0, 1)$ as,

$$C_i(x) = C_{nm}(x) = \begin{Bmatrix} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^k x - 2n + 1), \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0, \text{otherwise} \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots (1)$$

where

$$\alpha_m = \begin{Bmatrix} \sqrt{2}, m = 0 \\ 2, \text{otherwise} \end{Bmatrix}$$

with $i = n + 2^{k-1}m$, k is any positive integer, $n = 1, 2, \cdots, 2^{k-1}$, $m = 0, 1, 2, \cdots, M-1, M$ is the maximum degree of Chebyshev wavelets of first kind and x is the normalized time. $T_m(x)$ are Chebyshev polynomials of degree m which are orthogonal with respect to the weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$. The Chebyshev polynomials satisfy the following recurrence formula,

$$T_0(x) = 1, T_1(x) = x, T_{m+1} = 2xT_m(x) - T_{m-1}(x), \forall\, m = 1, 2, 3, \cdots.$$

The wavelet collocation points are defined as,

$$x_j = \frac{j - 0.5}{N}, \forall\, j = 1, 2, \cdots, N,$$

where $N = 2^{k-1} M$.

In order to solve the differential equations of second order, we require the following integrals.

$$P_i(x) = \int_0^x C_i(x)dx, \text{ and } Q_i(x) = \int_0^x P_i(x)dx.$$

2.1 Function Approximation

A function $f(x) \in L^2([0, 1))$ is represented as an infinite sum of Chebyshev wavelets in the form [4],

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{nm} C_{nm}(x), \ldots\ldots\ldots\ldots\ldots (2)$$

where

$$a_{nm} = \int_0^1 f(x) C_{nm}(x) \omega_n(x) dx. \dots\dots\dots\dots\dots (3)$$

Now, if we approximate the function $f(x)$ as piecewise constant in each subinterval, then we obtain

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm} C_{nm}(x), \dots\dots\dots\dots\dots (4)$$

where the Chebyshev wavelet coefficients a_{nm} are to be determined.

3 HAAR WAVELETS

The Haar wavelet family for $x \in [0,1]$ is defined as follows [5],

$$h_i\,(x) = \begin{Bmatrix} 1, \text{for } x \in [\xi_1, \xi_2) \\ -1, \text{for } x \in [\xi_2, \xi_3) \\ 0, \text{elsewhere} \end{Bmatrix} \dots\dots\dots\dots\dots (5)$$

where

$$\xi_1 = \frac{k}{m}, \xi_2 = \frac{k+0.5}{m}, \xi_3 = \frac{k+1}{m} \dots\dots\dots\dots\dots (6)$$

In the above definition $m = 2^d, d = 0, 1, \cdots, J$ indicates the level of the wavelet, $k = 0, 1, \cdots, m-1$ is the translation parameter, J is the maximum level of resolution and the index i in equation (5) is calculated by the formula $i = m + k + 1$. In the case of minimum values $m = 1$ and $k = 0$, we have $i = 2$. The maximum value of i is $i = 2M = 2^{J+1}$. For $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{Bmatrix} 1, \text{for } x \in [0,1) \\ 0, \text{elsewhere} \end{Bmatrix} \dots\dots\dots\dots\dots (7)$$

In order to solve differential equations of any order, we need the following integrals.

$$p_i(x) = \int_0^x h_i(x) dx = \begin{Bmatrix} x - \xi_1, \text{for } x \in [\,\xi_1, \xi_2) \\ \xi_3 - x, \text{for } x \in [\,\xi_2, \xi_3) \\ 0, \text{elsewhere} \end{Bmatrix} \dots\dots\dots\dots\dots (8)$$

$$q_i(x) = \int_0^x p_i(x) dx = \begin{Bmatrix} \dfrac{(x-\xi_1)^2}{2}, \text{for } x \in [\,\xi_1, \xi_2) \\ \dfrac{1}{4m^2} - \dfrac{(\xi_3 - x)^2}{2}, \text{for } x \in [\,\xi_2, \xi_3) \\ \dfrac{1}{4m^2}, \text{for } x \in [\,\xi_3, 1] \\ 0, \text{elsewhere} \end{Bmatrix} \dots\dots\dots\dots\dots (9)$$

