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On Some Results of Strong Γ-Subgroups of a Strong Γ −Group

Vikram Singh Kapil[1], Anil Kumar[2] and Tilak Raj Sharma[3]

[1]*Department of Mathematics,*
Government Degree College Ghumarwin (HP), India
[2]*Research Scholar,*
School of Sciences, Career Point University, Kota, Rajasthan, India
[3]*Department of Mathematics, Himachal Pradesh University,*
Regional Center, Khaniyara, Dharamshala (HP), India
[1]*Corresponding Author: vikramkapil6@gmail.com* [2]*planplus8@gmail.com*
[3]*trpangotra@gmail.com*

Abstract

In this paper, we study the concept of strong $\mathit{\Gamma}$ −group, strong $\mathit{\Gamma}$ −subgroup and their properties viz: Γ −cosets, Normal $\mathit{\Gamma}$ −subgroups, one-to-one correspondence between any two strong Γ −left(right) cosets of a strong $\mathit{\Gamma}$ −subgroup in a strong $\mathit{\Gamma}$ −group etc. Further if H and K be two strong Γ − subgroups of a strong Γ − group G. Then $H\Gamma K$ is a strong Γ −subgroup of a strong Γ −group G if and only if $H\Gamma K = K\Gamma H$.

Keywords: Strong $\mathit{\Gamma}$ −group, Strong $\mathit{\Gamma}$ −subgroup, Γ −cosets, Normal $\mathit{\Gamma}$ −subgroups.

1. INTRODUCTION

The notion of a ternary algebraic system was introduced by Lehmer [1] in 1932.As a speculation of ring, the notion of a Γ−ring was introduced by Nobusawa [7] in 1964. In 1981, Sen [8] introduced the notion of a Γ−semigroup as a generalization of semigroup. In 1995, Rao [2-5] introduced the notion of a Γ−semiring as a generalization of Γ−ring, ring, ternary semiring and semiring. Rao [6] introduced the notion of field Γ−semiring and Γ−field. Semi group, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. The formal study of semi groups begins in the early 20th century. Rao [5] studied ideals of Γ−semirings, semirings, semigroups and Γ−semigroups. Kumar [13] introduced the notion of a strong Γ −group as a generalisation of Γ −group and studied some of the properties of strong Γ −group.

2. PRELIMINARIES

Definition 2.1. [5] A semigroup is an algebraic system $(G,.)$ consisting of a non-empty set G together with an associative binary operation '.'

Definition 2.2. [5] An algebraic system $(G,.)$ consisting of a non-empty set G together with an associative binary operation '.' is called a group if it satisfies the following:
$(\boldsymbol{i})$ there exists $e \in G$, such that $x.e = e.x = x$ for all $x \in G$.
$(\boldsymbol{ii})$ if for each $x \in G$ there exists $y \in G$, such that $x.y = y.x = e$.

Definition 2.3. [5] Let G and Γ be non-empty sets. Then we call G a Γ −semigroup if there exists a mapping $G \times \Gamma \times G \to G$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in G, \alpha \in \Gamma$), such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$.

Definition 2.4. [5] A Γ −semigroup G is said to be commutative if $x\alpha y = y\alpha x$ for all $x, y \in G$ for all $\alpha \in \Gamma$.

Definition 2.5. [5] Let G be a Γ −semigroup. An element e ∈ G is said to be unity if for each $x \in G$ there exists α∈ Γ such that $x\alpha e = e\alpha x = x$.

Definition 2.6. [11] An element x of a Γ −semigroup G is said to be a strong Γ −idempotent if $x\gamma x = x$ for all $\gamma \in \Gamma$.

Definition 2.7. [11] A Γ −semigroup G is said to be strong Γ −idempotent if every element of G is strong Γ −idempotent.

Definition 2.8. [5] In a Γ −semigroup G with unity e, an element $x \in G$ is said to be left invertible (right invertible) if there exists $y \in G, \alpha \in \Gamma$ such that $y\alpha x = e$ $(x\alpha y = e)$.

