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Periodic solutions for an n-coupled FitzHugh–Rinzel model with delays

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Abstract

This paper investigates the dynamic behavior of an n-coupled FitzHugh Rinzel (FHR) model with multiple time delays. We extend the result in the literature from a two-coupled FHR model and three-coupled FHR model to an n-coupled FHR model. Using the method of mathematical analysis, the original system has been linearized. The unique equilibrium point of the original system has been changed to a trivial equilibrium point of the linearized system. The instability of the trivial equilibrium point of the linearized system implies the instability of the original equilibrium point. The instability of the unique equilibrium point and the boundedness of the solutions will force this system to generate a periodic solution. Some sufficient conditions to guarantee the oscillation of the solutions are provided, and computer simulations are given to support the present criteria.

Keywords and phrases: Coupled FHR model, Delay, Stability, Periodic solution.

AMS Mathematical Subject Classification: 34K13

1. INTRODUCTION

For the following FitzHugh–Rinzel (FHR) network model:

$$\begin{cases} v'(t) = v(t) - \frac{1}{3}v^3(t) - w(t) + y(t) + I, \\ w'(t) = \delta[a + v(t) - bw(t)], \\ y'(t) = \mu[c - v(t) - dy(t)], \end{cases} \tag{1}$$

where a, b, c, d, δ and μ are positive constants, I is a constant. Many researchers have studied various properties of the model (1). For example, Zemlyanukhin and Bochkarev studied the analytical properties of the solutions for model (1) [1]. The double mixed model oscillations, double canards and the feature transitions of the model were obtained numerically for a FitzHugh Nagumo Rinzel model [2].

The spatiotemporal dynamics of the FHR model with time delay and diffusion in a random network were investigated [3]. In [4]. the authors provided original numerical and theoretical analysis about FHR model. The complex dynamics such as canards, mixed-mode oscillations, Hopf-bifurcations and their spatially extended counterpart were discussed. In [5], the analytical properties of the solutions for model (1) were studied. This solution can be used to verify the results in a numerical study of the model. Kudryashov considered the integrability of the model (1), the exact solutions of the model (1) were given [6]. Tabekoueng et al. found that the model experiences self-excited firing activity [7]. The long-time behavior of the FHR neurons and the transition to instability were investigated in [8]. An analytical and theoretical investigation of the energy was done to support the various firing activities, such as bursting and spiking, captured in the model (1) [9]. Mondal et al. studied the following fractional-order FitzHugh-Rinzel bursting neuron model [10]:

$$\begin{cases} \frac{d^\alpha v}{dt^\alpha} = v(t) - \frac{1}{3}v^3(t) - w(t) + y(t) + I, \\ \frac{d^\alpha w}{dt^\alpha} = \delta[a + v(t) - bw(t)], \\ \frac{d^\alpha y}{dt^\alpha} = \mu[c - v(t) - dy(t)]. \end{cases} \tag{2}$$

The authors investigated how the implosion of the integer-order reaction varies with perturbation, with predictability and bifurcation interpretation based on the fractional-order. Qurashi et al. discussed dynamic prediction modeling and equilibrium stability of a fractional discrete biophysical neuron model [11-13]. On the other hand, many researchers have studied various coupled models since a coupled system can explain many diverse physical aspects in chemical theory, electrical engineering, and so on. For example, the multiple time delays coupled FitzHugh–Nagumo (FHN) models were employed to investigate the synchronization mode transition [14]. The synchronization mechanism of two coupled time-delayed FHN neurons with bidirectional coupling was extensively studied in [15]. The bifurcation structure and oscillation of coupled time-delayed FHN neurons were investigated in [16]. In [17], the authors discussed the stability of the solutions for a fractional-order coupled FHN model. Durairaj et al. investigated the bifurcation delay in a network of nonlocally coupled FHR model [18]. Recently, Hu et al. have investigated the following coupled FHR neuron system [19]:

$$\begin{cases} u_1'(t) = u_1(t) - \frac{u_1^3(t)}{3} - v_1(t) + w_1(t) + r\tanh(u_2(t-\tau_1)) + g(u_2(t-\tau_2) - u_1(t)), \\ v_1'(t) = \delta(a + u_1(t) - bv_1(t)), \\ w_1'(t) = \mu(c - u_1(t) - dw_1(t)), \\ u_2'(t) = u_2(t) - \frac{u_2^3(t)}{3} - v_2(t) + w_2(t) + r\tanh(u_1(t-\tau_1)) + g(u_1(t-\tau_2) - u_2(t)), \\ v_2'(t) = \delta(a + u_2(t) - bv_2(t)), \\ w_2'(t) = \mu(c - u_2(t) - dw_2(t)), \end{cases} \tag{3}$$

where δ is the parameter which controls the fast subsystem variables v_1 and v_2. $\tau_1, \tau_2 > 0$. In model (3), the Hopf bifurcation and fold bifurcation of limit cycles were deduced when choosing the strength of electrical coupling or chemical coupling as the bifurcation parameter, respectively. Motivated by [14-19], in this paper, we shall discuss the following general n-coupled FHR model with delays:

$$\begin{cases} u_1'(t) = u_1(t) - \frac{1}{3}u_1^3(t) - v_1(t) + w_1(t) + \sum_{i=2}^n r_{1i}\tanh(u_i(t-\tau_i)) \\ \qquad\quad + \sum_{i=2}^n k_{1i}(u_i(t-\tau_i) - u_1(t)), \\ v_1'(t) = \delta_1(a_1 + u_1(t) - b_1 v_1(t)), \\ w_1'(t) = \mu_1(c_1 - u_1(t) - d_1 w_1(t)), \\ u_2'(t) = u_2(t) - \frac{1}{3}u_2^3(t) - v_2(t) + w_2(t) + \sum_{i=1, i\neq 2}^n r_{2i}\tanh(u_i(t-\tau_i)) \\ \qquad\quad + \sum_{i=1, i\neq 2}^n k_{2i}(u_i(t-\tau_i) - u_2(t)), \\ v_2'(t) = \delta_2(a_2 + u_2(t) - b_2 v_2(t)), \\ w_2'(t) = \mu_2(c_2 - u_2(t) - d_2 w_2(t)), \\ u_3'(t) = u_3(t) - \frac{1}{3}u_3^3(t) - v_3(t) + w_3(t) + \sum_{i=1, i\neq 3}^n r_{3i}\tanh(u_i(t-\tau_i)) \\ \qquad\quad + \sum_{i=1, i\neq 3}^n k_{3i}(u_i(t-\tau_i) - u_3(t)), \\ v_3'(t) = \delta_3(a_3 + u_3(t) - b_3 v_3(t)), \\ w_3'(t) = \mu_3(c_3 - u_3(t) - d_3 w_3(t)), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n'(t) = u_n(t) - \frac{1}{3}u_n^3(t) - v_n(t) + w_n(t) + \sum_{i=1}^{n-1} r_{ni}\tanh(u_i(t-\tau_i)) \\ \qquad\quad + \sum_{i=1}^{n-1} k_{ni}(u_i(t-\tau_i) - u_n(t)), \\ v_n'(t) = \delta_n(a_n + u_n(t) - b_n v_n(t)), \\ w_n'(t) = \mu_n(c_n - u_n(t) - d_n w_n(t)), \end{cases} \tag{4}$$