3.1 Function Approximation

A square integrable function $f(x)$ on $(0,1)$ can be expressed as infinite sum of Haar wavelets as [5],

$$f(x)=\sum_{i=1}^{\infty} a(i)h_i(x), \dots\dots\dots\dots\dots (10)$$

where

$$a(i)=\int_0^1 f(x)h_i(x)dx. \dots\dots\dots\dots\dots (11)$$

If $f(x)$ is approximated as piecewise constant during each subinterval, then equation (10) will be terminated at finite terms, i.e.

$$f(x)=\sum_{i=1}^{2M} a(i)h_i(x), \dots\dots\dots\dots\dots (12)$$

where the wavelet coefficients $a(i), i=1,2,\cdots,2M$ are to be determined.

4 QUASILINEARIZATION TECHNIQUE

In order to solve single or systems of nonlinear ordinary and partial differential equations, Bellman and Kalaba [27] introduced the quasilinearization approach as a generalization of the Newton-Raphson method. This technique has quadratic rate of convergence.

Let the second order nonlinear differential equation be of the form,

$$y''=f(y(x),x), \dots\dots\dots\dots\dots (13)$$

with the boundary conditions,

$$y(a)=A, y(b)=B, a\le x\le b, \dots\dots\dots\dots\dots (14)$$

where f is a function of $y(x)$.

Choose initial approximations of the functions $y(x)$, say $y_0(x)=A$, for $a\le x\le b$.

Now expanding the function f about $y_0(x)$ using Taylor's series and by ignoring second and higher order terms, we get

$$f(y(x),x)=f(y_0(x),x)+\big(y(x)-y_0(x)\big)f_{y_0}(y_0(x),x), \dots\dots\dots\dots\dots (15)$$

Substituting equation (15) in equation (13), we obtain

$$y''(x)=f(y_0(x),x)+\big(y(x)-y_0(x)\big)f_{y_0}(y_0(x),x). \dots\dots\dots\dots\dots (16)$$

Equation (16) is solved for $y(x)$ and call the solution as $y_1(x)$. Again, expanding equation (13) about $y_1(x)$, we get

$$y''(x)=f(y_1(x),x)+\big(y(x)-y_1(x)\big)f_{y_1}(y_1(x),x). \dots\dots\dots\dots\dots (17)$$

Again solving (17) for $y(x)$ and call the solution as $y_2(x)$. When the problem converges, the process is continued till we get the desired accuracy. We obtain the recurrence relation as,

$$y''_{t+1}(x)=f(y_t(x),x)+\big(y_{t+1}(x)-y_t(x)\big)f_{y_t}(y'_t(x),y_t(x),x) + \big(y'_{t+1}(x)-y'_t(x)\big)f_{y'_t}(y'_t(x),y_t(x),x), \dots\dots\dots\dots\dots (18)$$

with the boundary conditions,

$$y_{t+1}(a)=A, y_{t+1}(b)=B, \dots\dots\dots\dots\dots (19)$$

where $t=0,1,2,\cdots$. is the iteration parameter.

Similarly, the same procedure can be applied for higher order nonlinear differential equations to obtain the recurrence relation of the form,

$$y_{t+1}^{n}(x) = f(y_t(x), y_t'(x), \ldots, y_t^{n-1}(x), x) + \sum_{j=0}^{n-1} \left(y_{t+1}^{j}(x) - y_t^{j}(x)\right) f_{y_t}^{j}(y_t(x), y_t'(x), \ldots, y_t^{n-1}(x), x). \ldots\ldots\ldots\ldots (20)$$

where n is the order of the differential equation [28, 29].

