International Journal of Pure and Applied Mathematical Sciences(IJPAMS).
ISSN 0972-9828 Volume 16, Number 1 (2023), pp. 31-41

https://dx.doi.org/10.37622/IJPAMS/16.1.2023.31-41

Completely Prime Fuzzy Ideals of $\Gamma-$ Semirings

Tilak Raj Sharma[*1] **and Ritu Sharma**[2]

[1,2]*Department of Mathematics, Himachal Pradesh University, Regional Centre, Khaniyara Dharamshala, Himachal Pradesh -176218, India.*

Abstract

We investigate some basic results of prime fuzzy ideals of semirings [3, 8] to Γ-semirings. Here, first we study some basic results of prime fuzzy ideals of a $\Gamma-$ semiring R and then characterize prime fuzzy ideals of R in terms of prime ideals of R. Finally, we characterize completely prime fuzzy ideals of R in terms of completely prime ideals of R.

Keywords:- $\Gamma-$ semiring, fuzzy ideals, prime ideals, prime fuzzy ideals.

1. INTRODUCTION

Semirings were first considered explicitly by Vandiver in 1934[15] in connection with the axiomatization of the arithmetic of the natural numbers. Many scholars have investigated semirings throughout the years, either independently or as part of an effort to branch out from ring theory or semi-group theory, or in connection with applications. The theory of rings and theory of semi-groups have considerable impact on the development of the theory of semirings.

The notion of Γ in algebra was introduced by N. Nobusawa [10] in 1964 as a generalization of the ring and further studied $\Gamma-$ ring. Rao [11] introduced the concept of $\Gamma-$ semiring as a generalization of $\Gamma-$ ring in the year 1995.

The theory of fuzzy sets was first introduced by L.A. Zadeh [16] in 1965. After that many mathematicians have applied the concept of fuzzy subsets to the theory of groups and rings in algebra and many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory,

*Corresponding Author's Email ID: trpangotra@gmail.com

real analysis, topology etc. For more information on fuzzification of group and ring theories, we refer [9]. In 1993, Ahsan et.al[1] first studied the fuzzy semirings and semi-modules. Later, Dutta and Biswas[4] together have extended this concept and introduced the fuzzy prime ideals of a semiring. In this paper, we characterize prime fuzzy ideals and completely prime fuzzy ideals in terms of prime ideals and completely prime ideals respectively of semirings [3, 8] to Γ-semirings.

2. PRELIMINARIES AND EXAMPLES

Recall from [2, 6, 7, 11, 13] that if (R, +) and (Γ, +) be two commutative semi-groups then R is called a Γ− semiring if there exists a mapping R × Γ × R → R denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying(i) $x\alpha(y+z) = x\alpha y + x\alpha z$. (ii) $(y+z)\alpha x = y\alpha x + z\alpha x$. (iii) $x(\alpha+\beta)z = x\alpha z + x\beta z$. (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Let A and B be semirings and $R = Hom(A, B)$ and $\Gamma = Hom(B, A)$ denote the sets of homomorphisms from A to B and B to A respectively. Then R is a Γ− semiring with operations of point wise addition and composition of mappings. Further, let M be a Γ− ring and let R be the set of ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a Γ− semiring. A Γ− semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$. A Γ− semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$. R is said to have an identity element if there exists $\gamma \in \Gamma$ such that $1\gamma x = x = x\gamma 1$ for all $x \in R$. R is said to have a strong identity element if for all $x \in R$, $1\alpha x = x = x\alpha 1$ for all $\alpha \in \Gamma$. An ideal P of R is prime if for any two ideals A and B of R, $A\Gamma B \subseteq P$ we have, either $A \subseteq P$ or $B \subseteq P$.
Let X be a non-empty set. A mapping μ : X → [0,1] is called a fuzzy subset of X.

Let A be a non empty subset of a Γ− semiring R. Then characteristic function of A is a fuzzy subset of R and is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

If $x \in X$ and $r \in (0, 1]$ then a fuzzy point x_r of X is a fuzzy subset of X, defined by

$$x_r(y) = \begin{cases} r & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

