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# Uniqueness of Linear q-shift Difference Polynomial of a Meromorphic Function

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**Abstract**

In this article,we prove the results on uniqueness of linear q-shift difference polynomial $L_k(f, E_q)$ for a transcendental meromorphic functions and Entire functions with zero order. The result obtained in this paper which extends and generalizes some recent results of Harina P Waghamore and Rajeshwari S [4].

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## 1. INTRODUCTION

In this paper, a meromorphic functions $f$ means meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the basic results and standard notations of Nevanlinna value distribution theory one can refer([6], [7], [15], [16]). If no poles occur, then $f$ reduces to an entire function. Let $f$ and $g$ be two non-constant meromorphic functions defined in the open complex plane. For $a \in \mathbb{C} \cup \{\infty\}$ and $k \in Z^+ \cup \{\infty\}$, the set $E(a, f) = \{z : f(z) - a = 0\}$ denotes all those $a-$points of $f$, where each $a-$point of $f$ with multiplicity is counted $k$ times in the set and the set $\overline{E}(a, f) = \{z : f(z) - a = 0\}$, denotes all those $a-$points of $f$, where the multiplicities is ignored. We say that $f$ and $g$ share $a$ $CM$ (counting multiplicities) if $f - a$ and

$g - a$ have the same zeros with same multiplicities and we have $E(a, f) = E(a, g)$. In addition, we say that $f$ and $g$ share $a$ $IM$ (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros with ignoring multiplicities and we will have $\overline{E}(a, f) = \overline{E}(a, g)$. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, then $f$ and $g$ share $\infty$ CM. While $N(r, f)$ denotes the counting function of poles of $f$ whose multiplicities are taken into account(respectively $\overline{N}(r, f)$ denotes the reduced counting function when multiplicities are ignored). Also we have $N(r, a; f) = N\left(r, \frac{1}{f-a}\right)$, denotes the counting function of $a$ points of $f(z)$ whose multiplicities are counted(respectively $\overline{N}(r, a; f)$ denotes for the reduced counting function when multiplicities are ignored).

In 2012, K. Liu, X. Liu and T. B. Cao in 2012 [8] proved the following result.

**Theorem 1** *[8] Let $f$ be a transcendental entire function of $\rho_2(f) < 1$. For $n \geq t(k+1)$, then $[P(f)f(z+c)]^{(k)} - \alpha(z)$ has finitely many zeros.*

**Theorem 2** *[8] Let $f$ be a transcendental meromorphic function of $\rho_2(f) < 1$, not a periodic function with period $c$. If $n \geq (t+1)(k+1)+1$, then $[f^n(\triangle_c f)^s]^{(k)} - \alpha(z)$ has finitely many zeros.*

**Theorem 3** *[8] Let $f$ be a transcendental entire function of $\rho_2(f) < 1$. For $n \geq t(k+1)+5$, then $[P(f)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

**Theorem 4** *[8] Let $f$ be a transcendental meromorphic function of $\rho_2(f) < 1$, not a periodic function with period $c$. If $n \geq (t+2)(k+1)+3+s$, then $[f^n(\triangle_c f)^s]^{(k)} - \alpha(z)$ has infinitely many zeros.*

**Theorem 5** *[8] Let $f$ and $g$ be a transcendental entire functions of $\rho_2(f) < 1$, not a periodic function with period $c$. If $n \geq 2k+m+6$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

**Theorem 6** *[8] Let $f$ and $g$ be a transcendental entire functions of $\rho_2(f) < 1$, not a periodic function with period $c$. If $n \geq 2k+m+6$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the 1 IM.*

In 2013, Harina P. Waghamore and Tanuja A.[5] extend Theorem 4 and Theorem 5 to meromorphic functions.