5 METHOD OF SOLUTION

In this section, an example for nonlinear initial and boundary value problems of second order are solved by using Chebyshev and Haar wavelets.

Example 1:

Consider the nonlinear ordinary differential equation with initial conditions,

$$y'' - 2y' - y^2 = -e^{4x}, \ldots\ldots\ldots\ldots\ldots (21)$$

$$y(0) = 1, y'(0) = 2. \ldots\ldots\ldots\ldots\ldots (22)$$

The exact solution is $y(x) = e^{2x}$.

Chebyshev Wavelet Collocation Method

The Chebyshev wavelet solution is given by,

$$y''(x) = \sum_{i=1}^{N} a_i C_i(x), \ \ldots\ldots\ldots\ldots\ldots (23)$$

where $a_i's, i = 1, 2, \ldots, N$ are Chebyshev wavelet coefficients to be determined. Integrating equation (23) twice with respect to x from 0 to x and using equation (22), we obtain

$$y'(x) = 2 + \sum_{i=1}^{N} a_i P_i(x), \ \ldots\ldots\ldots\ldots\ldots (24)$$

$$y(x) = 1 + 2x + \sum_{i=1}^{N} a_i Q_i(x). \ \ldots\ldots\ldots\ldots\ldots (25)$$

Using quasilinearization, equation (21) leads to

$$y''_{r+1} - 2y'_{r+1} - 2y_r y_{r+1} = -y_r^2 - e^{4x}. \ldots\ldots\ldots\ldots\ldots (26)$$

Substituting equations (23) and (25) in equation (26), we get

$$\sum_{i=1}^{N} a_i(C_i(x) - 2P_i(x) - 2y_r Q_i(x)) = -e^{4x} - y_r^2 + 4xy_r + 2y_r + 4. \ldots\ldots\ldots\ldots\ldots (27)$$

Taking the collocation points $x = x_j$ in equations (27) and (25), we get

$$\sum_{i=1}^{N} a_i(C_i(x_j) - 2P_i(x_j) - 2y_r Q_i(x_j)) = -e^{4x_j} - y_r^2 + 4x_j y_r + 2y_r + 4. \ldots\ldots\ldots\ldots\ldots (28)$$

$$y(x_j) = 1 + 2x_j + \sum_{i=1}^{N} a_i Q_i(x_j). \;\ldots\ldots\ldots\ldots\ldots (29)$$

The wavelet coefficients $a_i, i = 1, 2, \cdots, N$ are obtained by solving the N system of equations in equation (28). These coefficients are then substituted in equation (29) to obtain the Chebyshev wavelet solution at the collocation points $x_j, j = 1, 2, \cdots, N$.

Haar Wavelet Collocation Method

The Haar wavelet solution is given by,

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x), \ldots\ldots\ldots\ldots\ldots (30)$$

where $a_i's, i = 1, 2, \cdots, 2M$ are Haar wavelet coefficients to be determined.
Integrating equation (30) twice with respect to x from 0 to x and using equation (22), we obtain

$$y'(x) = 2 + \sum_{i=1}^{2M} a_i p_i(x), \ldots\ldots\ldots\ldots\ldots (31)$$

$$y(x) = 1 + 2x + \sum_{i=1}^{2M} a_i q_i(x). \ldots\ldots\ldots\ldots\ldots (32)$$

Now, quasilinearizing equation (21), we have

$$y''_{r+1} - 2y'_{r+1} - 2y_r y_{r+1} = -y_r^2 - e^{4x}. \ldots\ldots\ldots\ldots\ldots (33)$$

Substituting equations (30) and (32) in equation (33), we get

$$\sum_{i=1}^{2M} a_i(h_i(x) - 2p_i(x) - 2y_r q_i(x)) = -e^{4x} - y_r^2 + 4xy_r + 2y_r + 4. \ldots\ldots\ldots\ldots\ldots (34)$$