If A is a fuzzy subset of a non- empty set X and $x_r \in X$ is a fuzzy point then $x_r \in A$ means $x_r \subseteq A$. Let R_1 and R_2 be two Γ− semirings. Then $f : R_1 \to R_2$

is called a $\Gamma-$homomorphism if $f(x+y) = f(x)+f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in R_1$ and $\alpha \in \Gamma$. If f is both one-one and onto then f is a $\Gamma-$ isomorphism. Let μ be a fuzzy subset of R. Then μ is called a fuzzy $\Gamma-$ semiring if (i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$. (ii)$\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\}$. Let μ be a fuzzy subset of R. Then μ is called a fuzzy left (right) ideal of R if $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$ and $\mu(x\alpha y) \geq \mu(y)(\mu(x\alpha y) \geq \mu(x))$, for all $x, y \in R$ and $\alpha \in \Gamma$. A fuzzy ideal of R is a non empty fuzzy subset of R which is both fuzzy left and fuzzy right ideal of R. Let μ be a fuzzy subset of a $\Gamma-$ semiring R. Then μ is called a prime fuzzy ideal of R if (i)$\mu(x+y) \geq min\{\mu(x), \mu(y)\}$ and (ii) $\mu(x\alpha y) = max\{\mu(x), \mu(y)\}$ for all $x, y \in R$ and $\alpha \in \Gamma$. This definition can also be defined as: A fuzzy ideal μ of R is called prime fuzzy ideal if either $\mu = \chi_R$ or μ is a non constant function and for any two fuzzy ideals λ_1 and λ_2 of R, $\lambda_1\Gamma\lambda_2 \subseteq \mu$ implies that either $\lambda_1 \subseteq \mu$ or $\lambda_2 \subseteq \mu$.
A fuzzy ideal μ of a $\Gamma-$ semiring R is said to be completely prime fuzzy ideal if for all $x, y \in R, t \in (0,1], (x\alpha y)_t \in \mu$ implies that $x_t \in \mu$ or $y_t \in \mu$. Or
A fuzzy ideal μ of a $\Gamma-$ semiring R is completely prime fuzzy ideal if and only if either $\mu(x\alpha y) = \mu(x)$ or $\mu(x\alpha y) = \mu(y)$, for all $x, y \in R, \alpha \in \Gamma$.

Let μ_1, μ_2 be fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of R. Then product $\mu_1\Gamma\mu_2$ is defined as follows:

$$(\mu_1\Gamma\mu_2)(z) = \begin{cases} \sup\limits_{z=x\alpha y}\left[min\{\mu_1(x), \mu_2(y)\},\ x, y \in R, \alpha \in \Gamma\right] \\ 0 \quad if\ for\ any\ x, y \in R,\ for\ any\ \alpha \in \Gamma,\ z \neq x\alpha y \end{cases}$$

In case of product $\mu_1\Gamma\mu_2$ if R has a strong identity then the case $z \neq x\alpha y$ for any $x, y \in R, \alpha \in \Gamma$ does not exist.

Theorem 2.1. *[6] Let R be a* $\Gamma-$ *semiring and* λ *and* μ *are fuzzy ideals of R. Then* $\lambda\Gamma\mu$ *is a fuzzy ideal of R.*

Theorem 2.2. *[6] Let* R *be a* $\Gamma-$ *semiring and* λ, μ *be fuzzy left ideals(fuzzy right ideals, fuzzy ideals) of* R. *Then* $\lambda\Gamma\mu \subseteq \lambda \circ \mu$.

Proposition 2.3. *[6] Let* R *be a* $\Gamma-$ *semiring. Let* λ *and* μ *be fuzzy right and left ideals of* R *respectively. Then* $\lambda\Gamma\mu \subseteq \lambda \cap \mu$.

Lemma 2.4. *[5] If a and b are elements of a* $\Gamma-$ *semiring R then the following conditions on a prime ideal P of R are equivalent:*

(i) If $a\Gamma b \subseteq P$ *then either* $a \in P$ *or* $b \in P$.

(ii) If $a\Gamma b \subseteq P$ *then* $b\Gamma a \subseteq P$.

Theorem 2.5. *[5] An ideal P of a commutative $\Gamma-$ semiring R is prime if and only if $a\Gamma b \subseteq P$ implies that either $a \in P$ or $b \in P$.*

Remark 2.6. *Throughout this paper, R will denote a $\Gamma-$ semiring with zero element 0 and identity element 1 unless otherwise stated.*

3. PRIME FUZZY IDEALS OF A $\Gamma-$ SEMIRING

We start this section with the following example.

Example 3.1. *Let μ be a fuzzy subset of a $\Gamma-$ semiring N of non negative integers, defined by*

$$\mu(x) = \begin{cases} 1 & if x \in <3> \\ 0.5 & otherwise \end{cases} \tag{1}$$

Then μ is a prime fuzzy ideal of N.

Theorem 3.2. *Let R be a $\Gamma-$ semiring and λ be a fuzzy left ideal of R. Then $\lambda_0 = \{x \in R \mid \lambda(x) = \lambda(0)\}$ is a left ideal of R.*

Theorem 3.3. *Let R be a $\Gamma-$ semiring. Let I be an ideal of R and $a \leq b \neq 0$ be any two elements in $[0, 1]$. Then the fuzzy subset λ of R, defined by*

$$\lambda(x) = \begin{cases} b & \text{if } x \in I \\ a & \text{if } otherwise \end{cases}$$

is a fuzzy ideal of R .

