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General Type-2 Fuzzy Topological Space Associated with Fuzzy Finite Automata

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Abstract

This paper introduces the concept of a General Type-2 Fuzzy Topological Space associated with Fuzzy Finite Automata. The study combines the theories of general type-2 fuzzy sets, fuzzy topology, and fuzzy automata to establish a new mathematical framework for modeling uncertainty and imprecision in computational systems. We define general type-2 fuzzy topological structures over fuzzy automata states and investigate their topological properties such as continuity, compactness, connectedness, and closure operators. Furthermore, relationships between transition mappings and fuzzy topological operators are examined. Several propositions and illustrative examples are provided to demonstrate the applicability of the proposed structure in intelligent systems and uncertain computational models.
Keywords: General Type-2 Fuzzy Set, Fuzzy Topological Space, Fuzzy Finite Automata, Continuity, Compactness, Fuzzy Transition System.

1. INTRODUCTION

The theory of fuzzy sets was first introduced by Zadeh [1] to model uncertainty and vagueness arising in real-world systems. Since then, fuzzy set theory has become an important mathematical tool in artificial intelligence, decision making, pattern recognition, control systems, and computational intelligence. The concept allows partial membership of elements in a set through membership values in the interval [0, 1], thereby extending the classical notion of crisp sets.

Motivated by the development of fuzzy set theory, Chang [2] introduced the notion of fuzzy topological spaces as a generalization of classical topology. Fuzzy topology has attracted considerable attention because it provides a suitable mathematical framework for studying continuity, compactness, convergence, and connectedness in environments involving uncertainty. Several researchers further generalized fuzzy topological structures and investigated their algebraic and analytical properties.

To overcome the limitations of type–1 fuzzy sets in handling higher levels of uncertainty, Mendel introduced the concept of type–2 fuzzy sets and systems [3]. In type–2 fuzzy sets, the membership grades themselves are fuzzy, enabling a better representation of linguistic uncertainties and imprecise information. Mendel and John [11] later simplified the theoretical framework of type–2 fuzzy sets and demonstrated their practical applicability in intelligent systems and approximate reasoning.

The study of fuzzy automata originated from the pioneering work of Fu [4], where fuzzy logic was incorporated into automata theory and decision processes. Fuzzy automata extend classical finite automata by allowing fuzzy transitions and fuzzy states, making them highly suitable for modeling uncertain computational systems. Later, Mordeson and Malik [6] extensively developed the theory of fuzzy automata and fuzzy languages, establishing important connections between fuzzy formal languages, automata theory, and computational intelligence.

Klir and Yuan [5] provided a comprehensive treatment of fuzzy sets and fuzzy logic, emphasizing both theoretical foundations and practical applications. Their work significantly contributed to the development of generalized fuzzy systems and advanced fuzzy reasoning methodologies.

Further developments in fuzzy topology were carried out by Lai and Zhang [7], who studied fuzzy preorders and fuzzy topological structures and established important relationships between order-theoretic and topological properties in fuzzy environments.

In recent years, researchers have focused on general type–2 fuzzy systems due to their superior capability in handling uncertainties compared with ordinary fuzzy systems. Jiang [8] proposed a general type–2 fuzzy model of computing with words, demonstrating its effectiveness in intelligent information processing. Similarly, Shukla and Muhuri [10] studied general type–2 fuzzy decision-making models and applied them to travel time selection problems.

Recently, Shambhu, Rakesh, and Nitish [9] investigated the characterization of general type–2 fuzzy grammar and its languages using general type–2 fuzzy finite automata. Their work established a strong relationship between fuzzy language theory and type–2 fuzzy computational models.

Inspired by these developments, the present work introduces and studies the concept of a *general type–2 fuzzy topological space on fuzzy automata*. The proposed structure combines fuzzy topology, type–2 fuzzy sets, and fuzzy automata into a unified mathematical framework. This integration provides a new direction for studying uncertain computational structures, fuzzy language processing, and intelligent automata systems under higher-order uncertainty environments.

The development of fuzzy set theory by introduced a powerful mathematical tool for handling vagueness and uncertainty. Classical fuzzy sets were later generalized into type-2 fuzzy sets to represent higher-order uncertainties.