Taking the collocation points $x = x_j$ in equations (34) and (32), we get

$$\sum_{i=1}^{2M} a_i(h_i(x_j) - 2p_i(x_j) - 2y_r q_i(x_j))$$
$$= -e^{4x_j} - y_r^2 + 4x_j y_r + 2y_r + 4. \ldots\ldots\ldots\ldots\ldots (35)$$

$$y(x_j) = 1 + 2x_j + \sum_{i=1}^{2M} a_i q_i(x_j). \;\ldots\ldots\ldots\ldots\ldots (36)$$

The $2M$ system of equations in equation (35) are solved in order to determine the wavelet coefficients $a_i, i = 1, 2, \cdots, 2M$. Then the Haar wavelet solutions at the collocation points $x_j, j = 1, 2, \cdots, 2M$ are obtained by using the wavelet coefficients values in (36). The comparison of the results obtained by both the methods with exact solutions are shown in Table 1. Figure 1 depicts the numerical solutions of these methods.

ERROR ESTIMATE:

We define the error estimate as

$$\mu = \frac{1}{N} \| y(x) - y_{ex}(x) \|, \ldots\ldots\ldots\ldots\ldots (37)$$

where $y_{ex}(x)$ is the exact solution.
Tables 2, 3 and 4 shows the absolute, relative and wavelet error estimates respectively.

Example 2:
Consider the nonlinear ordinary differential equation with initial and boundary conditions,

$$y'' = -(1 + a^2 y'^2), \ldots\ldots\ldots\ldots\ldots (38)$$

$$y(0) = 0, y(1) = 0. \ldots\ldots\ldots\ldots\ldots (39)$$

The exact solution is $y(x) = \frac{ln\left(\frac{\cos\left(a\left(x-\frac{1}{2}\right)\right)}{\cos\left(\frac{a}{2}\right)}\right)}{a^2}$.

Chebyshev Wavelet Collocation Method
Consider the Chebyshev wavelet solution of the form,

$$y''(x) = \sum_{i=1}^{N} a_i C_i(x), \ \ldots\ldots\ldots\ldots\ldots (40)$$

where $a_i' s, i = 1, 2, \cdots, N$ are Chebyshev wavelet coefficients to be determined.
Integrating equation (40) twice with respect to x from 0 to x and using equation (39), we obtain

$$y'(x) = y'(0) + \sum_{i=1}^{N} a_i P_i(x), \ \ldots\ldots\ldots\ldots\ldots (41)$$

$$y(x) = xy'(0) + \sum_{i=1}^{N} a_i Q_i(x). \ \ldots\ldots\ldots\ldots\ldots (42)$$

Taking $x = 1$ in equation (42) and using equation (39), we get

$$y'(0) = -\sum_{i=1}^{N} a_i Q_i(1). \ \ldots\ldots\ldots\ldots\ldots (43)$$

Substituting equation (43) in equations (41) and (42), we get

$$y'(x) = \sum_{i=1}^{N} a_i (P_i(x) - Q_i(1)), \ldots\ldots\ldots\ldots\ldots (44)$$

$$y(x) = \sum_{i=1}^{N} a_i (Q_i(x) - xQ_i(1)). \ldots\ldots\ldots\ldots\ldots (45)$$

By quasilinearization, equation (38) reduces to

$$y''_{r+1} + 2a^2 y'_r y'_{r+1} = a^2 (y'_r)^2 - 1. \ldots\ldots\ldots\ldots\ldots (46)$$

Using equations (40) and (44), equation (46) gives

$$\sum_{i=1}^{N} a_i(C_i(x) + 2a^2 y_r' P_i(x) - 2a^2 y_r' Q_i(1)) = a^2(y_r')^2 - 1. \ldots\ldots\ldots\ldots\ldots (47)$$