Theorem 3.4. *Let P be an ideal of a $\Gamma-$ semiring R, $a \in [0, 1)$ and μ a fuzzy subset of R defined by*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in P \\ a & \text{if } x \notin P \end{cases}$$

Then μ is a prime fuzzy ideal of R if and only if P is a prime ideal of R.

Proof. Clearly by theorem 3.3, μ is non constant fuzzy ideal of R. Let μ_1 and μ_2 be two fuzzy ideals of R such that $\mu_1\Gamma\mu_2 \subseteq \mu$, $\mu_1 \nsubseteq \mu$ and $\mu_2 \nsubseteq \mu$. Then there exist x, $y \in R$ such that $\mu_1(x) > \mu(x)$ and $\mu_2(y) > \mu(y)$. This implies that $\mu(x) = \mu(y) = a$. Therefore, $x, y \notin P$. Since P is a prime ideal of R, there exists $r \in R, \alpha, \beta \in \Gamma$, such that $x\alpha r\beta y \notin P$ (By Theorem 2.4). Therefore, $\mu(x\alpha r\beta y) = a$. Now, $(\mu_1\Gamma\mu_2)(x\alpha r\beta y)$ $\geq min[\mu_1(x), \mu_2(r\beta y)] \geq min[\mu_1(x), \mu_2(y)] > min[\mu(x), \mu(y)] = a = \mu(x\alpha r\beta y)$, a contradiction. Hence μ is a prime fuzzy ideal of R. Conversely, let μ be a prime fuzzy ideal of R and B, C be two ideals of R such that $B\Gamma C \subseteq P$. If possible let $B \nsubseteq P$ and $C \nsubseteq P$. Then there exist $b \in B$ and $c \in C$ such that $b, c \notin P$. Let us define fuzzy subsets μ_1 and μ_2 of R as follows:

$$\mu_1(x) = \begin{cases} 1 & \text{if x} \in \text{B} \\ a & \text{if x} \notin \text{B} \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if x} \in \text{C} \\ a & \text{if x} \notin \text{C} \end{cases}$$

Then μ_1 and μ_2 are fuzzy ideals of R and $\mu_1 \nsubseteq \mu$ and $\mu_2 \nsubseteq \mu$ but $\mu_1\Gamma\mu_2 \subseteq \mu$, a contradiction since μ is a prime fuzzy ideal of R. Hence P is prime ideal of R.

□

Corollary 3.5. *Let P be an ideal of a $\Gamma-$semiring R. Then the characteristic function χ_P is a prime fuzzy ideal of R if and only if P is prime ideal of R*

Proof. The proof follows from the above theorem on substituting $a = 0$. □

Theorem 3.6. *Let R be a $\Gamma-$ semiring and μ be a prime fuzzy ideal of R. Then*

(i) $\mu(0) = 1$

(ii) $\mid Im\mu \mid = 2$

(iii) $\mu_0 = \{x \in R \mid \mu(x) = \mu(0)\}$ *is a prime ideal of R.*

Proof. (i) Let μ be a prime fuzzy ideal of R and $\mu(0) \neq 1$. Since μ is a non constant fuzzy ideal of R, so there exists $r \in R$ such that $\mu(0) > \mu(r)$. Define fuzzy subsets μ_1 and μ_2 of R as :

$$\mu_1(x) = \begin{cases} 1 & \text{if } \mu(x) = \mu(0), \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_2(x) = \mu(0)$ for all $x \in R$. Clearly, μ_1 and μ_2 are two fuzzy ideals of R. Now if x is such that $\mu(x) = \mu(0)$ then $\mu(x\alpha y) = \mu(0)$ for all $y \in R$, $\alpha \in \Gamma$.

So, $\mu(x\alpha y)$ = $\mu(0)$ = $min[\mu_1(x), \mu_2(y)]$. If $\mu(x) \neq \mu(0)$ then $\mu(x\alpha y) \geq 0 = min[\mu_1(x), \mu_2(y)]$. Thus, for all $x, y \in R$, min$[\mu_1(x), \mu_2(y)] \leq \mu(x\alpha y)$. Now, for any element $z \in R, (\mu_1\Gamma\mu_2)(z) = \underset{z=x\alpha y}{sup}\,[min[\mu_1(x), \mu_2(y)]] \leq \mu(z)$ Thus, $\mu_1\Gamma\mu_2 \subseteq \mu$. But $\mu_1(0) = 1 > \mu(0)$ and $\mu_2(r) = \mu(0) > \mu(r)$. This implies that $\mu_1 \nsubseteq \mu$ and $\mu_2 \nsubseteq \mu$, a contradiction. Hence, $\mu(0) = 1$