Definition 2.9. [12] In a Γ −semigroup G with unity e, an element $u \in G$ is said to be unit if there exists $v \in G$ and α ∈ Γ, such that $u\alpha v = e = v\alpha u$.

Definition 2.10. [12] A Γ −semigroup G with zero element 0 is said to hold cancellation laws if $x \neq 0, x\alpha y = x\alpha z, y\alpha x = z\alpha x$, where $x, y, z \in G, \alpha \in \Gamma$ then $y = z$.

Definition 2.11. [5] A Γ −semigroup G is said to be Γ −group if it satisfies the following
$(\boldsymbol{i})$ if there exists $e \in G$ and for each $x \in G$ there exists α∈ Γ, such that $x\alpha e = e\alpha x = x$.

(***ii***) if for each element $0 \neq x \in G$ there exists $y \in G, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = e$.

Every group G is a Γ −group if $\Gamma = G$ and ternary operation is $x\alpha y$ defined as the binary operation of the group.
The unity of a Γ −group may not be unique.

Example 2.12. Let G and Γ be the set of all rational numbers and the set of all natural numbers respectively. Define the ternary operation $G \times \Gamma \times G \to G$ by $(x, \alpha, y) \to x\alpha y$, using the usual multiplication. Then G is a Γ −group.

Example 2.13. Let $G = \{0,1\}$ and $\Gamma = \{\alpha, \beta\}$. We define operations with the following tables:

α	0	1
0	0	0
1	0	1

β	0	1
0	0	1
1	1	0

Then G is a Γ −group with unity 0 and 1.

Definition 2.14.[13] Let G be a $\Gamma - semigroup$.An element $e \in G$ is said to be strong identity e if for each $x \in G$,we have $x\alpha e = e\alpha x = x$ for all $\alpha \in \Gamma$.

Definition 2.15. [13] Let G be a $\Gamma - semigroup$ with strong identity $e \in G$.An element $x \in G$ is said to be have strong inverse in G if there exists $y \in G$ such that $x\alpha y = y\alpha x = e$ for all $\alpha \in \Gamma$.

Definition 2.16. [13] A $\Gamma - semigroup$ G is said to be a strong $\Gamma - group$ if it satisfies:
(i) If G has strong identity $e \in G$;
(ii) And every element has strong inverse in G.

Example 2.17. Let G be the set of positive rational numbers and Γ be the set of all real numbers whose square is 1. Define the ternary operation $G \times \Gamma \times G \longrightarrow G$ by $(x, \alpha, y) \to x.|\alpha|.y$, where '.' is the usual multiplication. Then G is a strong $\Gamma - group$.

3. PROPERTIES OF STRONG $\Gamma - GROUPS$

Definition 3.1. Let G be a strong $\Gamma - group$ and let H be a strong $\Gamma - subgroup$. For any $a \in G$ and $\alpha \in \Gamma$, the strong $\Gamma - left\ coset$ of H (determined by $'a'\ and\ '\alpha'$) is defined as the set $a\alpha H = \{a\alpha h: h \in H, \alpha \in \Gamma\}$.

Similarly, the strong $\Gamma - right\ coset$ of H is defined as the set $H\alpha a = \{h\alpha a: h \in H, \alpha \in \Gamma\}$.

Theorem 3.2. If G is strong $\Gamma - group$ which is commutative, then for any element $a \in G$ and $\alpha \in \Gamma$, the strong $\Gamma - left\ coset$ of H is equal to the corresponding strong $\Gamma - right\ coset$ of H in G.

Proof: Let $a \in G$ be any arbitrary element. Since G is abelian strong $\Gamma - group$. Therefore $x\alpha y = y\alpha x$ for all $x, y \in G$ and $\alpha \in \Gamma$. Hence $a\alpha H = \{a\alpha h: h \in H, \alpha \in \Gamma\} = \{h\alpha a: h \in H, \alpha \in \Gamma\} = H\alpha a$.