Taking $x = x_j$ in equations (47) and (45), we get

$$\sum_{i=1}^{N} a_i(C_i(x_j) + 2a^2 y_r' P_i(x_j) - 2a^2 y_r' Q_i(1)) = a^2(y_r')^2 - 1, \ldots\ldots\ldots\ldots\ldots (48)$$

$$y(x_j) = \sum_{i=1}^{N} a_i(Q_i(x_j) - x_j Q_i(1)). \ldots\ldots\ldots\ldots\ldots (49)$$

The wavelet coefficients $a_i, i = 1, 2, \cdots, N$ are obtained by solving the N system of equations in equation (48). These coefficients are then substituted in equation (49) to obtain the Chebyshev wavelet solution at the collocation points $x_j, j = 1, 2, \cdots, N$.

Haar Wavelet Collocation Method

The Haar wavelet solution is given by,

$$y''^{(x)} = \sum_{i=1}^{2M} a_i h_i(x), \ldots\ldots\ldots\ldots\ldots (50)$$

where $a_i's, i = 1, 2, \cdots, 2M$ are Haar wavelet coefficients to be determined.
Integrating equation (50) twice with respect to x from 0 to x and using equation (39), we get

$$y'(x) = y'(0) + \sum_{i=1}^{2M} a_i p_i(x), \ldots\ldots\ldots\ldots\ldots (51)$$

$$y(x) = xy'(0) + \sum_{i=1}^{2M} a_i q_i(x). \ldots\ldots\ldots\ldots\ldots (52)$$

Taking $x = 1$ in equation (52) and using equation (39), we get

$$y'(0) = \sum_{i=1}^{2M} a_i q_i(1). \ldots\ldots\ldots\ldots\ldots (53)$$

Substituting equation (53) in equations (51) and (52), we obtain

$$y'(x) = \sum_{i=1}^{2M} a_i(p_i(x) - q_i(1)), \ldots\ldots\ldots\ldots\ldots (54)$$

$$y(x) = \sum_{i=1}^{2M} a_i(q_i(x) - xq_i(1)). \ldots\ldots\ldots\ldots\ldots (55)$$

By employing quasilinearization, equation (38) becomes

$$y_{r+1}'' + 2a^2 y_r' y_{r+1}' = a^2(y_r')^2 - 1. \ldots\ldots\ldots\ldots\ldots (56)$$

Using equations (50) and (54) in equation (56), we get

$$\sum_{i=1}^{2M} a_i(h_i(x) + 2a^2 y_r' p_i(x) - 2a^2 y_r' q_i(1)) = a^2(y_r')^2 - 1. \ldots\ldots\ldots\ldots\ldots (57)$$

Taking $x = x_j$ in equations (57) and (55), we get

$$\sum_{i=1}^{2M} a_i(h_i(x_j) + 2a^2 y_r' p_i(x_j) - 2a^2 y_r' q_i(1)) = a^2 (y_r')^2 - 1. \dots\dots\dots\dots\dots (58)$$

$$y(x_j) = \sum_{i=1}^{2M} a_i(q_i(x_j) - x_j q_i(1)). \dots\dots\dots\dots\dots (59)$$

Further, equation (58) is solved to get the wavelet coefficients $a_i, i = 1, 2, \cdots, 2M$ which are used to obtain the Haar wavelet solutions at the collocation points $x_j, j = 1, 2, \cdots, 2M$ from equation (59). In this problem, since a being an integer we have taken $a = 1$ for computation purpose. The comparison of the results obtained by both the methods with exact solutions are shown in Table 5 and the behaviour of the solution is represented in Figure 2. The error estimates are obtained in Tables 6, 7 and 8.