where u_i stand the membrane potentials of neuron i, v_i signify the recovery variables in neuron i, and w_i imply the slow variables in neuron $i(i = 1, 2, \cdots, n)$. In model (4), the subsystem of the first neuron is formed by the first three equations. The subsystem of the second neuron is formed by the second three equations, and the last three equations crystallize the subsystem of the third neuron. Indeed, system (4) can be regarded as a slow-fast system with $2n$ fast variables $(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n)$ and n slow variables $(w_1, w_2, \cdots, w_n)$. The fast variables represent the membrane potentials of the neurons, which rapidly change during the generation of action potentials (spikes). The slow variables represent the recovery currents, which are responsible for the repolarization of the membranes after spikes and the subsequent refractory period. $a_i, b_i, d_i, \delta_i, (i = 1, 2, \cdots, n)$ are positive constants, $c_i, k_{ij}, r_{ij}(i, j = 1, 2, \cdots, n)$ are arbitrary constants, $\delta_i(i = 1, 2, \cdots, n)$ are the parameters which control the subsystem variables $v_1, v_2, \cdots, v_n$. These parameters govern the shape of the nonlinear function defining the fast variables, as well as the time scales and interactions between the fast and slow variables. These parameters allow the FHR model to exhibit a wide range of physiologically observed neuronal behaviors. Time delays $\tau_i \geq 0(i = 1, 2, \cdots, n)$, $0 < \mu_i(i = 1, 2, \cdots, n)$ are small parameters that adjudicate the pace of the slow variables $w_1, w_2, \cdots, w_n$. $(u_i(t-\tau_i) - u_j(t))(i, j = 1, 2, \cdots, n)$ are electrical coupling terms which indicate the current flowing into each neuron is proportional to the voltage difference at the point of three neurons connection, $k_{ij}(i, j = 1, 2, \cdots, n)$ represent the strength of the electrical coupling,

$\tanh(u_i)(i = 1, 2, \cdots, n)$ represent the electrical synaptic transfer functions. We discuss the oscillatory behavior for model (4) not only in theoretical analysis but also in numerical simulation. The result in [19] has been extended.

2. PRELIMINARIES

For model (4), we have the following lemmas:
Lemma 1 All solutions of the system (4) are bounded.
Proof To prove the boundedness, we construct a Lyapunov function $V(t) = \sum_{i=1}^{n} \frac{1}{2}(u_i^2 + v_i^2 + w_i^2)$. Noting that $|\tanh(u_i)| \leq 1$, then calculating the derivative of $V(t)$ through system (4) we have

$$V'(t)|_{(4)} \leq -\sum_{i=1}^{n}(\frac{1}{3}u_i^4 + \delta_i b_i v_i^2 + \mu_i d_i w_i^2) + \sum_{i,j=1}^{n}(a|u_i u_j| + b|u_i v_j| + c|u_i w_j|), \quad (5)$$

where a, b, and c are some positive constants. Noting that $u_i^4(i = 1, 2, \cdots, n)$ are higher order infinity than $|u_i u_j|, |u_i v_j|, |u_i w_j|$. Since δ_i, μ_i, b_i, and d_i are positive constants, so, there exists a suitably large positive number, say L, such that $V'(t)|_{(4)} < 0$ as u_i, v_i, and w_i are greater than L, implying that all solutions of system (4) are bounded.
Now suppose that $[u_1^*, v_1^*, w_1^*, u_2^*, v_2^*, w_2^*, \cdots, u_n^*, v_n^*, w_n^*]^T$ is an equilibrium point of the system (4), make the change of the variables $u_i(t) \to u_i(t) - u_i^*$, $v_i(t) \to v_i(t) - v_i^*$, $w_i(t) \to w_i(t) - w_i^*(i = 1, 2, \cdots, n)$, we have the Taylor expansion for function $\tanh(u_i)$ at the equilibrium point, and system (4) will be transfered as follows:

$$\begin{cases}
u_1'(t) = (1-(u_1^*)^2)u_1(t) - u_1^* u_1^2(t) - \frac{1}{3}u_1^3(t) - v_1(t) + w_1(t) + \sum_{i=2}^{n} r_{1i}\tanh'(u_i^*)u_i(t-\tau_i) \\
\qquad + \sum_{i=2}^{n} r_{1i}\sum_{n\geq 2}\frac{\tanh^{(n)}(u_i^*)}{n!}u_i(t-\tau_i)^n + \sum_{i=2}^{n} k_{1i}(u_i(t-\tau_i) - u_1(t)), \\
v_1'(t) = \delta_1(u_1(t) - b_1 v_1(t)), \\
w_1'(t) = \mu_1(-u_1(t) - d_1 w_1(t)), \\
u_2'(t) = (1-(u_2^*)^2)u_2(t) - u_2^* u_2^2(t) - \frac{1}{3}u_2^3(t) - v_2(t) + w_2(t) + \sum_{i=1, i\neq 2}^{n} r_{2i}\tanh'(u_i^*)u_i(t-\tau_i) \\
\qquad + \sum_{i=1, i\neq 2}^{n} r_{2i}\sum_{n\geq 2}\frac{\tanh^{(n)}(u_i^*)}{n!}u_i(t-\tau_i)^n + \sum_{i=1, i\neq 2}^{n} k_{2i}(u_i(t-\tau_i) - u_2(t)), \\
v_2'(t) = \delta_2(u_2(t) - b_2 v_2(t)), \\
w_2'(t) = \mu_2(-u_2(t) - d_2 w_2(t)), \\
u_3'(t) = (1-(u_3^*)^2)u_3(t) - u_3^* u_3^2(t) - \frac{1}{3}u_3^3(t) - v_3(t) + w_3(t) + \sum_{i=1, i\neq 3}^{n} r_{3i}\tanh'(u_i^*)u_i(t-\tau_i) \\
\qquad + \sum_{i=1, i\neq 3}^{n} r_{3i}\sum_{n\geq 2}\frac{\tanh^{(n)}(u_i^*)}{n!}u_i(t-\tau_i)^n + \sum_{i=1, i\neq 3}^{n} k_{3i}(u_i(t-\tau_i) - u_3(t)), \\
v_3'(t) = \delta_3(u_3(t) - b_3 v_3(t)), \\
w_3'(t) = \mu_3(-u_3(t) - d_3 w_3(t)). \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u_n'(t) = (1-(u_n^*)^2)u_n(t) - u_n^* u_n^2(t) - \frac{1}{3}u_n^3(t) - v_n(t) + w_n(t) + \sum_{i=1}^{n-1} r_{ni}\tanh'(u_i^*)u_i(t-\tau_i) \\
\qquad + \sum_{i=1}^{n-1} r_{ni}\sum_{n\geq 2}\frac{\tanh^{(n)}(u_i^*)}{n!}u_i(t-\tau_i)^n + \sum_{i=1}^{n-1} k_{ni}(u_i(t-\tau_i) - u_n(t)), \\
v_n'(t) = \delta_n(u_n(t) - b_n v_n(t)), \\
w_n'(t) = \mu_n(-u_n(t) - d_n w_n(t)),
\end{cases} \quad (6)$$