(ii) Let $|Im\mu| \geq 2$. Let x and y be two elements of R such that $1 > \mu(a) > \mu(b)$. Define fuzzy subset μ_1 and μ_2 of R as

$$\mu_1(x) = \begin{cases} 1 & \text{if } x \in < a > \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_2(x) = \mu(a)$ for all $x \in R, \alpha \in \Gamma$. Clearly μ_1, μ_2 are two fuzzy ideals of R. If $x \in < a >$, then $x\alpha y \in < a >$ for all $y \in R, \alpha \in \Gamma$. Therefore, $\mu(x\alpha y) \geq \mu(a)$ $= min[\mu_1(x), \mu_2(y)]$ If $x \notin < a >$ then $\mu(x\alpha y) \geq 0 = min[\mu_1(x), \mu_2(y)]$. Thus, for all $x, y \in R$, $\mu(x\alpha y) \geq min\{\mu_1(x), \mu_2(y)\}$. Hence, for any $z \in R$, $(\mu_1\Gamma\mu_2)(z) = \underset{z=x\alpha y}{sup}\,[min\{\mu_1(x), \mu_2(y)\}]] \leq \mu(z)$. Therefore, $\mu_1\Gamma\mu_2 \subseteq \mu$. But $\mu_1(a) = 1 > \mu(a)$. Thus, $\mu_1 \nsubseteq \mu$. Since μ is prime fuzzy ideal so $\mu_2 \subseteq \mu$. But $\mu_2(b) = \mu(a) > \mu(b)$, a contradiction. Thus, $| Im\mu | = 2$

(iii) Let μ and $\mu_0 = \{x \in R \mid \mu(x) = \mu(0)\}$ be prime fuzzy ideal of R. Also, $| Im\mu | = 2$, μ_0 is a proper ideal of R. Let A and B be two ideals of R such that $A\Gamma B \subseteq \mu_0$. Let λ_A and λ_B be two fuzzy ideals of R. If $(\lambda_A\Gamma\lambda_B)(x) = 0$ then $(\lambda_A\Gamma\lambda_B)(x) \leq \lambda_{\mu_0}(x)$. If $(\lambda_A\Gamma\lambda_B)(x) \neq 0$ then $\underset{x=u\alpha v}{sup}\,[min\{\lambda_A(u), \lambda_B(v)\}] \neq 0$ for some $u \in A, v \in B, \alpha \in \Gamma$. This implies that $min(\lambda_A(u), \lambda_B(v)) \neq 0$ for some $u, v \in R$ such that $x = u\alpha v$ for $u \in A$ and $v \in B$, $\alpha \in \Gamma$ and $(\lambda_A\Gamma\lambda_B)(x) = 1$. This implies that $x = u\alpha v \in A\Gamma B \in \mu_0$. Thus, $\lambda_{\mu_0}(x) = 1$. Hence, for any $x \in R, (\lambda_A\Gamma\lambda_B)(x) \leq \lambda_{\mu_0}(x)$. Thus, $\lambda_A\Gamma\lambda_B \subseteq \lambda_{\mu_0}$. Again, if $x \in \mu_0$, then $\lambda_{\mu_0}(x) = 1 = \mu(0) = \mu(x)$. If $x \notin \mu_0$ then $\lambda_{\mu_0}(x) = 0 \leq \mu(x)$. Thus, for any $x \in R$, $\lambda_{\mu_0(x)} \leq \mu(x)$. Therefore, $\lambda_{\mu_0} \subseteq \mu$, which implies that $\lambda_A\Gamma\lambda_B \subseteq \mu$. As μ is a prime fuzzy ideal of R, so either $\lambda_A \subseteq \mu$ or $\lambda_B \subseteq \mu$. If $\lambda_A \subseteq \mu$ then $x \in A$ implies that $\mu(x) \geq \lambda_A(x) = 1 = \mu(0)$. Hence, $\mu(x) = 1 = \mu(0)$. So $x \in \mu_0$. Thus, $A \subseteq \mu_0$. Similarly, if $\lambda_B \subseteq \mu$ then $B \subseteq \mu_0$. Hence, μ_0 is the prime ideal of R.