**Theorem 7** *[5] Let $f$ and $g$ be a transcendental meromorphic function with zero order. If $n \geq 4k+m+8$, $[f^n(f^m-1)f(qz+c)]^{(k)}$ and $[g^n(g^m-1)g(qz+c)]^{(k)}$ share the 1 CM, the $f = tg$, where $t^{n+1} = t^m = 1$.*

**Theorem 8** *[5] Let $f$ and $g$ be a transcendental meromorphic function with zero order. If $n \geq 5k+4m+17$, $[f^n(f^m-1)f(qz+c)]^{(k)}$ and $[g^n(g^m-1)g(qz+c)]^{(k)}$ share the $1$ IM, the $f = tg$, where $t^{n+1} = t^m = 1$.*

In 2017, Harina P. wghamore and Rajeshwari S [4], has extended the Theorem 7 and Theorem 8 to difference polynomials and obtain the following results.

**Theorem 9** *[4] Let $f$ and $g$ be transcendental meromorphic (res. entire) function with zero order. If $n \geq 4k+8(n \geq 2k+6)$, $[P(f)f(qz+c)]^{(k)}$ and $[P(g)g(qz+c)]^{(k)}$ share the $1\mathrm{CM}$, then*

1. *$f \equiv tg$ for a constant $t$ such that $t^d = 1$.*
2. *$f$ and $g$ satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1,w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$.*

**Theorem 10** *[4] Let $f$ and $g$ be a transcendental meromorphic(resp. entire) function with zero order. If $n \geq 10k+14(n \geq 5k+12)$, $[P(f)f(qz+c)]^{(k)}$ and $[P(g)g(qz+c)]^{(k)}$ share the $1\mathrm{IM}$, then the conclusion of Theorem 9 still holds.*

## 2. DEFINITIONS

**Definition 1** *[1] The order $\rho(f)$ of a meromorphic function $f(z)$ is defined as,*

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}.$$

**Definition 2** *[1] Let $k$ be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of $f$, where an $a$-point of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, then we say that $f$ and $g$ share the value $a$ with weight $k$.*

**Definition 3** *[1] Let $f$ and $g$ share the value $a$ IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$. Clearly, $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}(r,a;f) + \overline{N}(r,a;g)$.*

Now, let us recall the definitions of q-shift difference polynomial $L_k(f,E_q)$ and q-difference operator $L_k(f,\triangle)$ of a meromorphic function $f$.

**Definition 4** *[2] For a meromorphic function $f$ and $c, q(\neq 0) \in \mathbb{C}$, let us denote its q-shift, $E_{q,c}f$ and q-difference operator, $\Delta_{q,c}f$ respectively by $E_{q,c}f(z) = f(qz+c)$ and $\Delta_{q,c}f(z) = f(qz+c) - f(z)$, $\Delta_{q,c}^k f(z) = \Delta_{q,c}^{k-1}(\Delta_{q,c}f(z))$, for all $k \in \mathbb{N} - \{1\}$.*

In 2021, Haldar [2] defined the linear $q$-shift and linear $q$-difference operators denoted respectively by $L_k\left(f, E_q\right)$ and $L_k(f, \Delta)$, for a non-constant meromorphic function $f$ in a generalized way as follows,

**Definition 5** *[2] Let us define,*

$$L_k\left(f, E_q\right)=a_k f\left(q_k z+c_k\right)+a_{k-1} f\left(q_{k-1} z+c_{k-1}\right)+\ldots+a_0 f\left(q_0 z+c_0\right) \quad (2.1)$$

*and*

$$L_k(f, \Delta)=a_k \Delta_{q_k, c_k} f(z)+a_{k-1} \Delta_{q_{k-1}, c_{k-1}} f(z)+\ldots+a_0 \Delta_{q_0, c_0} f(z), \quad (2.2)$$

*where $a_0, a_1, \ldots, a_k ; q_0, q_1, \ldots, q_k ; c_0, c_1, \ldots, c_k$ are complex constants.*
*From above definition one can easily observe that*

$$L_k(f, \Delta)=L_k\left(f, E_q\right)-\sum_{j=0}^k a_j f(z).$$

*If we choose $q_j=q^j, c_j=c$ and $a_j=(-1)^{k-j}\begin{pmatrix}k \\ j\end{pmatrix}$ for $0 \leq j \leq k$, then $L_k(f, \Delta)$ reduces $\Delta_{q, c}^k f(z)$.*