Theorem 3.3. Let G be a strong $\Gamma - group$ and let H be a strong $\Gamma - subgroup$. Then $a \in H$ if and only if $a\alpha H = H$ for $\alpha \in \Gamma$.

Proof: Let $a \in H$ be any element. Let $x \in a\alpha H$, then $x = a\alpha h$ for some $h \in H$. Thus $x \in H, H$ being a strong $\Gamma - subgroup$ of G. Therefore $a\alpha H \subseteq H$. For the reverse inclusion let $x \in H$. Since $a \in H$ there exists $b \in H$ such that $a\delta b = e$, where 'e' is strong identity of G and for all $\delta \in \Gamma$. Now $x = e\beta x = (a\alpha b)\beta x = a\alpha(b\beta x) \in a\alpha H$. Hence $H \subseteq a\alpha H$.
Conversely, assume that $a\alpha H = H$ for $\alpha \in \Gamma$. Since H is a strong $\Gamma - subgroup$, therefore $a\alpha e = e\alpha a = a$ for all $\alpha \in \Gamma$, where 'e' is strong identity of H.Thus $a = a\alpha e \in a\alpha H = H$.

Theorem 3.4. Let G be a strong $\Gamma - group$ and let H be a strong $\Gamma - subgroup$. Then for any $a, b \in G$ and $\alpha, \beta \in \Gamma$, $a\alpha H = b\beta H$ if and only if $c\delta b \in H$ for $\delta \in \Gamma$, where 'c' is strong inverse of 'a' in G.

Proof: Let $a, b \in G$ and $\alpha, \beta \in \Gamma$. Assume that $a\alpha H = b\beta H$. Since $b = b\beta e \in b\beta H = a\alpha H$. This implies that $b = a\alpha h$ for some $h \in H$. Therefore $c\delta b = c\delta(a\alpha h) = (c\delta a)\alpha h = e\alpha h = h \in H$, where 'e' is strong identity of G.
Conversely, assume that $c\delta b \in H$ for $\delta \in \Gamma$. Now $c\delta b = h$ for some $h \in H$. Thus $b = e\delta b = (a\alpha c)\delta b = a\alpha(c\delta b) = a\alpha h$. Therefore $b\beta H = (a\alpha h)\beta H = a\alpha(h\beta H) = a\alpha H$.

Theorem 3.5. Any two strong $\Gamma - left(right)\ cosets$ are either disjoint or identical.

Proof: Let G be a strong $\Gamma - group$ and let H be a strong $\Gamma - subgroup$. Let $a\alpha H$ (determined by 'a' and 'α') and $b\beta H$ (determined by 'b' and 'β') be the two strong $\Gamma - left\ cosets$ of H in G, where $a, b \in G$ and $\alpha, \beta \in \Gamma$. If $a\alpha H \cap b\beta H = \emptyset$, then we are done. So, let $a\alpha H \cap b\beta H \neq \emptyset$, there exists $x \in a\alpha H \cap b\beta H$. This implies that $x \in a\alpha H$ and $x \in b\beta H$. Now $x = a\alpha h_1$ for some $h_1 \in H$ and $x = b\beta h_2$ for some $h_2 \in H$. Therefore $a\alpha h_1 = b\beta h_2$. Now $a = a\alpha e = a\alpha(h_1\delta\, h_3) = (a\alpha h_1)\delta h_3 = (b\beta h_2)\delta h_3 = b\beta(h_2\delta h_3) = b\beta h_4$, where $h_2\delta h_3 = h_4 \in H$. Hence $a\alpha H = (b\beta h_4)\alpha H = b\beta(h_4\alpha H) = b\beta H$.

Theorem 3.6. There is one to one correspondence between any two strong $\Gamma - left(right)\ cosets$ of H in G.

