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Contact CR and SCR-lightlike submanifolds of an indefinite Kenmotsu statistical manifold

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Abstract

This research article introduces the theoretical framework of lightlike submanifolds of an indefinite Kenmotsu statistical manifold and formulates various geometric results for contact CR and SCR-lightlike submanifolds. Several characterization theorems are established, focusing on the integrability of distributions and the geodesicity of these submanifolds. Relevant examples are also provided to illustrate the findings.
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1. INTRODUCTION

The theory of lightlike submanifolds in semi-Riemannian manifolds, introduced by [4], constitutes a vital area of research in differential geometry, with applications spanning multiple domains of mathematics and physics. The notion of contact Cauchy–Riemann (CR) and screen Cauchy–Riemann (SCR) lightlike submanifolds in indefinite Kenmotsu manifolds was initiated in [10], where the authors examined the integrability of distributions and the geodesic properties of such submanifolds. In a subsequent development, [11] investigated geometric conditions for the geodesicity of CR and SCR lightlike submanifolds, as well as geodesic transversal lightlike submanifolds in the same setting. These foundational studies have motivated further work in the geometric theory of lightlike submanifolds.

Statistical manifolds, viewed as Riemannian manifolds equipped with an affine connection, were introduced in [13] and later explored by [2], [7], [1], among others. These studies approached statistical inference through the lens of differential geometry. The statistical analogue of the Kenmotsu manifold, termed the Kenmotsu statistical manifold, was introduced in [9] and subsequently explored in [6], [3], and [14], with a focus on its geometric structure and statistical curvature. Statistical manifolds are prominently applied in areas such as neural networks, control systems, and information geometry.

Inspired by this line of research, the concept of an indefinite Kenmotsu statistical manifold, combining the statistical structure with the indefinite Kenmotsu metric has been introduced. Within this framework, the geometry of lightlike submanifolds, particularly contact CR and SCR-lightlike submanifolds is investigated and elaborated with suitable illustrations. The article also presents characterization theorems on mixed geodesicity, totally geodesicity, integrability of various distributions and on totally geodesic foliations in CR and SCR-lightlike submanifolds.

2. LIGHTLIKE GEOMETRY OF AN INDEFINITE STATISTICAL MANIFOLD

The theory of lightlike submanifolds, based on the frameworks established in [4] and [5], is outlined as follows:

Let $\mathbb{M}$ be a m-dimensional submanifold of a real $(m+n)$-dimensional semi-Riemannian manifold $(\bar{\mathbb{M}}, \bar{\mathfrak{g}})$ where $m, n \geq 1$. If $\mathbb{M}$ admits a degenerate metric $\mathfrak{g}$ induced from $\bar{\mathfrak{g}}$, whose Radical distribution $Rad(T\mathbb{M})$ of rank r $(1 \leq r \leq m)$ is defined by

$$Rad(T_x\mathbb{M}) = \{\zeta \in T_x\mathbb{M} : \mathfrak{g}_x(\zeta, X) = 0,\ X \in \Gamma(T\mathbb{M}),\ x \in \mathbb{M}\}$$

such that $Rad(T\mathbb{M}) = T\mathbb{M} \cap T\mathbb{M}^{\perp}$, where

$$T\mathbb{M}^{\perp} = \bigcup_{x \in \mathbb{M}} \{u \in T_x\bar{\mathbb{M}} : \bar{\mathfrak{g}}(u, v) = 0,\ \forall\ v \in T_x\mathbb{M}\},$$

then $\mathbb{M}$ is said to be lightlike submanifold of $\bar{\mathbb{M}}$.

Also, $T\mathbb{M} = Rad(T\mathbb{M}) \perp S(T\mathbb{M})$ where $S(T\mathbb{M})$ is a complementary distribution of $Rad(T\mathbb{M})$ in $T\mathbb{M}$ known as screen distribution and $T\mathbb{M}^{\perp} = Rad(T\mathbb{M}) \perp S(T\mathbb{M}^{\perp})$ where $S(T\mathbb{M}^{\perp})$ is a complementry vector bundle of $Rad(T\mathbb{M})$ in $T\mathbb{M}^{\perp}$ known as screen transversal vector bundle.