Now, it would be interesting to ask, if the $q-$ shift $f(qz+c)$ in Theorems 9 and 10, can be extended any further and what happens if we consider an intermediate sharing, between counting multiplicity and ignoring multiplicity sharing?.
In this paper, we try to solve these interesting question by considering Linear $q$-difference polynomial $L_k\left(f, E_q\right)$ as define in Definition 5, and we obtain Theorems 11 and 12, which extends and generalizes the Theorems 9 and 10 respectively.

**Theorem 11** *Let $f$ and $g$ be a transcendental meromorphic functions with zero order. Let $P(z)=a_n z^n+a_{n-1} z^{n-1}+\ldots+a_1 z+a_0$ be a non-constant polynomial with constant co-efficients $a_0, a_1, \ldots, a_{n-1}, a_n(\neq 0)$, and $m$ be the number of distinct zeros of $P(z)$. If $n \geq k+2(m+1)(l+2)+2$, $\left[P(f) L_k\left(f, E_q\right)\right]^{(l)}$ and $\left[P(g) L_k\left(g, E_q\right)\right]^{(l)}$ share the $1$ CM has infinitely many zeros, then the following conditions are satisfied.*

1. *$f \equiv t g$ for a constant $t$ such that $t^d=1$.*
2. *$f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$,*
   *where $R\left(w_1, w_2\right)=P\left(w_1\right) L_k\left(w_1, E_q\right)$-$P\left(w_2\right) L_k\left(w_2, E_q\right)$.*

**Theorem 12** *Let $f$ and $g$ be a transcendental meromorphic functions with zero order. Let $P(z)=a_n z^n+a_{n-1} z^{n-1}+\ldots+a_1 z+a_0$ be a non-constant polynomial with constant co-efficients $a_0, a_1, \ldots, a_{n-1}, a_n(\neq 0)$, and $m$ be the number of distinct zeros of $P(z)$. If $n \geq 5k+2m(l+2)+13$ , $\left[P(f) L_k\left(f, E_q\right)\right]^{(l)}$ and $\left[P(g) L_k\left(g, E_q\right)\right]^{(l)}$ share the $1$ IM has infinitely many zeros, then the conclusion of Theorem 11 holds.*

## 3. LEMMAS

We introduce some lemmas in this section which will be required later to support the main results.

**Lemma 1** *[3] Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1, \zeta < 1$, $\in$ is enough small number. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\zeta-\in}}\right) = S(r,f), \tag{2.3}$$

for all $r$ outside of a set of finite logarithmic measure. Combining the proof of [8] with Lemma 1, we can get the following Lemma 2.

**Lemma 2** *[8] Let $f$ be a non-constant meromorphic function. Then*

$$T(r, P_n(f)) = nT(r,f) + S(r,f). \tag{2.4}$$

**Lemma 3** *[2, 12] Let $f(z)$ be a transcendental meromorphic function of zero order and $L_k(r, E_q)$, be a linear q-shift polynomial defined in (2.1). Let $P(f) = a_n f^n + a_{n-1}f^{n-1} + \ldots + a_1 f + a_0$ be a polynomial of degree $n$. Then we have*

$$(n-k-1)T(r,f) + S(r,f) \leq T(r, P(f)L_k(f, E_q)) \leq (n+k+1)T(r,f) + S(r,f). \tag{2.5}$$

