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Hyers-Ulam-Rassias Stability for Certain Classes of Third-Order Nonlinear Functional Differential Equations

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Abstract

This paper investigates the Hyers-Ulam-Rassias (H-U-R) stability for two distinct classes of third-order nonlinear functional differential equations with initial conditions. Under appropriate structural assumptions on the nonlinearities and using the Cauchy formula for repeated integrals together with nonlinear Gronwall-Bellman-Bihari type inequalities, we establish sufficient conditions for H-U-R stability. Explicit formulas for the Hyers-Ulam-Rassias constants are derived. The special case $G \equiv 0$ is also analyzed. Two illustrative examples are provided to demonstrate the applicability of our theoretical results.

Keywords and phrases: Hyers-Ulam-Rassias stability, third-order nonlinear differential equations, functional differential equations, Gronwall-Bellman-Bihari inequality, integral inequalities.

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1. INTRODUCTION

The stability problem for functional equations originated from a question posed by Ulam [34] in 1940: "Under what conditions does an approximate solution of a functional equation remain close to an exact solution?" Hyers [17] provided the first affirmative answer for additive functions on Banach spaces, establishing what is now known as Hyers-Ulam stability. Subsequently, Rassias [33] introduced a significant generalization by allowing the bound to depend on a function, leading to the concept of Hyers-Ulam-Rassias (H-U-R) stability. Since these pioneering works, stability theory has been extensively developed for various classes of functional equations [3, 6, 7, 9, 10].

The extension of stability concepts to differential equations began with Obloza [27, 28], who investigated linear differential equations. Alsina and Ger [4] studied the stability of $y' = y$, while Jung [18, 19, 20] and Miura et al. [23, 24] established results for first-order linear equations. For higher-order linear equations, contributions include works by Abdollahpour et al. [1, 2], Li and Shen [21, 22], and Murali [25]. Nonlinear stability results have been obtained by Rus [31, 32], Qarawani [29, 30], and Fakunle and Arawomo [11, 12, 13, 14, 15, 16].

Despite these advances, the H-U-R stability of third-order nonlinear functional differential equations remains relatively unexplored. This paper addresses this gap by investigating two classes of such equations:

$$(R(u(t))u'(t))'' + q(t, u(t), u'(t), u''(t))u'(t) + y(t)r(u(t)) = G(t, u(t), u'(t)), \quad (1)$$

$$u'''(t) + \beta(t)h(u(t))u''(t) + \eta(t)g(u(t))u'(t) + y(t)r(u(t)) = G(t, u(t), u'(t)), \quad (2)$$

subject to the initial conditions

$$u(t_0) = u'(t_0) = u''(t_0) = 0, \quad t > 0. \quad (3)$$

The paper is organized as follows. Section 2 presents preliminary definitions and key inequalities. Section 3 establishes the existence and uniqueness of solutions. Section 4 presents the main stability theorems. Section 5 provides illustrative examples, and Section 6 offers concluding remarks.

2. PRELIMINARIES

Throughout this paper, we denote $= (0, \infty)$, $= [0, \infty)$, and $\mathbb{R} = (-\infty, \infty)$. The space of thrice continuously differentiable functions from to is denoted by $C^3(,)$.

Definition 2.1. Equation (1) is said to be *Hyers-Ulam-Rassias stable* if there exists a constant $C_\varphi > 0$ such that for every positive nondecreasing function $\varphi \in C(,)$ and every function $u \in C^3(,)$ satisfying

$$\left|(R(u)u')'' + q(t, u, u', u'')u' + y(t)r(u) - G(t, u, u')\right| \leq \varphi(t), \quad (4)$$

there exists an exact solution $u_0 \in C^3(,)$ of (1) such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t), \quad \forall t \in . \quad (5)$$

The constant C_φ is called a *Hyers-Ulam-Rassias constant*.

An analogous definition holds for equation (2).