Proof: Let G be a strong $\Gamma - group$ and let H be a strong $\Gamma - subgroup$. Let $a\alpha H$ (determined by 'a' and 'α') and $b\beta H$ (determined by 'b' and 'β') be the two strong $\Gamma - left\ cosets$ of H in G, where $a, b \in G$ and $\alpha, \beta \in \Gamma$. Define a map $f: a\alpha H \to b\beta H$ by $f(a\alpha h) = b\beta h$ for all $a\alpha h \in a\alpha H$. Let $x, y \in a\alpha H$ such that $f(x) = f(y)$. Therefore $x = a\alpha h_1$ and $y = a\alpha h_2$ for some $h_1 \in H$ and $h_2 \in H$. Now $f(x) = f(y)$ implies that $f(a\alpha h_1) = f(a\alpha h_2)$. Thus $b\beta h_1 = b\beta h_2$, by the left cancellation law in the strong $\Gamma - group\ G$, we have $h_1 = h_2$. Hence $a\alpha h_1 = a\alpha h_2$. Therefore $x = y$. This implies that f is one to one.
Let $y \in b\beta H$. Then $y = b\beta h$ for some $h \in H$. Now let $x = a\alpha h \in a\alpha H$ and $f(x) = f(a\alpha h) = b\beta h = y$. Thus f is onto.
Hence $a\alpha H$ and $b\beta H$ are in one-one correspondences.

Theorem 3.7. Let H be a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$. Then a relation R in G is defined by aRb if and only if $a\alpha c \in H$ for all $\alpha \in \Gamma$, where 'c' is strong inverse of 'b' in G is an equivalence relation.

Proof: Let $a \in G$ be any element. Then $a\beta d = d\beta a = e \in H$ for all $\beta \in \Gamma$, where 'd' is strong inverse of 'a' in G and 'e' is strong identity of G.Thus aRa and relation is reflexive. Let aRb then $a\alpha c \in H, \alpha \in \Gamma$, where 'c' is strong inverse of 'b' in G. Therefore $(a\alpha c)\beta(b\delta d) = a\alpha(c\beta b)\delta d = (a\alpha e)\delta d = a\delta d = e$ for $\alpha, \beta, \delta \in \Gamma$, where 'd' is strong inverse of 'a' in G. Hence $b\delta d$ is the strong inverse of $a\alpha c$. Since H is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$. This implies that $b\delta d \in H$. The relation is symmetric. Now, let aRb and bRc. Then $a\alpha d \in H$ and $b\beta f \in H$ for some $\alpha, \beta \in \Gamma$, where 'd' and 'f' are the strong inverses of 'b' and 'c' in $G. H$ being a strong $\Gamma - subgroup, (a\alpha d)\delta(b\beta f) \in H,$ for $\delta \in \Gamma$. Thus $a\alpha(d\delta b)\beta f = (a\alpha e)\beta f = a\beta f \in H$. Hence aRc and relation is transitive.

Theorem 3.8. The strong $\Gamma - group\ G$ is equal to the union of all strong left $\Gamma - cosets$ of H in G, where H is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$.

Proof: Let $e\alpha H = H, a\beta H, b\gamma H, \dots\dots\dots\dots\dots$ are all the strong left $\Gamma - cosets$ of H in G, where $\alpha, \beta, \dots\dots \in \Gamma$ and $a, b, \dots\dots . \in G$, 'e' is the strong identity of G. We shall show that $G = H \cup (a\beta H) \cup (b\gamma H) \dots\dots\dots\dots$
Let $x \in G$ be any element. Then $x\alpha H$ for $\alpha \in \Gamma$ is the strong left $\Gamma - coset$ of H in G. Since H is a strong $\Gamma - subgroup$, therefore $e \in H$ and $x = x\alpha e \in x\alpha H$. Thus $x \in H \cup (a\beta H) \cup (b\gamma H) \dots\dots\dots\dots (x\alpha H) \dots\dots\dots$
Conversely, assume that $x \in a\delta H$ for $a \in G$ and $\delta \in \Gamma$. Then $x = a\delta h$ for some $h \in H$. Since $h \in H$, therefore $h \in G$.Thus $x \in G$.