The linearized system of (6) is the following:

$$\begin{cases} u_1'(t) = a_{11}u_1(t) - v_1(t) + w_1(t) + \sum_{i=2}^{n} b_{1i}u_i(t-\tau_i), \\ v_1'(t) = \delta_1 u_1(t) - a_{22}v_1(t), \\ w_1'(t) = -\mu_1 u_1(t) - a_{33}w_1(t), \\ u_2'(t) = a_{44}u_2(t) - v_2(t) + w_2(t) + \sum_{i=1,i\neq 2}^{n} b_{2i}u_i(t-\tau_i), \\ v_2'(t) = \delta_2 u_2(t) - a_{55}v_2(t), \\ w_2'(t) = -\mu_2 u_2(t) - a_{66}w_2(t), \\ u_3'(t) = a_{77}u_3(t) - v_3(t) + w_3(t) + \sum_{i=1,i\neq 3}^{n} b_{3i}u_i(t-\tau_i), \\ v_3'(t) = \delta_3 u_3(t) - a_{88}v_3(t), \\ w_3'(t) = -\mu_3 u_3(t) - a_{99}w_3(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n'(t) = a_{3n-2,3n-2}u_n(t) - v_n(t) + w_n(t) + \sum_{i=1}^{n-1} b_{ni}u_i(t-\tau_i), \\ v_n'(t) = \delta_n u_n(t) - a_{3n-1,3n-1}v_n(t), \\ w_n'(t) = -\mu_n u_n(t) - a_{3n,3n}w_n(t), \end{cases} \tag{7}$$

where $a_{11} = 1-(u_1^*)^2 - \sum_{i=2}^{n} k_{1i}$,$a_{22} = \delta_1 b_1$,$a_{33} = \mu_1 d_1$, $a_{44} = 1-(u_2^*)^2 - \sum_{i=1,i\neq 2}^{n} k_{2i}$,$a_{55} = \delta_2 b_2$, $a_{66} = \mu_2 d_2$, $a_{77} = 1-(u_3^*)^2 - \sum_{i=1,i\neq 3}^{n} k_{3i}$,$a_{88} = \delta_3 b_3$,$a_{99} = \mu_3 d_3, \cdots, a_{3n-2,3n-2} = 1-(u_n^*)^2 - \sum_{i=1}^{n-1} k_{ni}$,$a_{3n-1,3n-1} = \delta_n b_n$,$a_{3n,3n} = \mu_n d_n$,$b_{1i} = k_{1i} + r_{1i}\tanh'(u_i^*)$,$b_{2i} = k_{2i} + r_{2i}\tanh'(u_i^*)$, $b_{3i} = k_{3i} + r_{3i}\tanh'(u_i^*)$, $\cdots, b_{ni} = r_{ni} + r_{ni}\tanh'(u_i^*)$. System (7) can be expressed in the following matrix form:

$$Z'(t) = AZ(t) + BZ(t-\tau), \tag{8}$$

where $Z(t) = [u_1(t), v_1(t), w_1(t), u_2(t), v_2(t), w_2(t), \cdots, u_n(t), v_n(t), w_n(t)]^T$, $Z(t-\tau) = [u_1(t-\tau_1), 0, 0, u_2(t-\tau_2), 0, 0, \cdots, u_n(t-\tau_n), 0, 0]^T$, A and B both are $3n\times 3n$ matrices,

$$A = \begin{pmatrix} a_{11} & -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \delta_1 & -a_{22} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\mu_1 & 0 & -a_{33} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 & -a_{55} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_2 & 0 & -a_{66} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{3n-2,3n-2} & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \delta_n & -a_{3n-1,3n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\mu_n & 0 & -a_{3n,3n} \end{pmatrix},$$

$$
B=\begin{pmatrix}
0 & 0 & 0 & b_{12} & 0 & 0 & b_{13} & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
b_{21} & 0 & 0 & 0 & 0 & 0 & b_{23} & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
b_{n1} & 0 & 0 & b_{n2} & 0 & 0 & b_{n3} & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
$$

Lemma
2 If matrix $P(=A+B)$ is a nonsingular matrix for selected parameters, then there exists a unique equilibrium point $Z^*=[u_1^*,v_1^*,w_1^*,u_2^*,v_2^*,w_2^*,\cdots,u_n^*,v_n^*,w_n^*]^T$ of the system (4).

Proof Obviously, the zero equilibrium point of the system (7) or (8) corresponds to the equilibrium point Z^* of the system (4). If X^* is an equilibrium point of the system (8), then we have

$$AX^*+BX^*=PX^*=\mathbf{0}. \tag{9}$$

According to the Cramer's Rule of linear algebraic theory, the system (9) has a unique trivial solution $X^*=\mathbf{0}$, since $P=A+B$ is a nonsingular matrix, implying that the system (4) has a unique equilibrium point. The proof is completed.

Based on Lemma 1 and Lemma 2, in the following, we provide two theorems to guarantee the existence of periodic oscillatory solutions.