Definition 2.2 (Cauchy Formula for Repeated Integrals)**.** For an integrable function f on $[t_0, t]$, the n-fold repeated integral is given by

$$\int_{t_0}^{t} \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_{n-1}} f(s_n)\, ds_n \cdots ds_1 = \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f(s)\, ds. \quad (6)$$

Lemma 2.1 (Bihari's Inequality [5]). *Let $u, f \in C([t_0, T],)$, $K > 0$, $M \geq 0$, and let $\omega :\to$ be a continuous, nondecreasing function with $\omega(u) > 0$ for $u > 0$. If*

$$u(t) \leq K + M \int_{t_0}^{t} f(s)\omega(u(s))\,ds, \quad t_0 \leq t < T, \tag{7}$$

then for all $t \in [t_0, T']$ where $T' \leq T$ is such that the following expression is defined,

$$u(t) \leq \Omega^{-1}\Big(\Omega(K) + M \int_{t_0}^{t} f(s)\,ds\Big), \tag{8}$$

where $\Omega(u) = \int_{u_0}^{u} \frac{d\tau}{\omega(\tau)}$, $u_0 > 0$, and Ω^{-1} denotes the inverse function.

Theorem 2.3 (Mean Value Theorem for Integrals [26]). *If $f, g \in C([t_0, t], \mathbb{R})$ and g does not change sign on $[t_0, t]$, then there exists $\xi \in [t_0, t]$ such that*

$$\int_{t_0}^{t} f(s)g(s)\,ds = f(\xi) \int_{t_0}^{t} g(s)\,ds. \tag{9}$$

Corollary 2.4 (Nonlinear Gronwall-Type Inequality [15, 16]). *Let ρ be a nonnegative, nondecreasing continuous function on , and let $u, r, h \in C(,)$. Suppose*

$$u(t) \leq \rho(t) + T \int_{t_0}^{t} r(s)\beta(u(s))\,ds + L \int_{t_0}^{t} h(s)\varpi(u(s))\,ds, \tag{10}$$

with $T, L > 0$ and β, ϖ continuous, nondecreasing, and positive on $(0, \infty)$. Then

$$\begin{aligned} u(t) \leq \rho(t)\, \Omega^{-1}\Big(\Omega(1) + L \int_{t_0}^{t} h(s)\varpi\big(F^{-1}\big(F(1) + T \int_{t_0}^{s} r(\alpha)\,d\alpha\big)\big)\,ds\Big) \\ \times F^{-1}\Big(F(1) + T \int_{t_0}^{t} r(s)\,ds\Big), \end{aligned} \tag{11}$$

where $F(u) = \int_{u_0}^{u} \frac{d\tau}{\beta(\tau)}$, $\Omega(u) = \int_{u_0}^{u} \frac{d\tau}{\varpi(\tau)}$, and $u_0 > 0$.

Theorem 2.5 (Generalized Gronwall Inequality [15, 16]). *Let $\rho \in C(,)$ be nondecreasing, and let $u, r, h, g \in C(,)$. Assume β, ϖ, γ are continuous, nondecreasing, positive on $(0, \infty)$, with γ submultiplicative. If*

$$u(t) \leq \rho(t) + A \int_{t_0}^{t} r(s)\beta(u(s))\,ds + B \int_{t_0}^{t} h(s)\varpi(u(s))\,ds + L \int_{t_0}^{t} g(s)\gamma(u(s))\,ds, \tag{12}$$

for $A, B, L > 0$, then

$$\begin{aligned} u(t) \leq \rho(t)\, \Upsilon^{-1}\Big[\Upsilon(1) + L \int_{t_0}^{t} g(s)\gamma\Big(\Omega^{-1}\Big(\Omega(1) + B \int_{t_0}^{s} h(\alpha)\varpi(T(\alpha))\,d\alpha\Big)T(s)\Big)\,ds\Big] \\ \times \Omega^{-1}\Big(\Omega(1) + B \int_{t_0}^{t} h(s)\varpi(T(s))\,ds\Big)T(t), \end{aligned} \tag{13}$$

where $T(t) = F^{-1}\big(F(1) + A \int_{t_0}^{t} r(s)\,ds\big)$, $\Upsilon(u) = \int_{u_0}^{u} \frac{d\tau}{\gamma(\tau)}$, and F, Ω are as in Corollary 2.4.

3. EXISTENCE AND UNIQUENESS

Before establishing stability results, we address the existence and uniqueness of solutions.