*If $f$ is a transcendental entire function of zero order, then*

$$T(r, P(f)L_k(r, E_q)) = (n+1)T(r,f) + S(r,f). \tag{2.6}$$

**Lemma 4** *[17] Let $f(z)$ be a transcendental meromorphic function of zero order. Then*

$$T(r, f(qz)) = T(r,f) + S(r,f) \tag{2.7}$$

*on a set of logarithmic density 1.*

**Lemma 5** *[1] Let $f(z)$ be a transcendental mermorphic function of finite order. Then*

$$T(r, f(z+c)) = T(r, f(z+c)) + S(r,f) \tag{2.8}$$

**Lemma 6** *[1] Let $f(z)$ be a transcendental meromorphic function of zero order. Then*

$$T(r, f(qz+c)) = T(r, f(z)) + S(r,f) \tag{2.9}$$

*on a set of logarithmic density 1.*

**Lemma 7** *[15] Let $F$ and $G$ be non constant meromorphic functions. If $F$ and $G$ share* $1\mathrm{CM}$, *then one of the following three cases holds:*
*(i)* $\max\{T(r,F),T(r,G)\} \le N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)+S(r,F)+S(r,G)$
*(ii)* $F=G$
*(iii)* $F.G=1$.

**Lemma 8** *[16] Let $F$ and $G$ be non constant meromorphic function sharing the value* 1 IM. *Let*

$$H=\frac{F''}{F'}-2\frac{F'}{F-1}-\frac{G''}{G'}+2\frac{G'}{G-1}$$

*If $H\neq 0$, then*

$$\begin{aligned}T(r,F)+T(r,G) \le\ & 2\left(N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right)\\ &+3\left(\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)\right)\\ &+S(r,F)+S(r,G).\end{aligned}$$

**Lemma 9** *[16] Let $f(z)$ be a non-constant meromorphic function and let $p,k$ be positive integers. Then*

$$\begin{aligned}&T\left(r,f^{(k)}\right)\le T(r,f)+k\overline{N}(r,f)+S(r,f).\\ &N_p\left(r,\frac{1}{f^{(k)}}\right)\le T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f),\\ &N_p\left(r,\frac{1}{f^{(k)}}\right)\le N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).\end{aligned}$$

**Lemma 10** *Let $f$ and $g$ be a transcendental meromorphic functions of zero order. If $n\ge 2k+2m+4$ and*

$$[P(f)L_k(f,E_q)]^{(l)}=[P(g)L_k(g,E_q)]^{(l)} \tag{2.10}$$

*then $f=tg$, where $t^{n+1}=t^m=1$, and $f$ and $g$ satisfy the algebraic equation*

$$R\left(w_1,w_2\right)=P\left(w_1\right)\left[L_k(w_1,E_q)\right]-P\left(w_2\right)\left[L_k(w_2,E_q)\right]$$

**Proof 1** *From* (2.10), *we have*

$$P(f)L_k(f,E_q)=P(g)L_k(g,E_q)+Q(z).$$

*where $Q(z)$ is a polynomial of degree atmost $k = 1$. $Q(z) \neq 0$, then we have*

$$\frac{P(f)L_k(f,E_q)}{Q(z)} = \frac{P(g)L_k(g,E_q)}{Q(z)} + 1$$

*From the second fundamental theorem of Nevanlinna theory and by Lemma 2, we have*

$$\begin{aligned}
&(n+k+1)T(r,f) = T\left(r, \frac{P(fL_k(f,E_q))}{Q(z)}\right) + S(r,f),\\
&\leq \overline{N}\left(r, \frac{P(f)L_k(f,E_q)}{Q(z)}\right) + \overline{N}\left(r, \frac{Q(z)}{P(f)L_k(f,E_q)}\right) + \overline{N}\left(r, \frac{Q(z)}{P(g)L_k(g,E_q)}\right)\\
&+ S(r,f),\\
&\leq \overline{N}(r,P(f)) + \overline{N}(r,L_k(f,E_q)) + \overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{L_k(f,E_q)}\right)\\
&+ \overline{N}\left(r, \frac{1}{P(g)}\right) + \overline{N}\left(r, \frac{1}{L_k(g,E_q)}\right) + S(r,f) + S(r,g),\\
&\leq (k+2)T(r,f) + (m+k+1)T(r,f) + (m+k+1)T(r,g) + S(r,f)\\
&+ S(r,g). \qquad (2.11)
\end{aligned}$$