6 RESULTS AND DISCUSSION

We justify the use of the Haar and Chebyshev wavelet collocation method by solving nonlinear initial and boundary value problems. In the case of boundary value problems, these methods are very convenient as it takes care of the boundary conditions automatically. Both methods convert the differential equations into a system of algebraic equations which can be solved easily. Lagrange's interpolation is used to obtain the solution at specified points. In Tables 1 and 5, we compare the numerical solutions with the exact solutions for the number of collocation points $N = 512$. In Figures 1 and 2, we see that the Chebyshev solutions are obtained for $N = 16$ whereas Haar solutions for $N = 128$. From this, we analyze that the accuracy of the Haar solutions increases with an increase in the number of grid points whereas the accuracy of the Chebyshev solutions is high for a smaller number of grid points.

Tables 2, 3 and 4 displays the absolute error, relative error, and wavelet error estimates of example 1. respectively. The error estimates of example 2. are presented in Tables 6, 7 and 8 at different N. We observe that the error values are much negligibly small in the Chebyshev wavelet solution than in the Haar wavelet solution which indicates that the Chebyshey wavelet solution is very close to the exact solution than the Haar wavelet solution. Thus, the Chebyshev wavelet method guarantees the necessary accuracy with a small number of grid points. Hence, Chebyshev wavelets are better than Haar wavelets which are simple, fast, and computationally efficient.

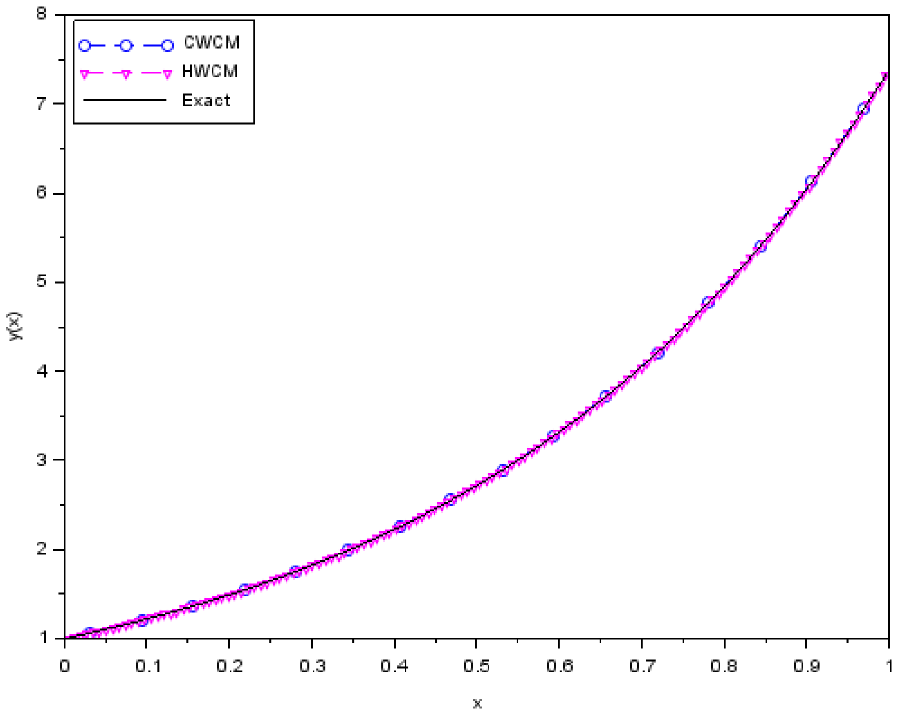

Figure 1: Comparison of HWCM, CWCM and exact solution of example 1.