3. THE EXISTENCE OF PERIODIC SOLUTIONS

Theorem 1 Assume that zero is the unique equilibrium point of the system (8) for selecting parameter values. Let $\alpha_1,\alpha_2,\cdots,\alpha_{3n}$ be characteristic values of matrix A, and $\beta_1,\beta_2,\cdots,\beta_{3n}$ be characteristic values of matrix B. If there exists one characteristic value, say α_k, such that $\alpha_k>0$, or $Re(\alpha_k)>0$, and $Re(\alpha_1)>max\{|\beta_1|,|\beta_2|,\cdots,|\beta_{3n}|\}$. Then the unique trivial solution of the system (8) is unstable, implying that there exists a periodic oscillatory solution in the system (4).

Proof According to the basic differential equation theory, if there exists one characteristic value, say α_k, such that $\alpha_k>0$, or $Re(\alpha_k)>0$, and $Re(\alpha_k)>max\{|\beta_1|,|\beta_2|,\cdots,|\beta_{3n}|\}$, then the trivial solution z(t) of the system (8) is unstable [20]. Indeed, setting $\tau_*=\min_{1\le i\le n}\{\tau_i\}$, and first consider the following system

$$Z'(t)=AZ(t)+BZ(t-\tau_*), \tag{10}$$

the characteristic equation associated with the system (10) can be written as follows:

$$\prod_{i=1}^{3n}(\lambda-\alpha_i-\beta_i e^{-\lambda\tau_*})=0. \tag{11}$$

Therefore, there is a characteristic equation from the system (11) as follows ($\beta_k = 0$):

$$\lambda - \alpha_k = 0, \tag{12}$$

or ($\beta_k \neq 0$)

$$\lambda - \alpha_k - \beta_k e^{-\lambda \tau_*} = 0. \tag{13}$$

If (12) holds, then the equation (10) has a positive characteristic value. If (13) holds, and $Re(\alpha_k) > 0$, $Re(\alpha_k) > max\{|\beta_1|, |\beta_2|, \cdots, |\beta_{3n}|\}$, which means that equation (10) has a positive real part characteristic value. Thus, the trivial solution of the system (10) is unstable. Based on the basic theory of delayed differential equation, the trivial solution of the system (8) is also unstable. Meanwhile, the nonlinear term of the system (6) is a higher-order infinitesimal as $u_i(t) \to 0$. Therefore, the instability of the trivial solution of the system (8) ensures the instability of the trivial solution of the system (6). This means that the unique equilibrium point $Z^* = [u_1^*, v_1^*, w_1^*, u_2^*, v_2^*, w_2^*, \cdots, u_n^*, v_n^*, w_n^*]^T$ of the system (4) is unstable. The instability of the unique equilibrium point together with the boundedness of the solutions will force the system (4) to generate a limit cycle, namely, there exists a periodic solution of the system (4) [21,22]. The proof is completed.

For simplify, setting $\parallel B \parallel = \max_{1 \leq j \leq 3n} |b_{ij}|, \mu(A) = \max_{1 \leq j \leq 3n}\{a_{jj} + \sum_{i=1, i \neq j}^{3n} |a_{ij}|\}$ [23]. Then we have

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following inequality is satisfied

$$\parallel B \parallel \tau_* e > \exp(\tau_* |\mu(A)|). \tag{14}$$

Then the trivial solution of the system (8) is unstable, implying that the system (4) has a periodic solution.

Proof To prove the instability of the trivial solution of the system (10), let $\Sigma(t) = \sum_{i=1}^{n}(|u_i(t)| + |v_i(t)| + |w_i(t)|)$. So $\Sigma(t) \geq 0$, and

$$\Sigma'(t) \leq \mu(A)\Sigma(t) + \parallel B \parallel \Sigma(t - \tau_*). \tag{15}$$

Specifically, consider a scalar delayed differential equation

$$Y'(t) = \mu(A)Y(t) + \parallel B \parallel Y(t - \tau_*). \tag{16}$$

We have $\Sigma(t) \leq Y(t)$. Now we prove that the trivial solution of the equation (16) is unstable. Indeed, the characteristic equation associated with the equation (16) is the following

$$\lambda = \mu(A) + \parallel B \parallel e^{-\lambda \tau_*}. \tag{17}$$

We claim that there exists a nonpositive root λ^* of (17) under the condition (14). Indeed, from (17) we have

$$|\lambda^*| \geq \parallel B \parallel e^{|\lambda \tau_*|} - |\mu(A)|. \tag{18}$$

Namely,

$$|\lambda^*| + |\mu(A)| \geq \parallel B \parallel e^{|\lambda \tau_*|} \tag{19}$$

Based on inequality $e^x \geq ex$, from (19) we get

$$1 \geq \frac{\parallel B \parallel e^{|\lambda \tau_*|}}{|\lambda^*| + |\mu(A)|} = \frac{\parallel B \parallel \tau_* e^{-\mu(A)\tau_*} \cdot e^{(|\lambda| + \mu(A))\tau_*}}{(|\lambda^*| + |\mu(A)|)\tau_*} \geq \parallel B \parallel \tau_* e e^{-\mu(A)\tau_*} \tag{20}$$

From (20) we have

$$\exp(\tau_* |\mu(A)|) \geq \parallel B \parallel \tau_* e \tag{21}$$

A contradiction with (14). Therefore, the trivial solution of equation (16) is unstable, implying that the trivial solution of (8) is unstable. Similar to Theorem 1, the system (4) has a periodic solution. The proof is completed.