Theorem 3.1 (Existence and Uniqueness for Equation (1))**.** *Consider the initial value problem* (1), (3)*. Assume the following conditions hold:*

1. *R, q, G, r, y are continuous in their respective domains.*
2. *There exist Lipschitz constants $L_R, L_q, L_G, L_r > 0$ such that for all arguments in a neighborhood of the initial data,*

$$
\begin{aligned}
|R(u_1) - R(u_2)| &\leq L_R|u_1 - u_2|, \\
|q(t, u_1, u_1', u_1'') - q(t, u_2, u_2', u_2'')| &\leq L_q(|u_1 - u_2| + |u_1' - u_2'| + |u_1'' - u_2''|), \\
|G(t, u_1, u_1') - G(t, u_2, u_2')| &\leq L_G(|u_1 - u_2| + |u_1' - u_2'|), \\
|r(u_1) - r(u_2)| &\leq L_r|u_1 - u_2|.
\end{aligned}
$$

3. *The functions y is bounded on .*
4. *The initial conditions* (3) *are given.*

Then there exists a unique solution $u \in C^3(,)$ to the initial value problem (1), (3) *on some interval $[t_0, t_0 + \delta]$.*

Proof. Define $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))^\top = (u(t), u'(t), u''(t))^\top$. The third-order equation (1) can be written as the first-order system

$$\mathbf{z}'(t) = \mathbf{F}(t, \mathbf{z}(t)), \quad \mathbf{z}(t_0) = \mathbf{0},$$

where $\mathbf{F}$ is Lipschitz in $\mathbf{z}$ under the given assumptions. By the Picard-Lindelöf theorem, there exists a unique local solution. □

A similar theorem holds for equation (2) under appropriate assumptions.

4. MAIN RESULTS

In this section, we establish sufficient conditions for the H-U-R stability of equations (1) and (2). The following assumptions are used throughout.

4.1. Assumptions

(H1) The function G has the separable form

$$G(t, u, u') = \phi(t)\, \alpha(u)\, (u')^n, \tag{14}$$

where $n \in \mathbb{N}$, $\alpha \in C(,)$, and $\phi \in C(,)$.

(H2) The function q satisfies

$$q(t,u,u',u'') = \kappa(t)\,\omega(u)\,(u'')^4\,(u')^n, \tag{15}$$

with $\kappa \in C(,)$ and $\omega \in C(,\mathbb{R})$.

(H3) There exist constants $\lambda, k_1 > 0$ such that

$$|u'(t)| \le \lambda, \quad |u''(t)| \le k_1, \quad \forall t \in . \tag{16}$$

(H4) The following integrals are bounded:

$$\lim_{t_0\to\infty}\int_{t_0}^{t}\phi(s)\,ds \le k_2, \qquad \lim_{t_0\to\infty}\int_{t_0}^{t}\kappa(s)\,ds \le k_3,$$
$$\lim_{t_0\to\infty}\int_{t_0}^{t}|u'(s)|\,ds \le k_4, \qquad \lim_{t_0\to\infty}\int_{t_0}^{t}q(s)\,ds \le k_5, \tag{17}$$

for some positive constants k_2, k_3, k_4, k_5.

(H5) There exists $\sigma > 0$ such that

$$\int_{t_0}^{t}\varphi(s)\,ds \le \sigma\varphi(t), \quad \forall t \in, \tag{18}$$

for the function φ appearing in the stability definition.

(H6) The function $y \in C(,)$ is nondecreasing and satisfies $y(t) \ge \delta > 0$ for some constant δ.

For equation (2), we additionally require:

(H'7) Define $D(u(t)) = \int_{u(t_0)}^{u(t)} g(s)\,ds$, with $g \in C(,)$.

(H'8) There exists $\psi > 0$ such that $\int_{t_0}^{t} |u'''(s)|\,ds \le \psi$.

(H'9) The following limits exist and are finite:

$$\lim_{t_0\to\infty}\int_{t_0}^{t}\beta(s)\,ds \le k_6, \qquad \lim_{t_0\to\infty}\int_{t_0}^{t}\eta(s)\,ds \le k_7, \tag{19}$$
$$\lim_{t_0\to\infty}\int_{t_0}^{t}y(s)\,ds \le k_8. \tag{20}$$

(H'10) $\eta \in C(,)$ is nondecreasing with $\eta(t) \ge \tau > 0$ for some constant τ.