*Similarly*

$$\begin{aligned}
&(n+k+1)T(r,g) \leq (k+2)T(r,g) + (m+k+1)T(r,g) + (m+k+1)T(r,f)\\
&+ S(r,g) + S(r,f). \qquad (2.12)
\end{aligned}$$

*Thus, we get*

$$(n+k+1)\left[T(r,f)+T(r,g)\right] \leq (3k+2m+4)\left[T(r,f)+T(r,g)\right] + S(r,f) + S(r,g), \qquad (2.13)$$

*which is a contradiction with $n \geq 2k+2m+4$. Hence, we get $Q(z) \equiv 0$, which implies that*

$$P(f)L_k(f,E_q) = P(g)L_q(g,E_q). \qquad (2.14)$$

*i.e.,*

$$\begin{aligned}
&\left(a_n f^n + a_{n-1}f^{n-1} + \cdots + a_0 f\right)\{a_k f(q_k z + c_k) + a_{k-1} f(q_{k-1}z + c_k) + \ldots, + a_0 f(q_0 z + c_0)\}\\
&= \left(a_n g^n + a_{n-1}g^{n-1} + \cdots + a_0 g\right)\{a_k g(q_k z + c_k) + a_{k-1} g(q_{k-1}z + c_k) + \cdots + a_0 g(q_0 z + c_0)\}
\end{aligned}$$

*Let $h(z) = \dfrac{f(z)}{g(z)}$, we break the rest of the proof into two cases.*

***Case 1.*** *Suppose $h(z)$ is a constant. Then by substituting $f = gh$ into* (2.14)*, we obtain*

$$\begin{aligned}
&\left(a_n h^n g^n + a_{n-1}h^{n-1}g^{n-1} + \cdots + a_0 hg\right)\left[a_k\left(h(q_k z + c_k)g(q_k z + c_k)\right)\right.\\
&\left. + a_{k-1}\left(h(q_{k-1}z + c_k)g(q_{k-1}z + c_k)\right) + \ldots + a_0(h(q_0 z + c_0)g(q_0 z + c_0))\right] \equiv 0
\end{aligned}$$

*which implies,*

$$a_ng^n\left[L_k\left(g,E_q\right)\right]\left(h^{n+1}-1\right)+a_{n-1}g^{n-1}\left[L_k\left(g,E_q\right)\left(h^n-1\right)+\ldots+a_0[L_k\left(g,E_q\right)](h-1)\right]\equiv 0. \quad (2.15)$$

*This implies $h^d=1$, where $d=GCD\left\{\lambda_j : j=0,1,\ldots,n\right\}$ and*

$$\lambda_j=\begin{cases} j+1,\ if\ a_j\neq 0,\\ n+1,\ if\ a_j=0.\end{cases}$$

*where $a_n(\neq 0), a_{n-1},\ldots,a_0$ are complex constants. By the fact that $g$ is a transcendental entire functions, we have $L_k\left(g,E_q\right)\not\equiv 0$. Hence, we obtain*

$$a_ng^n[L_k\left(g,E_q\right)]\left(h^{n+1}-1\right)+a_{n-1}g^{n-1}\left[L_k\left(g,E_q\right)\right]+\ldots+a_0[L_k\left(g,E_q\right)]\left(h-1\right)\equiv 0. \quad (2.16)$$

*Thus, $f\equiv tg$, where $t$ is a constant with $t^d=1$, where $d=GCD\left(\lambda_0,\lambda_1,\ldots,\lambda_n\right)$.*
***Case 2.*** *Suppose $h(z)$ is not a constant, then $f$ and $g$ satisfy the algebraic equation $R\left(f,g\right)=0$, where*

$$R\left(\omega_1,\omega_2\right)=P(\omega_1)(L_k\left(\omega_1,E_q\right))-P(\omega_2)(L_k\left(\omega_2,E_q\right)).$$