□

Corollary 3.7. *Let R be a $\Gamma-$ semiring and I be an ideal of R such that its characteristic function χ_I is a prime fuzzy ideal of R. Then I is a prime ideal of R.*

Proof. Let χ_I be a prime fuzzy ideal of R and I be an ideal of R. This implies that $(\chi_I)_0$ is a prime ideal of R. Now, $x \in I$ if and only if $\chi_I(x) = 1$ if and only if $\chi_I(x) = \chi_I(0)$ if and only if $x \in \chi_I(0)$. Thus, $\chi_I(0) = I$. Hence, I is a prime ideal of R. □

Theorem 3.8. *Let R be a $\Gamma-$ semiring and μ is a fuzzy subset of R then μ is a prime fuzzy ideal of R if and only if $Im\ \mu = \{1, a\}$, $a \in [0, 1)$, and μ_0 is a prime ideal of R.*

Proof. Proof follows from Theorems 3.4 and 3.6 □

Theorem 3.9. *Let R be a $\Gamma-$semiring and μ is a fuzzy subset of R. Then for any $x, y \in R$, $inf\{\mu(x\alpha r\alpha y) \mid r \in R\} = max\{\mu(x), \mu(y)\}$*

Proof. Let μ is a prime fuzzy ideal of R. Then $Im\mu = \{1, a\}$ where $a \in [0, 1)$ and $\mu_0 = \{x \in R| \ \mu(x) = \mu(0)\}$ is a prime ideal of R. If $max\{\mu(x), \mu(y)\} = 1$ then either $\mu(x) = 1$ or $\mu(y) = 1$, that is, $x \in \mu_0$ or $y \in \mu_0$. Thus, $x\alpha r\alpha y \in \mu_0$, for all $r \in R$. This implies that $\mu(x\alpha r\alpha y) = \mu(0) = 1$. Hence, $inf\{\mu(x\alpha r\alpha y) \mid r \in R\} = 1 = max\{\mu(x), \mu(y)\}$. Further, if $max[\mu(x), \mu(y)] = a$ then $\mu(x) = a$ and $\mu(y) = a$. This implies that $x \notin \mu_0$ and $y \notin \mu_0$. So, there exists an element $r \in R$ such that $x\alpha r\alpha y \notin \mu_0$. This implies that $\mu(x\alpha r\alpha y) = a$ This implies that $inf\{\mu((x\alpha r\alpha y) \mid r \in R\} = a = max\{\mu(x), \mu(y)\}$. Hence, $inf\{\mu(x\alpha r\alpha y) \mid r \in R\} = max\{\mu(x), \mu(y)\}$ □

Theorem 3.10. *Let R be a commutative $\Gamma-$ semiring. Let μ be a prime fuzzy ideal of R and $x, y \in R$ Then $\mu(x\alpha y) = max[\mu(x), \mu(y)]$ if and only if $\mu(x\alpha y) = \mu(y\alpha x)$.*

Proof. Let $\mu(x\alpha y) = max(\mu(x), \mu(y))$. This implies that, $\mu(y\alpha x) = max(\mu(y), \mu(x))$ $= max(\mu(x), \mu(y)) = \mu(x\alpha y)$. Conversely, let μ be a prime fuzzy ideal of R and $inf\{\mu(x\alpha r\alpha y) \mid r \in R\} = max\{\mu(x), \mu(y)\}$, for all $x, y \in R, \alpha \in \Gamma$. Thus, $\{\mu(x\alpha r\alpha y\} = \mu(r\alpha y\alpha x) \geq \mu(y\alpha x) = \mu(x\alpha y)$ for all $r \in R$. Hence, $max[\mu(x), \mu(y)] = inf\{\mu(x\alpha r\alpha y)\} \mid r \in R\} = inf\{\mu(r\alpha x\alpha y)\} \mid r \in R\}$ $\geq \mu(x\alpha y) \geq max[\mu(x), \mu(y)]$. Hence, $\mu(x\alpha y) = max[\mu(x), \mu(y)]$ □

4. COMPLETELY PRIME FUZZY IDEALS OF $\Gamma-$ SEMIRINGS

In this section, we characterize completely prime fuzzy ideals of a $\Gamma-$ semiring R in terms of completely prime ideals of R.
The following definition is analogous to the definition 4.8 in [4].

Definition 4.1. *An ideal P of a $\Gamma-$ semiring R is said to be completely prime if $a\alpha b \in P$ for a, b in R and $\alpha \in \Gamma$ implies that $a \in P$ or $b \in P$.*

Theorem 4.2. *Let R be a $\Gamma-$ semiring. A prime ideal P of R is completely prime if and only if $a\alpha b \in P$ implies that $b\alpha a \in P$.*

Theorem 4.3. *Let R be a $\Gamma-$semiring and μ a non constant fuzzy ideal of R. If for any two fuzzy points x_r and y_t of R, $x_r\Gamma y_t \subseteq \mu$ implies that either $x_r \subseteq \mu$ or $y_t \subseteq \mu$ then μ is completely prime fuzzy ideal of R*