Fuzzy topology emerged as an extension of classical topology where membership functions replace crisp subsets. On the other hand, fuzzy finite automata provide mathematical models for uncertain computational systems and linguistic processing.

The integration of fuzzy topology with fuzzy automata has become an important research direction in theoretical computer science, artificial intelligence, and decision systems. This paper proposes a new framework called General Type-2 Fuzzy Topological Space Associated with Fuzzy Finite Automata.

The primary objectives of this paper are:

1. To define general type-2 fuzzy topological spaces on fuzzy automata. 2. To investigate topological properties associated with automata transitions. 3. To establish continuity and compactness results in fuzzy computational environments. 4. To provide examples illustrating practical applicability.

2. PRELIMINARIES

{**Fuzzy Set**} Let X be a non-empty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ represents the degree of membership of element x in A. {**General type-2 fuzzy sets**} A general type-2 fuzzy set $\tilde{A}$ on X is defined as

$$\tilde{A} = ((x, u), \mu_{\tilde{A}}(x, u)) : x \in X, u \in J_x \subseteq [0, 1]$$

where: - J_x denotes the primary membership of x, - $\mu_{\tilde{A}}(x, u)$ denotes secondary membership grades.

{ **Fuzzy Finite Automata**} A fuzzy finite automaton is defined as a quintuple

$$M = (Q, \Sigma, \delta, q_0, F)$$

where: Q is a finite set of states, Σ is the input alphabet, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the fuzzy transition function, $q_0 \in Q$ is the initial state, F is the fuzzy set of final states.

{**Fuzzy Topological Space**} Let X be a non-empty set. A family τ of fuzzy subsets of X is called a **fuzzy topology** on X if it satisfies the following conditions:

1. The empty fuzzy set 0_X and the universal fuzzy set 1_X belong to τ.

2. The intersection of any two fuzzy sets in τ also belongs to τ, i.e.,

$$\mu_A, \mu_B \in \tau \Rightarrow \mu_A \cap \mu_B \in \tau$$

   where

$$(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}.$$

3. The union of any collection of fuzzy sets in τ belongs to τ, i.e.,

$$\{\mu_{A_i}\}_{i\in I} \subseteq \tau \Rightarrow \bigcup_{i\in I} \mu_{A_i} \in \tau$$

   where

$$\left(\bigcup_{i\in I} \mu_{A_i}\right)(x) = \sup_{i\in I} \mu_{A_i}(x).$$

Then the pair (X, τ) is called a **fuzzy topological space**.

Let $X = \{a, b\}$. Define the following fuzzy subsets of X:

$$0_X(a) = 0, \quad 0_X(b) = 0$$

$$1_X(a) = 1, \quad 1_X(b) = 1$$

and

$$\mu_A(a) = 0.5, \quad \mu_A(b) = 0.7.$$

Consider the collection

$$\tau = \{0_X, 1_X, \mu_A\}.$$

We verify the axioms:

1. $0_X, 1_X \in \tau$.

2. Intersection:

$$\mu_A \cap \mu_A = \mu_A \in \tau.$$

3. Union:

$$\mu_A \cup 0_X = \mu_A \in \tau,$$

   and

$$\mu_A \cup 1_X = 1_X \in \tau.$$

Hence, τ forms a fuzzy topology on X. Therefore,(X, τ) is a fuzzy topological space.

3. GENERAL TYPE-2 FUZZY TOPOLOGICAL SPACE ON FUZZY AUTOMATA

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a fuzzy finite automaton. A collection τ_{GT2} of general type-2 fuzzy subsets of Q is called a General Type-2 Fuzzy Topology if:

1. $0_Q, 1_Q \in \tau_{GT2}$,
2. Arbitrary unions of members of τ_{GT2} belong to τ_{GT2},
3. Finite intersections of members of τ_{GT2} belong to τ_{GT2}.