4. SIMULATION RESULT

This simulation is based on the system (4) and sets $n = 4$. We first select the parameters as in Table 1. Thus, the equilibrium point $[u_1^*, v_1^*, w_1^*]^T = (0.0124, 0.8152, 0.9367)^T$, $[u_2^*, v_2^*, w_2^*]^T = (0.0265, 1.6931, 0.9018)^T$, $[u_3^*, v_3^*, w_3^*]^T = (0.0402, 0.9578, 0.9752)^T$, $[u_4^*, v_4^*, w_4^*]^T = (0.0396, 0.8413, 0.9488)^T$. Then we have $a_{11} = 0.7496, a_{22} = 0.1195, a_{33} = 0.0181, a_{44} = 1.7875, a_{55} = 0.3010, a_{66} = 0.0432, a_{77} = 0.7298, a_{88} = 0.3626, a_{99} = 0.0396, a_{10,10} = 1.5078, a_{11,11} = 0.1680, a_{12,12} = 0.0375$. The characteristic values of matrix A are $1.2158, 1.1554, 0.3455, 0.2430, -0.2125, -0.1170, -0.0960, -0.0552, 0.3336 \pm 0.4522i, 0.2745 \pm 0.4852i$. Note that matrix A has a positive characteristic value of 1.2158. The conditions of Theorem 1 are satisfied. When time delays are selected as $\tau_1 = 0.5, \tau_2 = 0.8, \tau_3 = 0.7, \tau_4 = 0.4$, $\tau_1 = 1.5, \tau_2 = 1.8, \tau_3 = 1.7, \tau_4 = 1.4$, and $\tau_1 = 2.5, \tau_2 = 2.8, \tau_3 = 2.7, \tau_4 = 2.4$, respectively, there exist oscillatory solutions for the system (4) (see Figures 1-3). In Figure 4, we select the parameters as in Table 2. Thus, the equilibrium point $[u_1^*, v_1^*, w_1^*]^T = (-0.0344, -0.6375, -0.8924)^T$, $[u_2^*, v_2^*, w_2^*]^T = (-0.0553, -0.6844, -0.9012)^T$, $[u_3^*, v_3^*, w_3^*]^T = (-0.0525, -0.7176, -0.9926)^T$. $[u_4^*, v_4^*, w_4^*]^T = (-0.0412, -1.2768, -1.1779)^T$.Then we have $a_{11} = 2.1468, a_{22} = 0.3496, a_{33} = 0.0224, a_{44} = -0.5325, a_{55} = 0.5986, a_{66} = 0.0187, a_{77} = 3.1475, a_{88} = 0.3626, a_{99} = 0.0167, a_{10,10} = 0.2584, a_{11,11} = 0.2880, a_{12,12} = 0.0187, b_{12} = -0.5309, b_{13} = 1.2273, b_{14} = -0.2461, b_{21} = 1.9326, b_{23} = -0.1891, b_{24} = 1.1543, b_{31} = 1.1683, b_{32} = -0.5031, b_{34} = -1.6626, b_{41} = 0.7196, b_{42} = -0.0566, b_{43} = 0.7756$. Therefore, the characteristic values of matrix

A are $2.9753, 1.9031, 0.0586, 0.0242, -0.2313, -0.1869, -0.0688, -0.0579, 0.0102 \pm 0.6344i, -0.5460 \pm 0.9363i$. Since 2.9753 is a positive characteristic value of matrix A, the conditions of Theorem 1 are satisfied. We also have $\| B \| = 3.8205, \mu(A) = 2.1028$. When time delays are selected as $\tau_1 = 1.0, \tau_2 = 1.3, \tau_3 = 1.2, \tau_4 = 0.9$, then $\tau_* = 0.9$, we have $\| B \| \tau_* e = 3.8205 * 0.9e = 9.3644 > 6.6436 = e^{0.9*2.1028}$. The conditions of Theorem 2 are satisfied. there exists a periodic solution (see Fig.4). When we select time delays as $\tau_1 = 1.6, \tau_2 = 1.8, \tau_3 = 1.7, \tau_4 = 1.5$, then $\tau_* = 1.5$, we have $\| B \| \tau_* e = 3.8205 * 1.5e = 15.5773$, but $e^{1.5*2.1028} = 23.4266$, The conditions of Theorem 2 are not satisfied. However, there still exists a periodic solution (see Fig. 5), which means that Theorem 2 is a stronger sufficient condition. Then we consider the case of $n = 5$. The parameters are selected as in Table 3. Thus, the equilibrium point $[u_1^*, v_1^*, w_1^*]^T = (-0.0212, -0.6812, -0.8368)^T$, $[u_2^*, v_2^*, w_2^*]^T = (-0.0198, -0.6722, -0.8677)^T$, $[u_3^*, v_3^*, w_3^*]^T = (-0.0764, -0.7639, -1.0724)^T$, $[u_4^*, v_4^*, w_4^*]^T = (-0.0086, -1.4798, -1.2583)^T$,$[u_5^*, v_5^*, w_5^*]^T = (-0.0213, -1.2365, -1.1894)^T$. Then we have $a_{11} = 1.5996, a_{22} = 0.3496, a_{33} = 0.0244, a_{44} = -0.8319, a_{55} = 0.5986, a_{66} = 0.0187, a_{77} = 2.5684, a_{88} = 0.3626, a_{99} = 0.0667, a_{10,10} = 0.9387, a_{11,11} = 0.2880, a_{12,12} = 0.0187, a_{13,13} = 1.0297, a_{14,14} = 0.4680, a_{15,15} = 0.015.$, The characteristic values of matrix A are $2.3573, 1.2490, 0.1318, -0.0047, -0.0529, -0.0921, -0.1316, -0.1512, -0.2138,$ $0.3624 \pm 0.3583i, 0.3389 \pm 0.3942i, -0.69830 \pm 0.9282i$. Since 2.3573 is a positive characteristic value of matrix A, the conditions of Theorem 1 are satisfied. When time delays are selected as $\tau_1 = 0.6, \tau_2 = 0.9, \tau_3 = 0.8, \tau_4 = 0.7, \tau_5 = 0.4$, and $\tau_1 = 0.3, \tau_2 = 0.6, \tau_3 = 0.5, \tau_4 = 0.4, \tau_5 = 0.2$, respectively, there exist oscillatory solutions for the system (4) (see Figures 6-7). Also we have $b_{12} = -0.5303, b_{13} = 1.4597, b_{14} = 0.1396, b_{15} = 1.3297, b_{21} = 1.9398, b_{23} = -0.0397, b_{24} = 1.4997, b_{25} = -0.1203, b_{31} = 1.1698, b_{32} = -0.2802, b_{34} = -1.3402, b_{35} = 1.1798, b_{41} = 0.7198, b_{42} = -0.2698, b_{43} = 1.0998, b_{45} = -0.1802, b_{51} = 0.6998, b_{52} = -0.0301, b_{53} = 0.7998, b_{54} = -0.0502$. Then $\tau_* = 0.4$, and $\| B \| = 4.5292$, $\| B \| \tau_* e = 4.5292 * 0.4e = 4.9247 > 3.2968 = e^{0.4*2.9824}$. When $\tau_* = 0.2$, we also have $\| B \| \tau_* e = 4.5292 * 0.2e = 2.4623 > 1.8157 = e^{0.2*2.9824}$. Both of $\tau_* = 0.4$ and $\tau_* = 0.2$, the conditions of Theorem 2 are also satisfied.