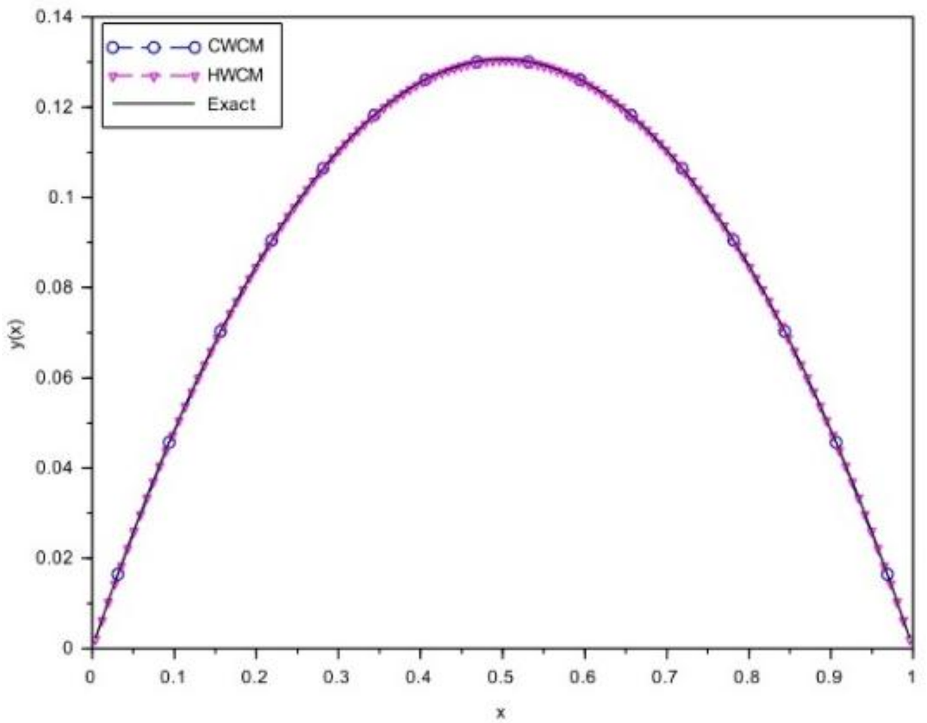

Figure 2: Comparison of HWCM, CWCM and exact solution of example 2.

Table 1: Comparison of HWCM, CWCM and exact solution of example 1.

x	HWCM	CWCM	Exact
0.0	0.999999999757092	0.999999999764024	0.999999999764242
0.1	1.221403069891411	1.221402758176221	1.221402758176199
0.2	1.491825467783094	1.491824697639115	1.491824697640115
0.3	1.822120240786024	1.822118800394171	1.822118800396144
0.4	2.225543349809520	2.225540928486843	2.225540928489836
0.5	2.718285695382164	2.718281828456547	2.718281828460525
0.6	3.320122946406960	3.320116922727224	3.320116922735194
0.7	4.055209264544810	4.055199966832925	4.055199966845915
0.8	4.953046803523455	4.953032424320688	4.953032424342416
0.9	6.049669945215069	6.049647464746510	6.049647464781701

Table 2: Absolute Error in the solution of example 1.

N	HWCM		CWCM	
	l_2	l_∞	l_2	l_∞
4	4.273E-01	4.061E-01	2.272E-02	2.104E-02
8	1.397E-01	1.151E-01	1.278E-03	1.049E-03
16	4.867E-02	3.203E-02	8.552E-05	5.753E-05
32	1.714E-02	8.538E-03	6.940E-06	3.574E-06
64	6.058E-03	2.209E-03	5.997E-07	2.272E-07
128	2.141E-03	5.624E-04	5.270E-08	1.440E-08
256	7.570E-04	1.419E-04	4.652E-09	9.080E-10
512	2.676E-04	3.564E-05	4.110E-10	5.701E-11

Table 3: Relative Error in the solution of example 1.

N	HWCM		CWCM	
	l_2	l_∞	l_2	l_∞
4	7.425E-02	7.056E-02	3.949E-03	3.656E-03
8	2.142E-02	1.765E-02	1.960E-04	1.609E-04
16	7.011E-03	4.615E-03	1.232E-05	8.288E-06
32	2.394E-03	1.192E-03	9.690E-07	4.990E-07
64	8.328E-04	3.037E-04	8.244E-08	3.124E-08
128	2.920E-04	7.671E-05	7.189E-09	1.965E-09
256	1.028E-04	1.928E-05	6.320E-10	1.233E-10
512	3.629E-05	4.832E-06	5.573E-11	7.731E-12

Table 4: Wavelet Error in the solution of example 1.