Proof. Let σ and τ be fuzzy ideals of R and $\sigma\Gamma\tau \subseteq \mu$. Let $\sigma \nsubseteq \mu$. Then there exists $x \in R$ such that $\sigma(x) \geq \mu(x)$. Let $\sigma(x) = r$ and $\tau(y) = t$ for some $x, y \in R$. If $z = x\alpha y$, for some $\alpha \in \Gamma$ then $(x_r\Gamma y_t)(z) = min(r, t)$. Therefore, $\mu(z) = \mu(x\alpha y) \geq \sigma\Gamma\tau(x\alpha y) \geq min\{\sigma(x), \tau(y)\} \geq min\{r, t\} = (x_r\Gamma y_t)(z)$. Therefore, $x_r\Gamma y_t \subseteq \mu$. This implies that $x_r \subseteq \mu$ or $y_t \subseteq \mu$. This implies that $r \leq \mu(x)$ or $t \leq \mu(y)$. This implies that $\tau(y) = t \leq \mu(y)$, since $r \nleq \mu(x)$. This implies that $\tau \subseteq \mu$. Hence, μ is a completely prime fuzzy ideal of R.

□

Theorem 4.4. *Let X be a non-empty set and $x_r, y_r \in X$ then $x_r\Gamma y_r = (x\alpha y)_{min(r,s)}$, for all $\alpha \in \Gamma$.*

Theorem 4.5. *Let R be a $\Gamma-$ semiring. Then every completely prime fuzzy ideal of R is a prime fuzzy ideal of R. Conversely, if R is commutative with strong identity then a prime fuzzy ideal of R is completely prime fuzzy ideal.*

Proof. Let μ be a completely prime fuzzy ideal of R and μ_1, μ_2 be two fuzzy ideals of R such that $\mu_1\Gamma\mu_2 \subseteq \mu$. Let $\mu_1 \nsubseteq \mu$. Then there exists $x \in R$ such that $\mu_1(x) > \mu(x)$. This implies that $x_{\mu_{1(x)}} \notin \mu$ and $x_{\mu_1(x)}(x) \in \mu_1$. Let y be an arbitrary element of R, then $y_{\mu_2(y)}) \leq \mu(y) \in \mu_2$. Let $\alpha \in \Gamma$. Now, $(x_{\mu_{1(x)}}\Gamma y_{\mu_{2(y)}})(z) = (x\alpha y)_{min(\mu_1(x),\mu_2(y))}(z)$

$$= \begin{cases} min[\mu_1(x), \mu_2(y)] & \text{if z} = x\alpha y \\ 0 & \text{if } z \neq x\alpha y \end{cases}$$

Again, $(\mu(z) \geq \mu_1\Gamma\mu_2)(z) = \sup_{z=x\alpha y} [min[\mu_1(x), \mu_2(y)]] \geq min[\mu_1(x), \mu_2(y)]$, $z = x\alpha y$ $= (x_{\mu_{1(x)}}\Gamma y_{\mu_{2(y)}})(z)$. If $z \neq x\alpha y$ for any $x, y \in R$, then $\mu(z) \geq (x_{\mu_1(x)}\Gamma y_{\mu_2(y)})(z)$. Thus, $x_{\mu_1(x)}\Gamma y_{\mu_2(y)} \subseteq \mu$ Since μ is completely prime fuzzy ideal and $x_{\mu_1(x)} \notin \mu$ then $y_{\mu_2(y)}) \in \mu$. Thus, $y_{\mu_2(y)}) \leq \mu(y)$. This implies that $\mu_2 \subseteq \mu$. Hence, μ is a fuzzy prime ideal of R. Conversely, let μ is a prime fuzzy ideal of a commutative $\Gamma-$ semiring R. Let x_r and y_s are two fuzzy points of R such that $x_r\Gamma y_s \subseteq \mu$. Then $(x_r\Gamma y_s)(x\alpha y) \leq \mu(x\alpha y)$, $\alpha \in \Gamma$. This implies that $min(r, s) \leq \mu(x\alpha y)$. Let μ_3 and μ_4 be two fuzzy subset of R defined by

$$\mu_3(t) = \begin{cases} r & \text{if } t \in < x > \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_4(t) = \begin{cases} s & \text{if } t \in < y > \\ 0 & \text{otherwise} \end{cases}$$