*This completes the proof of Lemma 10.*

**Lemma 11** *[1] Let $f$ and $g$ be transcendental entire function of finite order. If $n\geq k+2m+2$, $[P(f)L_k\left(f,E_q\right)]^{(l)}=[P(g)L_k\left(g,E_q\right)]^{(l)}$ then the condition of Lemma 10 holds.*

**Proof 2** *Similarly, we can prove the result for th entire functions using $N\left(r,f\right)=\overline{N}\left(r,f\right)=S\left(r,f\right), N\left(r,g\right)=\overline{N}\left(r,g\right)=S\left(r,g\right)$, and proceeding as in the proof of Lemma 10, we get Lemma 11.*

## 4. PROOF OF THE MAIN RESULTS

**Proof of Theorem 11:** Let $F=[P(f)L_k(f,E_q)]^{(l)}, F_1=P(f)L_k(f,E_q)$ and $G=[P(g)L_k(g,E_q)]^{(l)}, G_1=P(g)L_k(g,E_q)$. Thus $F$ and $G$ share the value 1 CM. From lemma 9 and $f$ is a transcendental meromorphic function, then

$$T(r,F)\leq T(r,P(f)L_k(f,E_q))+k\overline{N}(r,f)+S(r,P(f)L_k(f,E_q)). \quad (2.17)$$

On combining (2.17) with Lemma 3, we have $S(r,F) = S(r,f)$. We also have $S(r,G) = S(r,g)$, from the same reason as above, from lemma 9 we obtain

$$N_2\left(r,\frac{1}{F}\right) = N_2\left(r,\frac{1}{[P(f)L_k(f,E_q)]^{(l)}}\right)$$
$$\leq T(r,F) - T(r,P(f)L_k(f,E_q)) + N_{l+2}\left(r,\frac{1}{P(f)L_k(f,E_q)}\right) + S(r,f). \tag{2.18}$$

Thus, from Lemma 3 and (2.18) we get

$$(n+k+1)T(r,f) = T(r,P(f)L_k(f,E_q)) + S(r,f)$$
$$\leq T(r,F) - N_2\left(r,\frac{1}{F}\right) + N_{l+2}\left(r,\frac{1}{P(f)L_k(f,E_q)}\right) + S(r,f). \tag{2.19}$$

From Lemma 9 , we obtain

$$N_2\left(r,\frac{1}{F}\right) \leq N_{l+2}\left(r,\frac{1}{P(f)L_k(f,E_q)}\right) + S(r,f),$$
$$\leq m(l+2)N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{L_k(f,E_q)}\right) + l\overline{N}(r,f) + S(r,f),$$
$$\leq m(l+2)T(r,f) + (k+1)T(r,f) + lT(r,f) + S(r,f),$$
$$N_2\left(r,\frac{1}{F}\right) \leq (k+(m+1)(l+2)-1)T(r,f) + S(r,f). \tag{2.20}$$

Similarly for $G$ we have,

$$(n+k+1)T(r,g) = T(r,P(g)L_k(g,E_q)) + S(r,g)$$
$$\leq T(r,G) - N_2\left(r,\frac{1}{G}\right) + N_{l+2}\left(r,\frac{1}{P(g)L_k(g,E_q)}\right) + S(r,f). \tag{2.21}$$

From Lemma 9, we obtain

$$N_2\left(r,\frac{1}{G}\right) \leq N_{l+2}\left(r,\frac{1}{P(g)L_k(g,E_q)}\right) + S(r,g),$$
$$\leq m(l+2)N\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{L_k(g,E_q)}\right) + l\overline{N}(r,g) + S(r,g),$$
$$\leq m(l+2)T(r,g) + (k+1)T(r,g) + lT(r,g) + S(r,g),$$
$$N_2\left(r,\frac{1}{G}\right) \leq (k++(m+1)(l+2)-1)T(r,g) + S(r,g). \tag{2.22}$$