4.2. Stability for Equation (1)

Theorem 4.1. *Under assumptions (H1)-(H6), equation* (1) *is Hyers-Ulam-Rassias stable. Moreover, a Hyers-Ulam-Rassias constant is given by*

$$\begin{aligned} C_{\varphi,1} = \sigma\frac{\lambda}{\delta}\,\Upsilon^{-1}\Bigg[\Upsilon(1) + \frac{k_2\lambda^{n+1}}{\delta}\,\alpha\Bigg(\Omega^{-1}\Bigg(\Omega(1) + k_3\frac{k_1^4\lambda^{n+2}}{\delta}\,\omega(T^*)\Bigg)T^*\Bigg)\Bigg] \\ \times\,\Omega^{-1}\Bigg(\Omega(1) + k_3\frac{k_1^4\lambda^{n+2}}{\delta}\,\omega(T^*)\Bigg)T^*, \end{aligned} \tag{21}$$

where $T^* = F^{-1}\big(F(1)+k_4\frac{\lambda}{\delta}\big)$*, and* F, Ω, Υ *are defined as in Section 2 with* $\beta(u) = \alpha(u)$*,* $\varpi(u) = \omega(u)$*, and* $\gamma(u) = \alpha(\Omega^{-1}(\cdots))$*, respectively.*

Proof. Let $u \in C^3(,)$ satisfy inequality (4). Then

$$-\varphi(t) \le (R(u)u')'' + q(t,u,u',u'')u' + y(t)r(u) - G(t,u,u') \le \varphi(t). \tag{22}$$

Multiplying by $u'(t)$ (taking absolute values carefully) and integrating from t_0 to t, we obtain

$$\begin{aligned} \Big|\int_{t_0}^{t}(R(u)u')''u'\,ds\Big| + \int_{t_0}^{t}|q(s,u,u',u'')|(u')^2\,ds + \int_{t_0}^{t}y(s)|r(u)u'|\,ds \\ + \int_{t_0}^{t}|G(s,u,u')u'|\,ds \le \int_{t_0}^{t}\varphi(s)\,ds. \end{aligned} \tag{23}$$

Applying the mean value theorem for integrals (Theorem 2.3) to the first term, there exists $\eta \in [t_0,t]$ such that

$$\int_{t_0}^{t}(R(u)u')''u'\,ds = u'(\eta)\int_{t_0}^{t}(R(u)u')''\,ds. \tag{24}$$

Integrating by parts twice and using the initial conditions (3), we have

$$\int_{t_0}^{t}(R(u)u')''\,ds = \int_{t_0}^{t}R(u)u'\,ds. \tag{25}$$

Applying the Cauchy formula (Definition 2.2) and assumptions (H1)-(H2), after simplification we obtain

$$\begin{aligned} |u(t)| \le \frac{\lambda}{\delta}\int_{t_0}^{t}\varphi(s)\,ds + \frac{\lambda}{\delta}\int_{t_0}^{t}R(|u|)|u'|\,ds + \frac{k_1^4\lambda^{n+2}}{\delta}\int_{t_0}^{t}\kappa(s)\omega(|u|)\,ds \\ + \frac{\lambda^{n+1}}{\delta}\int_{t_0}^{t}\phi(s)\alpha(|u|)\,ds, \end{aligned} \tag{26}$$

where we have used $|y(t)r(u)u'| \geq \delta|u'||r(u)|$ from (H6) and the relation $|\Phi(t)| \geq |u(t)|$ with $\Phi(u) = \int_{u(t_0)}^{u} r(s)\,ds$.
Now applying the generalized Gronwall inequality (Theorem 2.5) with

$$\begin{aligned}
&\rho(t) = \frac{\lambda}{\delta}\int_{t_0}^{t}\varphi(s)\,ds, \quad A = \frac{\lambda}{\delta}, \quad B = \frac{k_1^4\lambda^{n+2}}{\delta}, \quad L = \frac{\lambda^{n+1}}{\delta},\\
&r(s) = |u'(s)|, \quad \beta(u) = R(u), \quad h(s) = \kappa(s), \quad \varpi(u) = \omega(u),\\
&g(s) = \phi(s), \quad \gamma(u) = \alpha(u),
\end{aligned}$$

we obtain

$$\begin{aligned}
|u(t)| \leq \frac{\lambda}{\delta}\int_{t_0}^{t}\varphi(s)\,ds\,\Upsilon^{-1}\Bigg[&\Upsilon(1) + \frac{\lambda^{n+1}}{\delta}\int_{t_0}^{t}\phi(s)\\
&\times\alpha\left(\Omega^{-1}\left(\Omega(1) + \frac{k_1^4\lambda^{n+2}}{\delta}\int_{t_0}^{s}\kappa(\alpha)\omega(T(\alpha))\,d\alpha\right)T(s)\right)ds\Bigg]\\
&\times\Omega^{-1}\left(\Omega(1) + \frac{k_1^4\lambda^{n+2}}{\delta}\int_{t_0}^{t}\kappa(s)\omega(T(s))\,ds\right)T(t),
\end{aligned} \tag{27}$$

where $T(t) = F^{-1}\big(F(1) + \frac{\lambda}{\delta}\int_{t_0}^{t}|u'(s)|\,ds\big)$.
Finally, using assumptions (H4)-(H5), we arrive at the estimate

$$|u(t)| \leq C_{\varphi,1}\varphi(t), \tag{28}$$

with $C_{\varphi,1}$ given by (21). Since an exact solution u_0 exists by Theorem 3.1, we have $|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi,1}\varphi(t)$, completing the proof. □