It is clear that μ_3 and μ_4 are fuzzy ideals of R. If $z = u\alpha v$ for any $u \in < x >, v \in < y >$ then $(\mu_3\Gamma\mu_4)(z) = 0$ otherwise $(\mu_3\Gamma\mu_4)(z) = \sup_{z=u\alpha v} [min[\mu_3(u), \mu_4(v)]] = min(r, s)$. Since R is commutative with strong identity, $u \in < x >$ implies that $u = r_1\alpha x$ for some $r_1 \in R$, $\alpha \in \Gamma$ and $v \in < y >$ implies that $v = r_2\alpha y$ for some $r_2 \in R$, $\alpha \in \Gamma$. Then $u\alpha v = (r_1\alpha x)\alpha(r_2\alpha y) = (r_1\alpha r_2\alpha x\alpha y) = r_3\alpha x\alpha y, r_3 = r_1\alpha r_2 \in R$. Now $\mu(u\Gamma v) = \mu(r_3\alpha x\alpha y) \geq \mu(x\alpha y) \geq min(r, s)$. Thus, $\mu_3\Gamma\mu_4 \subseteq \mu$. Since μ is prime fuzzy ideal, so we have either $\mu_3 \subseteq \mu$ or $\mu_4 \subseteq \mu$. This implies that either $\mu_3(x) \leq \mu(x)$ or $\mu_4(y) \leq \mu(y)$. Thus, either $x_r \in \mu$ or $y_s \in \mu$. Hence, μ is a completely fuzzy prime ideal of R. □

Theorem 4.6. *Let R be a* $\Gamma-$ *semiring and* μ *a completely prime fuzzy ideal of* R*. Then* $\mu_0 = \{x \in R \mid \mu(x) = \mu(0)\}$ *is a completely prime ideal of* R*.*

Proof. Let μ be a fuzzy ideal of R, then μ_0 is an ideal of R. Let $x, y \in R$, $\alpha \in \Gamma$ and $x\alpha y \in \mu_0$. Then $\mu(x\alpha y) = \mu(0)$. Since μ is a completely prime fuzzy ideal of R, therefore μ is a prime fuzzy ideal of R. Hence $\mu(0) = 1$ and $Im\mu = \{1, a\}$ where $a \in [0, 1)$. So $\mu(x\alpha y) = 1$, that is, $(x\alpha y)_1 \in \mu$. This implies that $x_1\Gamma y_1 \in \mu$. This implies that either $x_1 \in \mu$ or $y_1 \in \mu$. Thus, either $\mu(x) \geq 1$ or $\mu(y) \geq 1$, that is, $\mu(x) = 1$ or $\mu(y) = 1$. This implies that either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$. Thus, either $x \in \mu_0$ or $y \in \mu_0$. Hence, μ_0 is a completely prime ideal of R. □

Theorem 4.7. *Let* μ *be a fuzzy subset of a* $\Gamma-$*semiring* R *such that* $Im\mu = \{1, a\}$ *where* $a \in [0, 1)$ *and* μ_0 *is completely prime ideal of* R*. Then* μ *is a completely prime fuzzy ideal of* R*.*

Proof. Let μ be a fuzzy ideal of R. Let x_r, y_s are two fuzzy points of R such that $x_r \notin \mu$ and $y_s \notin \mu$. This implies that $r > \mu(x)$ and $s > \mu(y)$. Thus, $\mu(x) = \mu(y) = a$. This implies that $x \notin \mu_0$ and $y \notin \mu_0$. This implies that $x\alpha y \notin \mu_0, \alpha \in \Gamma$. Thus, $\mu(x\alpha y) = a$, Therefore, $x_r\Gamma y_s \notin \mu$. For, $x_r\Gamma y_s \in \mu$, we have $(x\alpha y)_{min(r,s)} \in \mu$ Thus, $a = \mu(x\alpha y) \geq min(r, s) > min[\mu(x), \mu(y)] = a$, which is a contradiction . Hence, μ is a completely fuzzy prime ideal. □

Theorem 4.8. *Let R be a $\Gamma-$ semiring and μ be a fuzzy ideal of R such that Im $\mu = \{1, a\}, a \in [0, 1)$. Then μ is a completely prime fuzzy ideal of R if and only if its only proper level ideal μ_1 is a completely prime ideal of R.*

Proof. Follows from Theorem 4.7 and Theorem 4.6. □

Theorem 4.9. *Let R be a $\Gamma-$ semiring. A prime fuzzy ideal μ of R is completely prime fuzzy ideal if and only if for two fuzzy points $x_r, y_s \in R$, $x_r\Gamma y_r \in \mu$ implies that $y_s\Gamma x_r \in \mu$.*

Proof. Let μ of R be a completely prime fuzzy ideal of R. Let $x_r, y_s \in R$ be such that, $x_r\Gamma y_s \in \mu$. This implies that $(x\alpha y)_{min(r,s)} \in \mu, \alpha \in \Gamma$. This implies that, $\mu(x\alpha y) \geq min(r, s)$. Since μ is prime fuzzy ideal, so $Im\mu = \{1, a\}, a \in [0, 1)$ and $\mu(0) = 1$

Case I If $\mu(x\alpha y) = 1$ then $x\alpha y \in \mu_1$. Since μ is completely prime fuzzy ideal, soi μ_1 is completely prime. This implies that $y\alpha x \in \mu_1$. This implies that $\mu(y\alpha x)$ = 1 = $\mu(x\alpha y)$. So,$\mu(y\alpha x) \geq min(s, r)$. Thus, $(y\alpha x)_{min(r,s)} \in \mu$. Hence, $y_s\Gamma x_r \in \mu$.