5. CONCLUSION

This paper discusses the periodic solutions for an n-coupled FHR model with delays. Two sufficient conditions to guarantee the existence of periodic solutions are given. Computer simulations are provided to indicate the correctness of the criteria. In neuroscience, periodic solutions can represent rhythmic firing patterns in neurons, like repetitive spiking or bursting activity. In our simulation, we found that time delays affect the periodic oscillatory frequency. When delays are increased, the oscillatory frequency is slightly down. The values of a_i and $c_i (i = 1, 2, \cdots, n)$ can be selected as positive or negative real numbers, which is different from the result in [19]. When a_i and $c_i (i = 1, 2, \cdots, n)$ are selected

Table 1: The parameter values in Figure 1.

r_{12}	r_{13}	r_{14}	r_{21}	r_{23}	r_{24}	r_{31}	r_{32}	r_{34}	r_{41}	r_{42}	r_{43}
0.62	0.64	0.66	0.52	0.55	0.58	0.45	0.47	0.48	0.32	0.45	0.38
k_{12}	k_{13}	k_{14}	k_{21}	k_{23}	k_{24}	k_{31}	k_{32}	k_{34}	k_{41}	k_{42}	k_{43}
0.35	−0.52	0.42	−0.72	0.62	−0.68	0.40	−0.65	0.52	−0.60	0.71	−0.62
a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4
0.30	0.35	0.38	0.32	0.46	0.43	0.44	0.42	0.25	0.24	0.26	0.28
d_1	d_2	d_3	d_4	δ_1	δ_2	δ_3	δ_4	μ_1	μ_2	μ_3	μ_4
0.18	0.4	0.22	0.25	0.26	0.70	0.29	0.32	0.12	0.14	0.18	0.15

Table 2: The parameter values in Figure 4.

r_{12}	r_{13}	r_{14}	r_{21}	r_{23}	r_{24}	r_{31}	r_{32}	r_{34}	r_{41}	r_{42}	r_{43}
0.72	0.74	0.76	0.62	0.65	0.68	0.55	0.57	0.58	0.42	0.45	0.48
k_{12}	k_{13}	k_{14}	k_{21}	k_{23}	k_{24}	k_{31}	k_{32}	k_{34}	k_{41}	k_{42}	k_{43}
−1.25	0.72	−0.62	1.32	−0.61	0.82	0.62	−0.85	−1.92	0.30	−0.18	0.62
a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4
−0.45	−0.40	−0.48	−0.82	0.76	0.73	0.74	0.72	−0.35	−0.34	−0.36	−0.38
d_1	d_2	d_3	d_4	δ_1	δ_2	δ_3	δ_4	μ_1	μ_2	μ_3	μ_4
0.28	0.24	0.22	0.25	0.46	0.82	0.49	0.40	0.080	0.078	0.076	0.075

as negtive real numbers, the equilibrium $[u_1^*, u_2^*, \cdots, u_n^*]^T$ may be negtive values. When a_i and $c_i (i = 1, 2, \cdots, n)$ are selected as positive values, the equilibrium $[u_1^*, u_2^*, \cdots, u_n^*]^T$ may be positive values. The values of $k_{ij}(i, j = 1, 2, \cdots, n)$ can be selected as positive values or negative values.

Competing Interests
The author has declared that no competing interests exist.

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Table 3: The parameter values in Figure 6.

r_{12}	r_{13}	r_{14}	r_{15}	r_{21}	r_{23}	r_{24}	r_{25}	r_{31}	r_{32}	r_{34}	r_{35}
0.72	0.74	0.76	0.78	0.62	0.65	0.68	0.70	0.55	0.57	0.58	0.60
r_{41}	r_{42}	r_{43}	r_{45}	r_{51}	r_{52}	r_{53}	r_{54}	k_{12}	k_{13}	k_{14}	k_{15}
0.42	0.45	0.48	0.50	0.32	0.35	0.38	0.40	-1.25	0.72	-0.62	0.55
k_{21}	k_{23}	k_{24}	K_{25}	k_{31}	k_{32}	k_{34}	K_{35}	k_{41}	k_{42}	k_{43}	k_{45}
1.32	-0.61	0.82	-0.82	0.62	-0.85	-1.92	0.58	0.30	-0.18	0.62	-0.68
k_{51}	k_{52}	k_{53}	k_{54}	a_1	a_2	a_3	a_4	a_5	b_1	b_2	b_3
0.38	-0.38	0.42	-0.45	-0.40	-0.48	-0.82	-0.72	-0.75	0.76	0.73	0.74
b_4	b_5	c_1	c_2	c_3	c_4	c_5	d_1	d_2	d_3	d_4	d_5
0.72	0.78	-0.35	-0.34	-0.36	-0.38	-0.42	0.28	0.24	0.22	0.25	0.22
δ_1	δ_2	δ_3	δ_4	δ_5	μ_1	μ_2	μ_3	μ_4	μ_5		
0.46	0.82	0.49	0.40	0.6	0.080	0.078	0.076	0.075	0.072		

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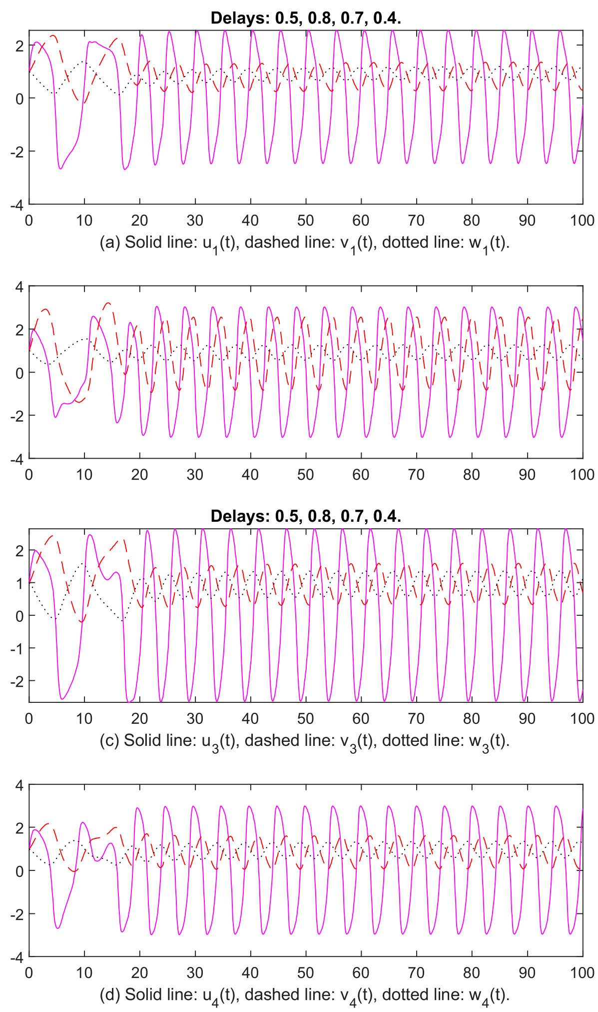


Figure 1: Oscillation of the solutions, delays: $0.5, 0.8, 0.7, 0.4$.