Definition 3.9. (Order of a strong $\Gamma - group$)
Let G be a strong $\Gamma - group$.Then the number of elements in G is defined as the order of G.

Theorem 3.10. (Lagrange's theorem for strong $\Gamma - group$).
Let G be a strong $\Gamma - group\ G$ of order n. Let Γ be a finite set and H be a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$ of order m. Then order of H is a divisor of order of G.

Proof: Suppose $h_1, h_2, \ldots \ldots \ldots \ldots . . h_m$ be m distinct elements of H and $\Gamma = \{\alpha_1, \ldots \ldots . . \alpha_s\}$. Let $a \in G, \alpha \in \Gamma$. Then $a\alpha H$ is the strong left $\Gamma - coset$ of H in G and we have $a\alpha H = \{a\alpha h_1, \ldots \ldots \ldots . a\alpha h_m\}$. Thus $a\alpha H$ has m distinct members, since $a\alpha h_i = a\alpha h_j, 1 \leq i \leq m, 1 \leq j \leq m;\ i \neq j$ implies $h_i = h_j$, a contradiction by left cancellation law in strong $\Gamma - group\ G$. Therefore each strong left $\Gamma - coset$ of H in G has m distinct members. Moreover, any two distinct strong left $\Gamma - cosets$ of H in G are disjoint. Since G is a finite strong $\Gamma - group$, the number of distinct strong left $\Gamma - cosets$ of H in G will be finite, say equal to k. The union of these k distinct strong left $\Gamma - cosets$ of H in G is equal to G.
Thus the number of elements in $G =$ number of elements in $a_1\alpha_1 H + \ldots \ldots \ldots \ldots \ldots \ldots . +$ number of elements in $a_1\alpha_s H +$.............................+ number of elements in $a_2\alpha_1 H +$..................... + number of elements in $a_2\alpha_s H +$.......................+ number of elements in $a_n\alpha_1 H +$...................................+ number of elements in $a_n\alpha_s H$, where $G = \{a_1, \ldots \ldots \ldots . . a_n\}$.
Therefore $n = k.m$ and m is a divisor of n. Hence order of H is a divisor of order of G.

Definition 3.11. (Strong normal $\Gamma - subgroup$)
Let G be a strong $\Gamma - group$ and let H be a strong $\Gamma - subgroup$. Then H is Strong normal $\Gamma - subgroup$ of strong $\Gamma - group\ G$ if $a\alpha H = H\alpha a$ for all $\alpha \in \Gamma$ and for all $a \in G$.

Theorem 3.12. Let G be a strong $\Gamma - group$ which is commutative. Then every strong $\Gamma - subgroup$ of G is strong normal $\Gamma - subgroup$.

Proof: Let H be a strong $\Gamma - subgroup$ of strong $\Gamma - group\ G$. Then for $a \in G$ and $\alpha \in \Gamma, a\alpha H = \{a\alpha h : h \in H\}$. Since G is commutative, therefore $x\beta y = y\beta x$ for all $x, y \in G$ and for all $\beta \in \Gamma$. Thus $a\alpha H = \{a\alpha h : h \in H\} = \{h\alpha a : h \in H\} = H\alpha a$. Hence H is strong normal $\Gamma - subgroup$.

Theorem 3.13. Let G be a strong $\Gamma - group$ and H be a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$. Then H is a strong normal $\Gamma - subgroup$ of a strong $\Gamma - group\ G$ if and only if $(g\alpha h)\beta g' \in H$ for all $\alpha, \beta \in \Gamma$ and $h \in H, g \in G$, where g' is strong inverse of g in G.