4.3. Stability for Equation (2)

Theorem 4.2. *Under assumptions (H1)-(H6) and (H1')-(H4'), equation* (2) *is Hyers-Ulam-Rassias stable. Moreover, a Hyers-Ulam-Rassias constant is given by*

$$\begin{aligned}
C_{\varphi,2} = \sigma\frac{1+\psi}{\tau}\,\Upsilon^{-1}&\left[\Upsilon(1) + k_2\frac{\lambda^n}{\tau}\,\alpha\left(\Omega^{-1}\left(\Omega(1) + k_7\frac{1}{\tau}\,r(T_2^*)\right)T_2^*\right)\right]\\
&\times\Omega^{-1}\left(\Omega(1) + k_7\frac{1}{\tau}\,r(T_2^*)\right)T_2^*,
\end{aligned} \tag{29}$$

where $T_2^* = F^{-1}\big(F(1) + k_6\frac{k_1}{\tau}\big)$.

Proof. The proof follows a similar pattern to Theorem 4.1, starting from inequality (2) and using assumption (H1') to handle the term involving $g(u)u'$. The key steps involve triple integration, application of the mean value theorem, and the generalized Gronwall inequality. The details are omitted for brevity. □

4.4. The Homogeneous Case $G \equiv 0$

When $G \equiv 0$, we obtain simpler stability constants.

Corollary 4.3. *Under assumptions (H2)-(H6) with* $G \equiv 0$*, equation* (1) *is H-U-R stable with constant*

$$C_{\varphi,3} = \sigma \frac{1}{\delta}\, \Omega^{-1}\Big(\Omega(1) + k_3 \frac{k_1^4 \lambda^{n+1}}{\delta}\, \omega\big(F^{-1}(F(1) + k_4/\delta)\big)\Big) F^{-1}(F(1) + k_4/\delta). \quad (30)$$

Corollary 4.4. *Under assumptions (H3)-(H6) and (H1')-(H4') with* $G \equiv 0$*, equation* (2) *is H-U-R stable with constant*

$$\begin{aligned} C_{\varphi,4} = \sigma\, \Upsilon^{-1}\Big[\Upsilon(1) + k_8 r\Big(\Omega^{-1}\big(\Omega(1) + k_7 \lambda g(T_4^*)\big) T_4^*\Big)\Big] \\ \times\, \Omega^{-1}\big(\Omega(1) + k_7 \lambda g(T_4^*)\big) T_4^*, \end{aligned} \quad (31)$$

where $T_4^* = F^{-1}(F(1) + k_1 k_6)$.

5. ILLUSTRATIVE EXAMPLES

Example 5.1. Consider the third-order nonlinear differential equation

$$u'''(t) + \frac{1}{1+t^2} u(t) u''(t) + e^{-t} u'(t) + \frac{t}{1+t} u(t) = \frac{e^{-2t}}{10} u'(t)^2 u(t), \quad (32)$$

with initial conditions $u(0) = u'(0) = u''(0) = 0$.
This equation is of the form (2) with

$$\begin{aligned} \beta(t) &= \frac{1}{1+t^2}, \quad h(u) = u, \quad \eta(t) = e^{-t}, \quad g(u) = 1, \\ y(t) &= \frac{t}{1+t}, \quad r(u) = u, \quad G(t, u, u') = \frac{e^{-2t}}{10} u'^2 u. \end{aligned}$$

We verify the assumptions:

- (H1): $G(t, u, u') = \phi(t)\alpha(u)(u')^n$ with $\phi(t) = e^{-2t}/10$, $\alpha(u) = u$, $n = 2$.
- (H2): The q term is not present, so this assumption is vacuous.
- (H3): For a given approximate solution, we may have bounds $|u'| \le \lambda$, $|u''| \le k_1$.
- (H4): $\int_0^t \phi(s)\, ds = \frac{1}{10}\int_0^t e^{-2s}\, ds = \frac{1}{20}(1 - e^{-2t}) \le \frac{1}{20} =: k_2$.
- (H5): Choose $\varphi(t) = 1$ on $[0, T]$. Then $\int_0^t 1\, ds = t \le t \cdot 1 = \sigma\varphi(t)$ with $\sigma = t$.
- (H6): For $t \ge 1$, $y(t) = t/(1+t) \ge 1/2$, so $\delta = 1/2$.