If the $(i)$ of Lemma 7 is satisfied implies that,

$$\max\{T(r,F),T(r,G)\} \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G).$$

Now, combining above with (2.19) and (2.22) we obtain,

$$(n+k+1)\{T(r,f)+T(r,g)\} \leq 2[N(r,f)+N(r,g)]+2N_{l+2}\left(r,\frac{1}{P(f)L_k\left(f,E_q\right)}\right)$$
$$+2N_{l+2}\left(r,\frac{1}{P(g)L_k(g,E_q)}\right)+S(r,f)+S(r,g),$$
$$\leq (k+(m+1)(l+2)-1)T(r,f)+2[N(r,f)+N(r,g)]+(k+(m+1)(l+2)-1)T(r,g)$$
$$+S(r,f)+S(r,g),$$
$$\leq 2(k+(m+1)(l+2)+1)[T\left(r,f\right)+T\left(r,g\right)]+S\left(r,f\right)+S\left(r,g\right),$$
$$\leq 2k+2(m+1)(l+1)+2.$$

Which is in contradiction with $n>k+2(m+1)(l+2)+1$.
Hence $F=G$ or $FG=1$.
From Lemma 10, we get $f=tg$ for $t^m=t^{n+1}=1$ and $f$ and $g$ satisfy the algebraic equation $R(f,g)=0$, where $R\left(w_1,w_2\right)=P\left(w_1\right)L_k(w_1,E_q)-p\left(w_2\right)L_k(w_2,E_q)$.

**Corollary 1** *Let $f$ and $g$ be a transcendental entire functions with zero order. Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ be a non-constant polynomial with constant co-efficients $a_0,a_1,\ldots,a_{n-1},a_n(\neq 0)$, and $m$ be the number of distinct zeros of $P(z)$. If $n\geq k+2(m+1)(l+2)+4$, $[P(f)L_k(f,E_q)]^{(l)}$ and $[P(g)L_k(g,E_q)]^{(l)}$ share the* 1 CM *has infinitely many zeros, then the conclusion of Theorem 11 holds.*

**Proof of Theorem 12:** Let $F=[P(f)L_k(f,E_q)]^{(l)}, \quad G=[P(g)L_k(g,E_q)]^{(l)}$.

From lemma 4, assume that $H\neq 0$, we get

$$T(r,F)+T(r,G)\leq 2\left[N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)\right]$$
$$+3\left[\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right]+S(r,F)+S(r,G). \quad (2.23)$$

Combining above with (2.19)-(2.22) and from lemma 3, we obtain

$$(n+k+1)[T(r,f)+T(r,g)]\leq T(r,F)+T(r,G)-N_2\left(r,\frac{1}{F}\right)-N_2\left(r,\frac{1}{G}\right)$$
$$+N_{l+2}\left(r,\frac{1}{P(f)L_k(f,E_q)}\right)+N_{l+2}\left(r,\frac{1}{P(g)L_k(g,E_q)}\right)+S(r,f)+S(r,g),$$
$$\leq 2\left(N_2(r,F)+N_2(r,G)\right)+2N_{l+2}\left(r,\frac{1}{P(f)L_k(f,E_q)}\right)+2N_{l+2}\left(r,\frac{1}{P(g)L_k(g,E_q)}\right)$$
$$+3\left[\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right]+S(r,f)+S(r,g),$$
$$\leq 2(2k+4)\{T(r,f)+T(r,g)\}+2(m(l+2)+k+1)\{T(r,f)+T(r,g)\}$$
$$+3\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$
$$\leq (6k+2m(l+2)+13)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \quad (2.24)$$

which is a contradiction with $n \geq 5k + 2m(l+2) + 13$. Thus we get $H \equiv 0$.
The following proof is trivial, we give the complete proof. By integration for $H$ twice, we obtain

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \quad G = \frac{(a-b-1) - (a-b)F}{Fb - (b+1)}. \tag{2.25}$$

which implies that $T(r,F) = T(r,G) + O(1)$. Since

$$\begin{aligned} T(r,F) &\leq T(r, P(f)L_k(f,E_q)) + S(r,f), \\ &\leq (n+k+1)T(r,f) + S(r,f). \end{aligned} \tag{2.26}$$

then $S(r,F) = S(r,f)$. So $S(r,G) = S(r,g)$ is. We distinguish into three cases as follows.