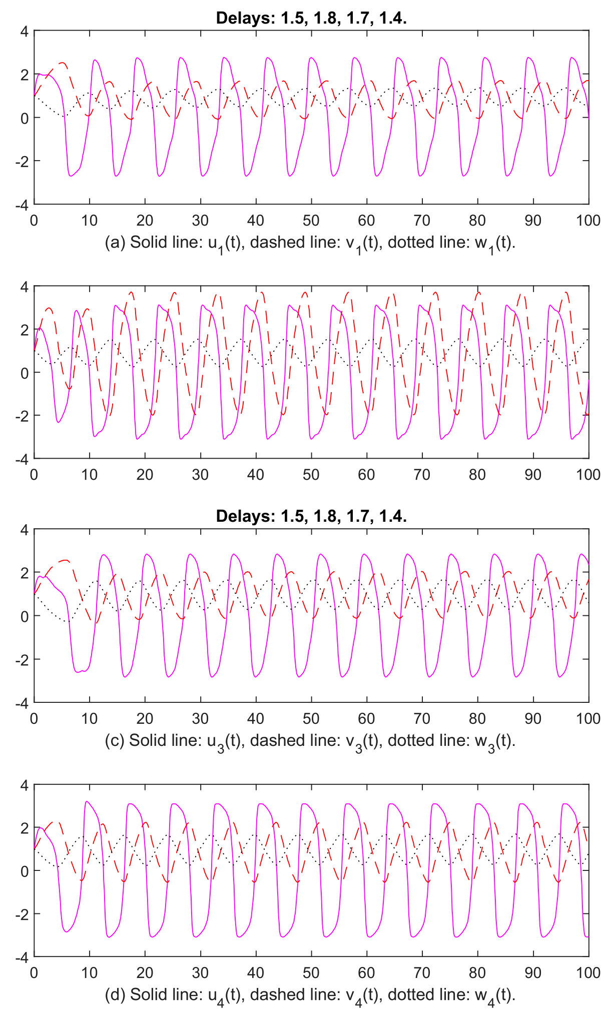


Figure 2: Oscillation of the solutions, delays: $1.5, 1.8, 1.7, 1.4$.

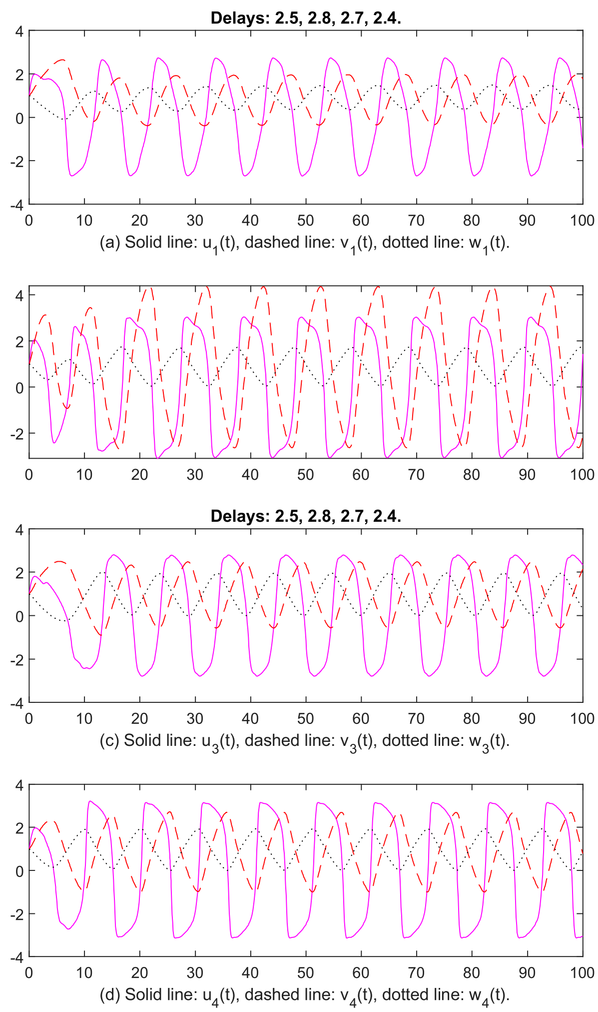
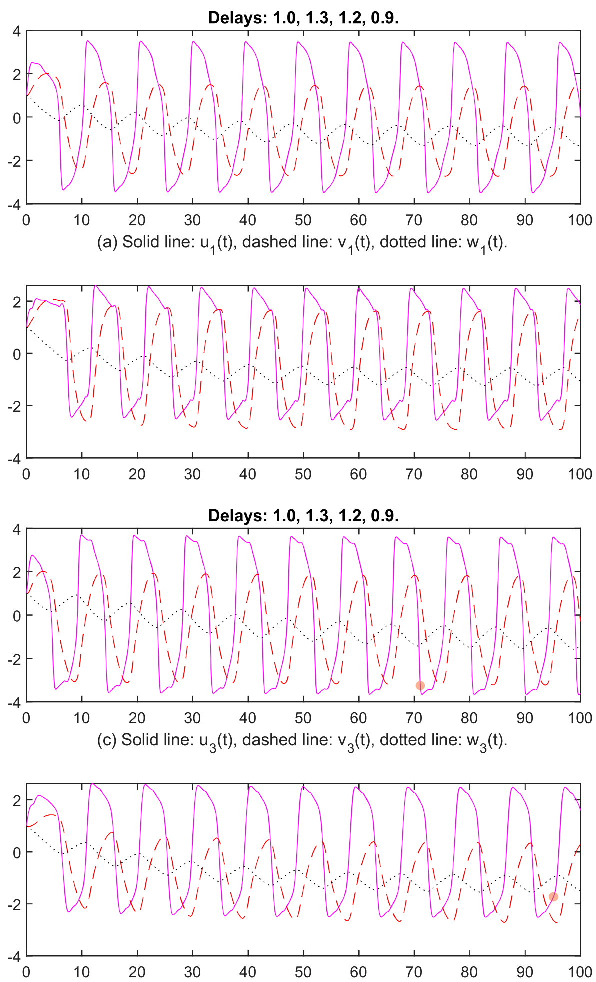


Figure 3: Oscillation of the solutions, delays: $2.5, 2.8, 2.7, 2.4$.



(a) Solid line: $u_1(t)$, dashed line: $v_1(t)$, dotted line: $w_1(t)$.

(c) Solid line: $u_3(t)$, dashed line: $v_3(t)$, dotted line: $w_3(t)$.