Furthermore, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(T\mathbb{M}^{\perp})$ in $(S(T\mathbb{M}))^{\perp}$ such that $\bar{\mathfrak{g}}(\zeta_i, N_j) = \delta_{ij}$ and $\bar{\mathfrak{g}}(N_i, N_j) = 0$ for any local basis $\{\zeta_i\}$ of $Rad(T\mathbb{M})$. It follows that there exists a lightlike transversal vector bundle $ltr(T\mathbb{M})$ locally spanned by $\{N_i\}$ where $i = 1, 2, ..., r$.

Let $tr(T\mathbb{M})$ be the complementary (but not orthogonal) vector bundle to $T\mathbb{M}$ in $T\bar{\mathbb{M}} \mid_{\mathbb{M}}$.

Then

$$tr(T\mathbb{M}) = ltr(T\mathbb{M}) \perp S(T\mathbb{M}^{\perp}),$$
$$T\bar{\mathbb{M}} \mid_{\mathbb{M}} = S(T\mathbb{M}) \perp \{Rad(T\mathbb{M}) \oplus ltr(T\mathbb{M})\} \perp S(T\mathbb{M}^{\perp}).$$

Let $\tilde{\mho}$ be the Levi-Civita connection on $\bar{\mathbb{M}}$, then the Gauss and Weingarten formulae are given by

$$\tilde{\mho}_X Y = \mho_X Y + h(X,Y), \quad \forall\ X, Y \in \Gamma(T\mathbb{M}) \tag{1}$$

$$\tilde{\mho}_X V = -A_V X + \mho_X^{\perp} V, \quad \forall\ X \in \Gamma(T\mathbb{M}),\ V \in \Gamma(tr(T\mathbb{M})) \tag{2}$$

where $\mho_X Y,\ A_V X \in \Gamma(T\mathbb{M})$ and $h(X,Y),\ \mho_X^{\perp} V \in \Gamma(tr(T\mathbb{M}))$. Here, $\mho$ and $\mho^{\perp}$ are the linear connections on $\mathbb{M}$ and vector bundle $tr(T\mathbb{M})$, respectively.

Consider the projection morphisms

$$L : tr(T\mathbb{M}) \to ltr(T\mathbb{M}) \quad and \quad S : tr(T\mathbb{M}) \to S(T\mathbb{M}^{\perp})$$

then, (1) and (2) become

$$\begin{aligned} \tilde{\mho}_X Y &= \mho_X Y + h^l(X,Y) + h^s(X,Y), \\ \tilde{\mho}_X V &= -A_V X + D_X^l V + D_X^s V. \end{aligned} \tag{3}$$

In particular, we have

$$\begin{aligned} \tilde{\mho}_X N &= -A_N X + \mho_X^l N + D^s(X,N), \\ \tilde{\mho}_X W &= -A_W X + \mho_X^s W + D^l(X,W) \end{aligned} \tag{4}$$

where $X, Y \in \Gamma(T\mathbb{M}),\ N \in \Gamma(ltr(T\mathbb{M}))$ and $W \in \Gamma(S(T\mathbb{M}^{\perp}))$.

Here, $\mho_X^l N,\ D^l(X,W) \in \Gamma(ltr(T\mathbb{M})), \quad \mho_X^s W,\ D^s(X,N) \in \Gamma(S(T\mathbb{M}^{\perp}))$, $\mho_X Y,\ A_N X,\ A_W X \in \Gamma(T\mathbb{M})$ and $h^l(X,Y) = Lh(X,Y), h^s(X,Y) = Sh(X,Y)$, $D_X^l V = L(\mho_X^{\perp} V),\ D_X^s V = S(\mho_X^{\perp} V)$; wherein h^l and h^s are respectively known as the lightlike second fundamental form and the screen second fundamental form on $\mathbb{M}$.