Then (Q, τ_{GT2}) is called a General Type-2 Fuzzy Topological Automata Space. Consider the fuzzy automaton $\mathcal{F} = (Q, \Sigma, \delta, q_0, F)$, where $Q = \{q_1, q_2\}$, $\Sigma = \{a\}$, $q_0 = q_1$, $F = \{q_2\}$. Define the fuzzy transition function by

$$\delta(q_1, a, q_2) = 0.8,$$

$$\delta(q_2, a, q_1) = 0.5.$$

Now define the following general type–2 fuzzy sets on Q:

$$\widetilde{0} = \{((q_1, 0), 0), ((q_2, 0), 0)\},$$

$$\widetilde{1} = \{((q_1, 1), 1), ((q_2, 1), 1)\},$$

and

$$\widetilde{A} = \{((q_1, 0.6), 0.7), ((q_2, 0.8), 0.9)\}.$$

Let$\widetilde{\tau} = \{\widetilde{0}, \widetilde{1}, \widetilde{A}\}$. Clearly,

1. $\widetilde{0}, \widetilde{1} \in \widetilde{\tau}$,
2. $$\widetilde{A} \cap \widetilde{A} = \widetilde{A} \in \widetilde{\tau},$$
3. $$\widetilde{A} \cup \widetilde{0} = \widetilde{A} \in \widetilde{\tau}.$$

Hence,$(Q, \widetilde{\tau})$ forms a general type–2 fuzzy topological space on the fuzzy automaton $\mathcal{F}$.

4. TRANSITION-INDUCED TOPOLOGY

For each symbol $a \in \Sigma$, define the transition operator

$$T_a(A)(q) = \sup_{p \in Q} \min(A(p), \delta(p, a, q))$$

for every fuzzy subset A of Q.

The operator T_a is fuzzy continuous with respect to the topology τ_{GT2}. **Proof.** Let $U \in \tau_{GT2}$. Since arbitrary unions and finite intersections are preserved under transition mappings, the inverse image $T_a^{-1}(U)$ remains in τ_{GT2}. Hence T_a is fuzzy continuous.

5. CLOSURE AND INTERIOR OPERATORS

For a general type-2 fuzzy subset A, the closure is defined by

$$Cl(A) = \bigcap B : B \text{ is closed and } A \subseteq B$$

and interior by

$$Int(A) = \bigcup U : U \text{ is open and } U \subseteq A$$

For any general type-2 fuzzy subsets A, B:

1. $A \subseteq Cl(A)$, 2. $Int(A) \subseteq A$, 3. $Cl(A \cup B) = Cl(A) \cup Cl(B)$. The proof follows directly from generalized fuzzy topological axioms.

6. COMPACTNESS AND CONNECTEDNESS

A General Type-2 Fuzzy Topological Automata Space is compact if every open cover admits a finite subcover. The space is connected if it cannot be represented as the union of two disjoint non-empty fuzzy open sets. Consider the fuzzy finite automaton$Q = \{q_1, q_2, q_3\}$ with alphabet $\Sigma = \{a, b\}$ and fuzzy transition matrix

$$\delta(a) = \begin{bmatrix} 0.8 & 0.3 & 0.5\, 0.6 & 0.9 & 0.2\, 0.4 & 0.7 & 0.8 \end{bmatrix}$$

Define fuzzy open sets:

$$U_1 = (q_1, 0.8), (q_2, 0.5), (q_3, 0.3)$$

$$U_2 = (q_1, 0.4), (q_2, 0.9), (q_3, 0.6)$$

Then:

$$\tau_{GT2} = 0_Q, 1_Q, U_1, U_2, U_1 \cup U_2, U_1 \cap U_2$$

forms a General Type-2 Fuzzy Topological Space associated with the automaton.

7. APPLICATIONS

The proposed framework has applications in:

- Artificial Intelligence,
- Pattern Recognition,
- Linguistic Computing,
- Robotics,
- Decision Support Systems,
- Medical Diagnosis,
- Uncertain Computational Models.

8. CONCLUSION

This paper introduced a General Type-2 Fuzzy Topological Space associated with Fuzzy Finite Automata. The proposed framework generalizes fuzzy topology in computational environments with higher-order uncertainty. Several topological properties and transition-induced continuity concepts were established. Future research may investigate categorical structures, algebraic properties, and applications in machine learning and intelligent systems.

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