Case II If $\mu(x\alpha y) = a$ then $x\alpha y \in \mu_a = R$ This implies that $y\alpha x \in \mu_a$. So, $\mu(y\alpha x) \geq a = \mu(x\alpha y) \geq min(r, s)$. Therefore, $(y\alpha x)_{min(r,s)} \in \mu$. This implies that $(y\alpha x)_{min(s,r)} \in \mu$. Hence, $y_s\Gamma x_r \in \mu$. Conversely, assume that μ is a prime fuzzy ideal of R and for any two fuzzy points x_r, y_s in R, $x_r\Gamma y_s \in \mu$ implies that $y_s\Gamma x_r \in \mu$. Again, μ is a prime fuzzy ideal, $Im\mu = \{1, a\}, a \in [0, 1]$ and μ_0 is a prime ideal of R. So let $x\alpha y \in \mu_0$. This implies that $\mu(x\alpha y) = 1$. This implies that $x_1\Gamma y_1 \in \mu$. Therefore, $y_1\Gamma x_1 \in \mu$. So, $y\alpha x \in \mu_0$. Hence, by Theorem 4.2, μ_0 is a completely prime ideal of R. By Theorem 4.7, it follows that μ is a completely prime fuzzy ideal of R. □

REFERENCES

[1] Ahsan, J., Saifullah, K. and Khan, M.F., (1993), Fuzzy Semirings, Fuzzy Sets and Systems(1993), pp. 302-309.

[2] Bhargavi, Y. and Eswarlal, T. , (2015), Fuzzy $\Gamma-$ Semirings, International Journal of Pure and Applied Mathematics 98, pp. 339-349.

[3] Biswas, B. B. and Biswas, B. K. , (2000), Ph.D. thesis, Fuzzy Ideals of Semirings and Near-Ring.

[4] Dutta, T. K. and Biswas, B. K. , (1994), Fuzzy prime ideals of a semiring., Bulletin of the Malaysian Mathematical Sciences Society 17, pp.9-16.

[5] Dutta, T. K. and Sardar, S. K. , (2002,) On the Operator Semirings of a $\Gamma-$ Semiring, South-east Asian Bull. of Math , pp.203-213.

[6] Dutta, T. K., Sardar, S. K. and Goswami, Sarbani ,Operations on fuzzy ideals of $\Gamma-$ semirings, Proceedings of national seminar on algebra,analysis and discrete mathematics arXiv: 1101. 4791v1 [math. RA]

[7] Gupta, S. , (2018), Ph.D. Thesis, On some problems in a Γ -Semiring and their Ideals.

[8] Joseph, R. , (2008), Ph.D.thesis, Some Results on Ideals of Semiring and their Smash products.

[9] Mordson J. N. and Malik D. S. ,(1998) Fuzzy Commutative Algebra, World Scientific.

[10] Nobusawa, N. , (1964) On a generalization of the ring theory,Osaka J. math 1, pp. 81-89.

[11] Rao, M. M. K, (1995), $\Gamma-$ semirings-I, South-East Asian Bull. of Math., 19, pp. 49-54.

[12] Rao, M. M. K , (2016), On fuzzy k-ideals, k-fuzzy ideals and fuzzy 2-prime ideals in a $\Gamma-$ Semiring, J. Appl. math and informatics 34, pp. 405-419.

[13] Rosenfeld, A. , (1971), Fuzzy Groups, J. Math Analysis Applications 35, pp. 512-517.

[14] Sharma, Tilak Raj , Sharma, Ritu and Sharma, Anuj , (2022), Fuzzy Ideals of a $\Gamma-$ semiring, published in AIP conference proceedings 2451 , pp. 020016-1 to 020016-7. https://doi.org/10.1063/5.0095244.

[15] Vandiver, H. S. , (1934), Note on a simple type of algebra in which the cancellation law of addition does not hold, bull., Amer. Math. Soc., 40, pp. 914-920

[16] Zadeh, L. A. , 1965, Fuzzy sets, inform and control, 8, pp. 330-353.