Let P be the projection morphism of $T\mathbb{M}$ on $S(T\mathbb{M})$, then we consider the following decompositions:

$$\mho_X PY = \mho'_X PY + h'(X,PY), \quad \mho_X \zeta = -A'_{\zeta} X + {\mho'_X}^{\perp} \zeta$$

where $\{\mho'_X Y, -A'_{\zeta} X\} \in \Gamma(S(T\mathbb{M}))$ and $\{h'(X,PY), {\mho'_X}^{\perp} \zeta\} \in \Gamma(Rad(T\mathbb{M}))$, respectively; $\mho'$ and ${\mho'}^{\perp}$ are linear connections on $S(T\mathbb{M})$ and $Rad(T\mathbb{M})$, respectively.

From the above equations, we obtain

$$\bar{\mathfrak{g}}(h^l(X,PY),\zeta) = \mathfrak{g}(A'_{\zeta} X, PY), \quad \bar{\mathfrak{g}}(h'(X,PY),N) = \mathfrak{g}(A_N X, PY),$$

$$\bar{\mathfrak{g}}(h^l(X,\zeta),\zeta)=0, \quad \mathfrak{g}(A'_\zeta PX, PY)=\mathfrak{g}(PX, A'_\zeta PY),$$
$$\bar{\mathfrak{g}}(A'_\zeta X, N)=0, \quad \bar{\mathfrak{g}}(A_N X, N)=0, \quad A'_\zeta\zeta=0$$

for any $X, Y\in\Gamma(T\mathbb{M}),\ \zeta\in\Gamma(Rad(T\mathbb{M}))$ and $N\in\Gamma(ltr(T\mathbb{M}))$.

Definition 2.1. A pair $(\bar{\mho},\bar{\mathfrak{g}})$ is called an indefinite statistical structure on a semi-Riemannian manifold $\bar{\mathbb{M}}$ where $\bar{\mathfrak{g}}$ is a semi-Riemannian metric of constant index $q\geq 1$ on $\bar{\mathbb{M}}$, if

$$\bar{\mho}_X Y-\bar{\mho}_Y X=[X,Y]$$
$$\text{and} \quad (\bar{\mho}_X\bar{\mathfrak{g}})(Y,Z)=(\bar{\mho}_Y\bar{\mathfrak{g}})(X,Z)$$

hold for any $X,Y,Z\in\Gamma(T\bar{\mathbb{M}})$.

Moreover, there exists $\bar{\mho}^*$ which is a dual connection of $\bar{\mho}$ with respect to $\bar{\mathfrak{g}}$, such that

$$X\bar{\mathfrak{g}}(Y,Z)=\bar{\mathfrak{g}}(\bar{\mho}_X Y,Z)+\bar{\mathfrak{g}}(Y,\bar{\mho}^*_X Z) \qquad \forall\ X,Y,Z\in\Gamma(T\bar{\mathbb{M}}). \tag{5}$$

If $(\bar{\mho},\bar{\mathfrak{g}})$ is an indefinite statistical structure on $\bar{\mathbb{M}}$, then so is $(\bar{\mho}^*,\bar{\mathfrak{g}})$. Hence, the indefinite statistical manifold is denoted by $(\bar{\mathbb{M}},\bar{\mho},\bar{\mho}^*,\bar{\mathfrak{g}})$.