Proof: Let $g \in G, h \in H$ and $\alpha, \beta \in \Gamma$. Then $g\alpha h \in g\alpha H$. Since H is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$.Therefore $g\alpha H = H\alpha g$ for all $\alpha \in \Gamma$ and $g \in G$. Now $g\alpha h \in H\alpha g$ implies that $g\alpha h = h_1\alpha g$ for some $h_1 \in H$. Thus $(g\alpha h)\beta g' = (h_1\alpha g)\beta g' = h_1\, \alpha(g\beta g') = h_1\, \alpha e = h_1 \in H$, where 'e' is strong identity of G.
Conversely, assume that $(g\alpha h)\beta g' \in H$ for all $\alpha, \beta \in \Gamma$ and $h \in H, g \in G$, where g' is strong inverse of g in G. Let $x \in g\alpha H$ for $g \in G$ and $\alpha, \beta, \gamma \in \Gamma$ Then $x = g\alpha h$ for $h \in H$. Therefore $x = g\alpha h = (g\alpha h)\beta e = (g\alpha h)\beta(g'\alpha g) = \big((g\alpha h)\beta g'\big)\alpha g \in H\alpha g$. Thus $g\alpha H \subseteq H\alpha$g. Similarly, the reverse inclusion Hαg ⊆ gαH follows. Thus $g\alpha H = H\alpha g$. Hence H is a strong normal $\Gamma - subgroup$.

Theorem 3.14. The Center of a strong $\Gamma - group\ G, C(G)$ is a strong normal $\Gamma - subgroup$ of a strong $\Gamma - group\ G$.

Proof: We know that $C(G)$ is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$. Let $h \in C(G)$ and $g \in G$ Since $h \in C(G), h\alpha g = g\alpha h$ for all $g \in G$ and $\alpha \in \Gamma$.Therefore $(g\alpha h)\beta g' = (h\alpha g)\beta g' = h\alpha(g\beta g') = h\alpha e = h \in C(G)$, for $\alpha\ \beta \in \Gamma$, where g' is strong inverse of g in G and 'e' is the strong identity of G. Hence $C(G)$ is a strong normal $\Gamma - subgroup$ of a strong $\Gamma - group\ G$.

Theorem 3.15. The intersection of two strong normal $\Gamma - subgroup$s of a strong $\Gamma - group$ is again a strong normal $\Gamma - subgroup$.

Proof: Let G be a strong $\Gamma - group$ and let H_1 and H_2 be two strong normal $\Gamma - subgroup$s of a strong $\Gamma - group\ G$. Then $H_1 \cap H_2$ is a strong $\Gamma - subgroup$. Let $h \in H_1 \cap H_2$ and $g \in G$. Therefore $h \in H_1$ and $h \in H_2$. Thus $(g\alpha h)\beta g' \in H_1$ and $(g\alpha h)\beta g' \in H_2$ for all $\alpha, \beta \in \Gamma$, where g' is strong inverse of g in G. Therefore $(g\alpha h)\beta g' \in H_1 \cap H_2$ and $H_1 \cap H_2$ is a strong normal $\Gamma - subgroup$ of strong $\Gamma - group\ G$.

Theorem 3.16. Let G be a strong $\Gamma - group$ and H be a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$. Then H is a strong normal $\Gamma - subgroup$ of a strong $\Gamma - group\ G$ if and only if $(a\alpha H)\delta(b\beta H) = (a\alpha b)\delta H$ for all $a, b \in G$ and for all $\alpha, \beta, \delta \in \Gamma$.

Proof: Let $a, b \in G$ and $\alpha, \beta, \delta \in \Gamma$. Since H is a strong normal $\Gamma - subgroup$ of a strong $\Gamma - group\ G$, therefore $b\delta H = H\delta b$, for all $\delta \in \Gamma$.Thus $(a\alpha H)\delta(b\beta H) = \big(a\alpha(H\delta b)\big)\beta H = \big(a\alpha(b\delta H)\big)\beta H = (a\alpha b)\delta(H\beta H) = (a\alpha b)\delta H$. Conversely, assume that $(a\alpha H)\delta(b\beta H) = (a\alpha b)\delta H$ for all $a, b \in G$ and for all $\alpha, \beta, \delta \in \Gamma$. Let $g \in G$, then there exists $g' \in G$ such that $g\gamma g' = g'\gamma g = e$ for all $\gamma \in \Gamma$, where 'e' is the strong identity of G. Thus $g\alpha H, g'\beta H$ are the strong left $\Gamma - cosets$ of H in G. Now $(g\alpha H)\delta(g'\beta H)$ is a strong left $\Gamma - coset$ of H in G. Since $e \in H$, therefore $e = (g\alpha e)\delta(g'\beta e) \in (g\alpha H)\delta(g'\beta H)$. Thus $H \cap \{(g\alpha H)\delta(g'\beta H)\} \neq \emptyset$. The strong $\Gamma - left(right)\ cosets$ of H in G are either identical or disjoint**,** and hence $(g\alpha H)\delta(g'\beta H) = H$. Let $h \in H$ be any element, then $(g\alpha h)\delta(g'\beta h) \in$