(d) Solid line: $u_4(t)$, dashed line: $v_4(t)$, dotted line: $w_4(t)$.

Figure 4: Oscillation of the solutions, delays: $1.0, 1.3, 1.2, 0.8$.

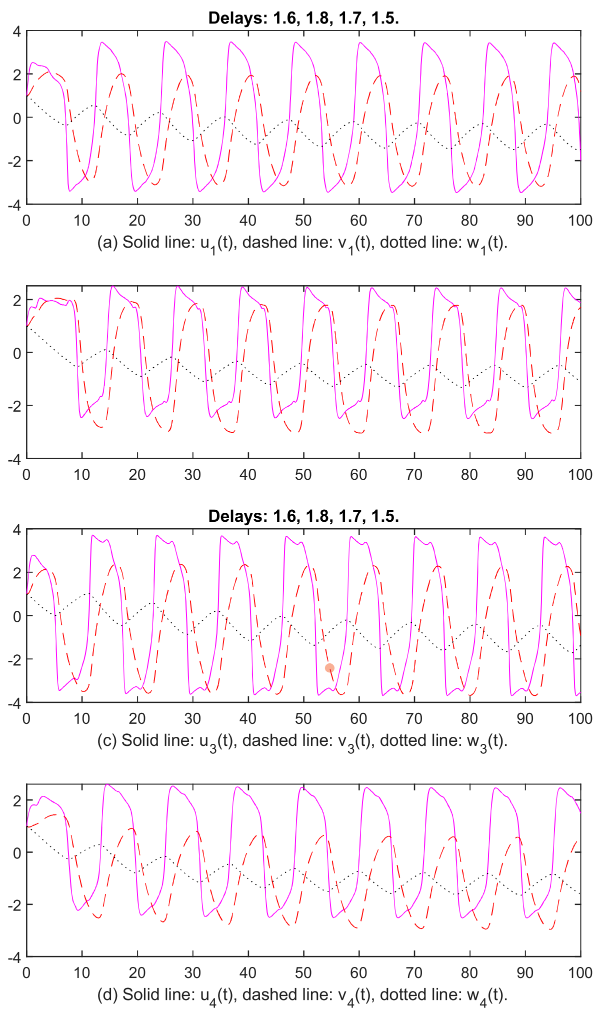


Figure 5: Oscillation of the solutions, delays: $1.6, 1.8, 1.7, 1.5$.

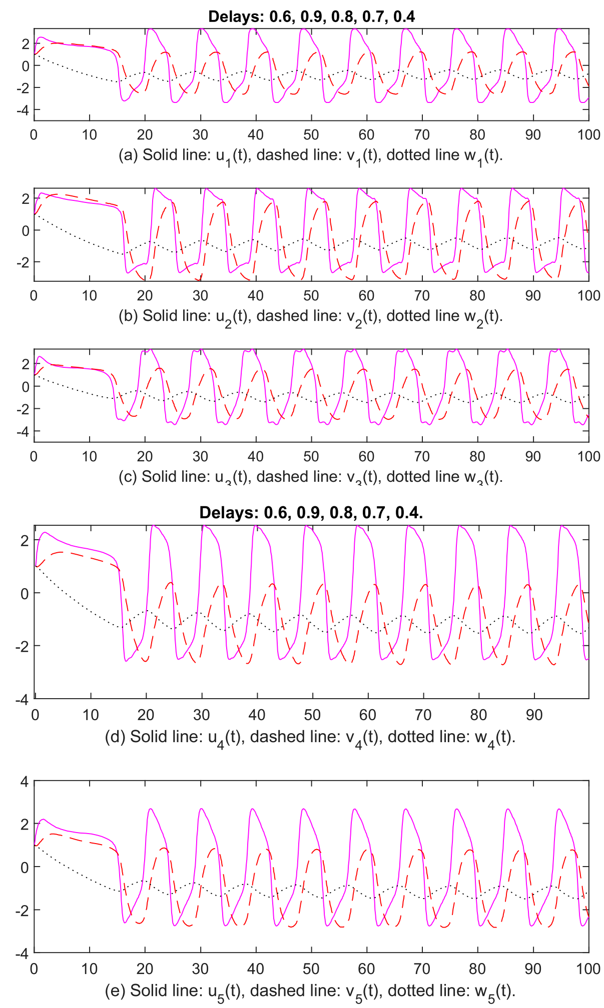


Figure 6: Oscillation of the solutions, delays: $0.6, 0.9, 0.8, 0.7, 0.4$.

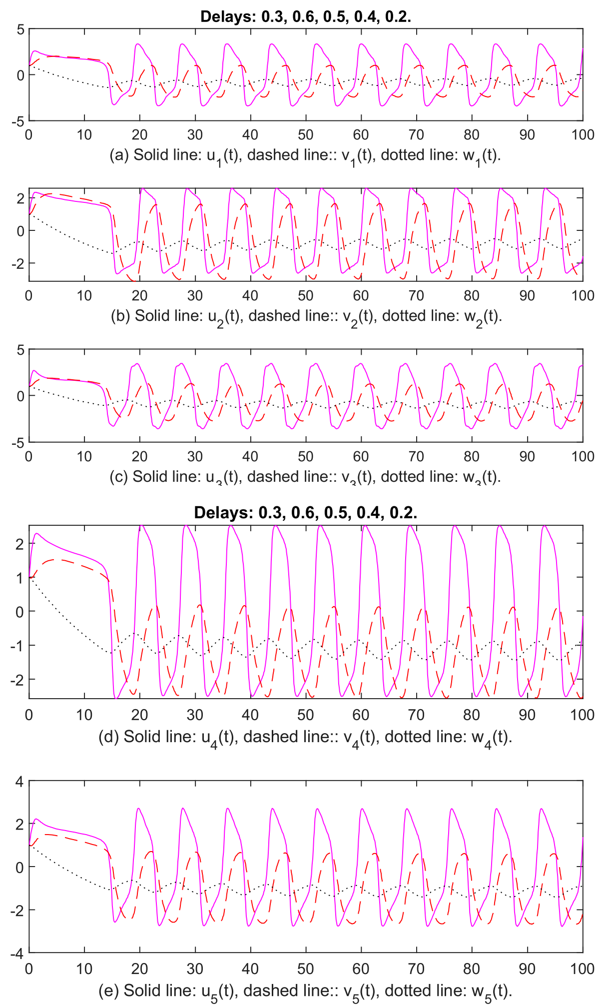


Figure 7: Oscillation of the solutions, delays: $0.3, 0.6, 0.5, 0.4, 0.2$.