N	HWCM		CWCM	
	l_2	l_∞	l_2	l_∞
4	1.068E-01	1.015E-01	5.681E-03	5.260E-03
8	1.746E-02	1.438E-02	1.597E-04	1.311E-04
16	3.042E-03	2.002E-03	5.345E-06	3.596E-06
32	5.359E-04	2.668E-04	2.168E-07	1.116E-07
64	9.465E-05	3.452E-05	9.370E-09	3.551E-09
128	1.673E-05	4.394E-06	4.117E-10	1.125E-10
256	2.957E-06	5.543E-07	1.817E-11	3.547E-12
512	5.227E-07	6.961E-08	8.028E-13	1.113E-13

Table 5: Comparison of HWCM, CWCM and exact solution of example 2.

x	HWCM	CWCM	Exact
0.0	0.000000001135899	0.000000001136426	0.000000001136426
0.1	0.048355245620469	0.048355221296794	0.048355221297174
0.2	0.084892624279420	0.084892584522836	0.084892584523442
0.3	0.110449516562801	0.110449467369135	0.110449467369870
0.4	0.125575939120570	0.125575884830284	0.125575884831088
0.5	0.130584296336657	0.130584240435533	0.130584240436359
0.6	0.125575939116509	0.125575884826223	0.125575884827026
0.7	0.110449516579132	0.110449467385464	0.110449467386200
0.8	0.084892624456717	0.084892584700129	0.084892584700735
0.9	0.048355244435639	0.048355220111996	0.048355220112376

Table 6: Absolute Error in the solution of example 2.

N	HWCM		CWCM	
	l_2	l_∞	l_2	l_∞
4	1.236E-03	7.958E-04	4.141E-05	2.789E-05
8	4.801E-04	2.217E-04	2.193E-05	1.194E-05
16	1.735E-04	5.679E-05	2.544E-06	8.551E-07
32	6.168E-05	1.428E-05	2.343E-07	5.402E-08
64	2.183E-05	3.575E-06	2.088E-08	3.380E-09
128	7.723E-06	8.94E-07	1.849E-09	2.112E-10
256	2.731E-06	2.236E-07	1.635E-10	1.320E-11
512	9.655E-07	5.590E-08	1.445E-11	8.254E-13

Table 7: Relative Error in the solution of example 2.

N	HWCM		CWCM	
	l_2	l_∞	l_2	l_∞
4	1.007E-02	6.483E-03	3.374E-04	2.272E-04
8	3.732E-03	1.723E-03	1.705E-04	9.286E-05
16	1.333E-03	4.365E-04	1.955E-05	6.573E-06
32	4.728E-04	1.094E-04	1.796E-06	4.141E-07
64	1.672E-04	2.739E-05	1.599E-07	2.589E-08
128	5.915E-05	6.848E-06	1.416E-08	1.618E-09
256	2.091E-05	1.712E-06	1.252E-09	1.011E-10
512	7.394E-06	4.280E-07	1.107E-10	6.321E-12

Table 8: Wavelet Error in the solution of example 2.

N	HWCM		CWCM	
	l_2	l_∞	l_2	l_∞
4	3.091E-04	1.989E-04	1.035E-05	6.973E-06
8	6.001E-05	2.771E-05	2.742E-06	1.493E-06
16	1.084E-05	3.549E-06	1.590E-07	5.344E-08
32	1.927E-06	4.463E-07	7.323E-09	1.688E-09
64	3.412E-07	5.587E-08	3.263E-10	5.281E-11
128	6.034E-08	6.986E-09	1.444E-11	1.650E-12
256	1.066E-08	8.734E-10	6.388E-13	5.158E-14
512	1.885E-09	1.091E-10	2.823E-14	1.612E-15

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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