$(g\alpha H)\delta(g'\beta H) = H$. This implies that $((g\alpha h)\delta g')\beta h \in H$. Therefore $((g\alpha h)\delta g')\beta h)\gamma h' \in H\gamma h' = H$ for all $\alpha, \beta, \delta \in \Gamma$. Thus $(g\alpha h)\delta g' \in H$, where h' is strong inverse of h in H. Hence H is a strong normal $\Gamma - subgroup$ of a strong $\Gamma - group\ G$.

Definition 3.17. (Product of two strong $\Gamma - subgroups$)
Let H and K be two strong $\Gamma - subgroups$ of a strong $\Gamma - group\ G$. Then the product of two strong $\Gamma - subgroups$ is defined as the set of all elements $h\alpha k$, for all $h \in H, \alpha \in \Gamma, k \in K$. It is denoted by $H\Gamma K$.

Theorem 3.18. Let H and K be two strong $\Gamma - subgroups$ of a strong $\Gamma - group\ G$. Then $H\Gamma K$ is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$ if and only if $H\Gamma K = K\Gamma H$.

Proof: Let $x \in H\Gamma K$ be any element. Since $H\Gamma K$ is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$, therefore $x' \in H\Gamma K$, where x' is the strong inverse of x in $H\Gamma K$. Then $x' = h'\gamma k'$ for some $h' \in H, \gamma \in \Gamma, k' \in K$. Also $x\beta x' = x'\beta x = e$ for all $\beta \in \Gamma$. Then $x = k\alpha h$, for some $h \in H, \alpha \in \Gamma, k \in K$. Thus $x = k\alpha h \in K\Gamma H$. Similarly, the reverse inclusion $K\Gamma H \subseteq H\Gamma K$ follows. Conversely, assume that $H\Gamma K = K\Gamma H$. Let $x, y \in H\Gamma K$ be arbitrary elements. Then $x = h_1\alpha_1 k_1\ and\ y = h_2\alpha_2 k_2$ for some $h_1, h_2 \in H$ and $k_1, k_2 \in K, \alpha_1, \alpha_2 \in \Gamma$. Let ${k_2}'$ and ${h_2}'$be the strong inverses of k_2 and h_2 in K and H respectively. Also let $z = {k_2}'\alpha_3 {h_2}'$ for $\alpha_3 \in \Gamma$. Now $x\beta z = (h_1\alpha_1 k_1)\beta({k_2}'\alpha_3 {h_2}') = h_1\alpha_1(k_1\beta {k_2}')\alpha_3 {h_2}'$. Since K is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$, therefore $k_1\beta {k_2}' \in K$. Then $(k_1\beta {k_2}')\alpha_3 {h_2}' \in K\Gamma H = H\Gamma K$ implies that $(k_1\beta {k_2}')\alpha_3 {h_2}' = h_3\alpha_4 k_3$ for some $h_3 \in H, \alpha_4 \in \Gamma, k_3 \in K$. Thus $x\beta z = h_1\alpha_1(h_3\alpha_4 k_3) = (h_1\alpha_1 h_3)\alpha_4 k_3 \in H\Gamma K$. Hence $H\Gamma K$ is a strong $\Gamma - subgroup$ of a strong $\Gamma - group\ G$.

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