Thus, Theorem 4.2 guarantees H-U-R stability on $[1, \infty)$ with an explicitly computable constant.

Example 5.2. Consider the homogeneous equation

$$u'''(t) + \frac{1}{t^2}u(t)u''(t) + \frac{1}{t^3}u'(t) + \frac{1}{t^2}u(t) = 0, \quad t > 1, \tag{33}$$

with $u(1) = u'(1) = u''(1) = 0$.
This corresponds to (2) with $G \equiv 0$, $\beta(t) = t^{-2}$, $h(u) = u$, $\eta(t) = t^{-3}$, $g(u) = 1$, $y(t) = t^{-2}$, $r(u) = u$.
We compute the necessary bounds:

$$\int_1^t \beta(s)\,ds = \int_1^t s^{-2}\,ds = 1 - \frac{1}{t} \leq 1 =: k_6,$$
$$\int_1^t \eta(s)\,ds = \int_1^t s^{-3}\,ds = \frac{1}{2}\Big(1 - \frac{1}{t^2}\Big) \leq \frac{1}{2} =: k_7,$$
$$\int_1^t y(s)\,ds = \int_1^t s^{-2}\,ds = 1 - \frac{1}{t} \leq 1 =: k_8.$$

Taking $\varphi(t) = t^{-2}$, we have $\int_1^t s^{-2}\,ds = 1 - t^{-1} \leq 1 = t^2 \cdot t^{-2} = t^2\varphi(t)$, so $\sigma = t^2$ on $[1, T]$. Assuming bounds $|u'| \leq \lambda$, $|u''| \leq k_1$, and $\int_1^t |u'''|\,ds \leq \psi$, Corollary 4.4 applies with

$$F(u) = \int_{u_0}^u \frac{d\tau}{\tau} = \ln\frac{u}{u_0}, \quad F^{-1}(x) = u_0 e^x,$$
$$\Omega(u) = \int_{u_0}^u \frac{d\tau}{\tau} = \ln\frac{u}{u_0}, \quad \Omega^{-1}(x) = u_0 e^x,$$
$$\Upsilon(u) = \int_{u_0}^u \frac{d\tau}{\tau} = \ln\frac{u}{u_0}, \quad \Upsilon^{-1}(x) = u_0 e^x.$$

Then $T_4^* = F^{-1}(F(1) + k_1 k_6) = e^{k_1}$ (taking $u_0 = 1$), and the stability constant becomes

$$C_{\varphi,4} = \sigma \exp\Big(k_8 e^{k_7 \lambda e^{k_1}}\Big) \exp\big(k_7 \lambda e^{k_1}\big) e^{k_1}.$$

This provides an explicit bound ensuring that any approximate solution remains close to the exact solution.

6. CONCLUSION

In this paper, we have established Hyers-Ulam-Rassias stability for two classes of third-order nonlinear functional differential equations. Using the Cauchy formula for repeated integrals together with nonlinear Gronwall-Bellman-Bihari type inequalities, we derived explicit stability constants under appropriate structural assumptions. The results extend the existing literature on H-U-R stability to higher-order nonlinear equations and provide a systematic methodology for analyzing such problems. The illustrative examples demonstrate the applicability of our theoretical findings. Future work may focus on relaxing the structural assumptions or extending the results to boundary value problems and partial differential equations.

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DECLARATION

The authors declare that they have no competing interests.

REFERENCES

[1] M. R. Abdollahpour and A. Najati, Stability of linear differential equations of third order, *Appl. Math. Lett.* **24** (2011), 1827–1830.

[2] M. R. Abdollahpour and A. Najati, Hyers-Ulam stability of a differential equation of third order, *Int. J. Math. Anal.* **6**(59) (2012), 2943–2948.

[3] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.

[4] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function, *J. Inequal. Appl.* **2** (1998), 373–380.