**Case 1.** $b \neq 0, -1$. If $a-b-1 \neq 0$, then by (2.25), we get

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{F - \frac{a-b-1}{b+1}}\right). \tag{2.27}$$

By the Nevanlinna second main theorem, and lemma 3, we have

$$\begin{aligned} &(n+k+1)T(r,g) \leq T(r,G) + N_{l+2}\left(r, \frac{1}{P(g)L_k(g,E_q)}\right) - N\left(r, \frac{1}{G}\right) + S(r,g), \\ &\leq (l+1)T(r,g) + (m(l+2)+k+1)T(r,f) + S(r,f) + S(r,g). \end{aligned} \tag{2.28}$$

Similarly, we get

$$(n+k+1)T(r,f) \leq (l+1)T(r,f) + (m(l+2)+k+1)T(r,g) + S(r,f) + S(r,g). \tag{2.29}$$

Thus from equations (2.22) and (2.29), we have

$$\begin{aligned} &(n+k+1)\{T(r,f) + T(r,g)\} \leq [2(l+1) + 2m(l+2) + 2k + 2]\{T(r,f) + T(r,g)\} \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{2.30}$$

Which is a contradiction with $n \geq 5k + 2m(l+2) + 13$. Thus $a-b-1 = 0$, then

$$F = \frac{(b+1)G}{bG+1}. \tag{2.31}$$

Using the same method as above, we get

$$\begin{aligned} &(n+k+1)T(r,g) \leq T(r,G) + N_l\left(r, \frac{1}{P(g)L_k(g,E_q)}\right) - N\left(r, \frac{1}{G}\right) + S(r,g), \\ &\leq N_l\left(r, \frac{1}{P(g)L_k(g,E_q)}\right) + \overline{N}\left(r, \frac{1}{G + \frac{1}{b}}\right) + S(r,g), \\ &\leq (ml+k+1)T(r,g) + S(r,g). \end{aligned} \tag{2.30}$$

Which is a contradiction.
**Case 2.** $b = 0, a \neq 1$. From (2.25), we have

$$F = \frac{G + a - 1}{a}.$$

Similarly, we also can get a contradiction. Thus $a = 1$ follows, it implies that $F = G$.
**Case 3.** $b = -1, a \neq -1$. From (2.25), we obtain

$$F = \frac{a}{a + 1 - G}.$$

Similarly, we can get a contradiction, $a = -1$ follows. Thus, we get $F.G = 1$. From Lemma 5, we get $f = tg$ for $t^m = t^{n+1} = 1$, and $f$ and $g$ satisfy the algebraic expression $R(f, g) = 0$, where

$$R(w_1, w_2) = P(w_1) L_k(w_1, E_q) - P(w_2) L_k(w_2, E_q).$$

Thus, we have completed the proofs.

**Corollary 2** *Let $f$ and $g$ be a transcendental entire functions with zero order. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a non-constant polynomial with constant co-efficients $a_0, a_1, \ldots, a_{n-1}, a_n (\neq 0)$, and $m$ be the number of distinct zeros of $P(z)$. If $n \geq 2k + 2m(l + 2) + 5, [P(f)L_k(f, E_q)]^{(l)}$ and $[P(g)L_k(g, E_q)]^{(l)}$ share the* 1 IM *has infinitely many zeros, then the conclusion of Theorem 12 holds.*

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There is no conflict of interest from authors.

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