[5] I. Bihari, Researches of the boundedness and stability of the solutions of nonlinear differential equations, *Acta Math. Acad. Sci. Hung.* **7** (1957), 278–291.

[6] N. Brillouet-Belluot, J. Brzdek, and K. Cieplinski, On some recent developments in Ulam's type stability, *Abstr. Appl. Anal.* **2012** (2012), Art. ID 716936, 41 pp.

[7] D. G. Bourgin, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.* **57** (1951), 223–237.

[8] J. Cronin, *Differential Equations: Introduction and Qualitative Theory*, Dekker, New York, 1980.

[9] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore, 2002.

[10] S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 59–64.

[11] I. Fakunle and P. O. Arawomo, Hyers-Ulam stability of a perturbed generalised Lienard equation, *Int. J. Appl. Math.* **32**(3) (2019), 479–489.

[12] I. Fakunle and P. O. Arawomo, Hyers-Ulam stability of certain class of nonlinear second order differential equations, *Int. J. Pure Appl. Math. Sci.* **11**(1) (2018), 55–65.

[13] I. Fakunle and P. O. Arawomo, On Hyers-Ulam stability of nonlinear second order ordinary and functional differential equations, *Int. J. Differ. Equ. Appl.* **17**(1) (2018), 77–88.

[14] I. Fakunle and P. O. Arawomo, Hyers-Ulam stability theorems for second order nonlinear damped differential equations with forcing term, *J. Nigerian Math. Soc.* **42** (2023), 19–35.

[15] I. Fakunle and P. O. Arawomo, Hyers-Ulam-Rassias stability of nonlinear second order of a perturbed ordinary differential equation, *Proyecciones J. Math.* **42**(5) (2023), 1157–1175.

[16] I. Fakunle and P. O. Arawomo, On Hyers-Ulam stability of a perturbed nonlinear second differential using Gronwall-Bellman-Bihari inequality, *Nigerian J. Math. Appl.* **32**(1) (2022), 189–201.

[17] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA* **27** (1941), 222–224.

[18] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order III, *J. Math. Anal. Appl.* **311**(1) (2005), 137–146.

[19] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order II, *Appl. Math. Lett.* **19**(9) (2006), 854–858.

[20] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order, *Appl. Math. Lett.* **17**(10) (2004), 1135–1140.

[21] Y. Li and S. Yan, Hyers-Ulam stability of nonhomogeneous linear differential equations of second order, *Int. J. Math. Math. Sci.* **2009** (2009), Art. ID 576852, 7 pp.

[22] Y. Li and S. Yan, Hyers-Ulam stability of linear differential equations of second order, *Appl. Math. Lett.* **23** (2010), 306–309.

[23] T. Miura, S. Miyajima, and S. E. Takahasi, A characterization of Hyers-Ulam stability of first order linear differential operators, *J. Math. Anal. Appl.* **286**(1) (2003), 136–146.

[24] T. Miura, S. Miyajima, and S. Takahasi, Hyers-Ulam stability of linear differential operators with constant coefficients, *Math. Nachr.* **258** (2003), 90–96.

[25] R. Murali, Hyers-Ulam-Rassias stability for the linear ordinary differential equations of third order, *Kragujevac J. Math.* **42**(2) (2018), 579–590.

[26] R. S. Murray, *Schum's Outline of Theory and Problems of Calculus*, SI (Metric) Edition, McGraw-Hill, 1974.

[27] M. Obloza, Connection between Hyers-Ulam and Lyapunov stability of the ordinary differential equations, *Rocznik Nauk.-Dydakt. Prace Math.* **11** (1997), 141–146.

[28] M. Obloza, Hyers-Ulam stability of the linear differential equations, *Rocznik Nauk.-Dydakt. Prace Math.* **13** (1993), 259–270.

[29] M. N. Qarawani, Hyers-Ulam stability of linear and nonlinear differential equations of second order, *Int. J. Appl. Math. Res.* **1**(4) (2012), 422–432.

[30] M. N. Qarawani, Hyers-Ulam stability of a generalised second order nonlinear differential equations, *Appl. Math.* **3** (2012), 1857–1861.

[31] I. A. Rus, Ulam stability of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26**(1) (2010), 103–107.

[32] I. A. Rus, Ulam stability of ordinary differential equations, *Studia Univ. Babes-Bolyai Math.* **54**(4) (2010), 306–309.

[33] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72**(2) (1978), 297–300.

[34] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, New York, 1960.