The lightlike theory available so far leads to the Gauss and Weingarten formulae for the structure of a lightlike submanifold $(\mathbb{M},\mathfrak{g})$ of an indefinite statistical manifold $(\bar{\mathbb{M}},\bar{\mathfrak{g}},\bar{\mho},\bar{\mho}^*)$ as follows:

$$\begin{aligned}\bar{\mho}_X Y&=\mho_X Y+h^l(X,Y)+h^s(X,Y),\\ \bar{\mho}^*_X Y&=\mho^*_X Y+h^{*l}(X,Y)+h^{*s}(X,Y),\end{aligned} \tag{6}$$

$$\begin{aligned}\bar{\mho}_X V=-A_V X+D^l_X V+D^s_X V, &\quad \bar{\mho}^*_X V=-A^*_V X+D^{*l}_X V+D^{*s}_X V,\\ \bar{\mho}_X N=-A_N X+\mho^l_X N+D^s(X,N), &\quad \bar{\mho}^*_X N=-A^*_N X+\mho^{*l}_X N+D^{*s}(X,N),\\ \bar{\mho}_X W&=-A_W X+\mho^s_X W+D^l(X,W),\\ \bar{\mho}^*_X W&=-A^*_W X+\mho^{*s}_X W+D^{*l}(X,W)\end{aligned} \tag{7}$$

for any $X,Y\in\Gamma(T\mathbb{M})$, $V\in\Gamma(tr(T\mathbb{M}))$, $N\in\Gamma(ltr(T\mathbb{M}))$ and $W\in\Gamma(S(T\mathbb{M}^\perp))$.

Considering the corresponding projection morphism P of tangent bundle $T\mathbb{M}$ to the screen distribution, the following decompositions w.r.t $\mho$ and $\mho^*$ hold:

$$\begin{aligned}\mho_X PY=\mho'_X PY+h'(X,PY), &\quad \mho^*_X PY=\mho^{*\prime}_X PY+h^{*\prime}(X,PY),\\ \mho_X\zeta=-A'_\zeta X+\mho'^{\perp}_X\zeta, &\quad \mho^*_X\zeta=-A^{*\prime}_\zeta X+\mho^{*\prime\perp}_X\zeta\end{aligned} \tag{8}$$

for any $X,Y\in\Gamma(T\mathbb{M})$, $\zeta\in\Gamma(Rad(T\mathbb{M}))$.

Thus, we have

$$\bar{\mathfrak{g}}(h^l(X,PY),\zeta)=\mathfrak{g}(A^{*\prime}_\zeta X,PY), \quad \bar{\mathfrak{g}}(h^{*l}(X,PY),\zeta)=\mathfrak{g}(A'_\zeta X,PY),$$
$$\bar{\mathfrak{g}}(h'(X,PY),N)=\mathfrak{g}(A^*_N X,PY), \quad \bar{\mathfrak{g}}(h^{*\prime}(X,PY),N)=\mathfrak{g}(A_N X,PY).$$

3. INDEFINITE KENMOTSU STATISTICAL MANIFOLD

Let $\widehat{\bar{\mho}}$ be the Levi-Civita connection of $\bar{\mathfrak{g}}$. Therefore, by [8], we consider the decomposition $\widehat{\bar{\mho}} = \frac{1}{2}(\bar{\mho} + \bar{\mho}^*)$, where $\bar{\mho}^*$ is called the dual connection of $\bar{\mho}$ with respect to $\bar{\mathfrak{g}}$.

For a statistical structure $(\bar{\mho}, \bar{\mathfrak{g}})$ on $\bar{\mathbb{M}}$, we set

$$K_X Y = K(X,Y) = \bar{\mho}_X Y - \widehat{\bar{\mho}}_X Y \tag{9}$$

for any $X, Y \in \Gamma(T\mathbb{M})$ where K is the difference (1,2) type tensor of a torsion free affine connection $\bar{\mho}$ and the Levi-Civita connection $\widehat{\bar{\mho}}$.

Since $\bar{\mho}$ and $\widehat{\bar{\mho}}$ are torsion free, then we have

$$K_X Y = K_Y X, \quad \bar{\mathfrak{g}}(K_X Y, Z) = \bar{\mathfrak{g}}(Y, K_X Z) \qquad \forall\, X, Y, Z \in \Gamma(T\bar{\mathbb{M}}). \tag{10}$$

Also,

$$K(X,Y) = \widehat{\bar{\mho}}_X Y - \bar{\mho}^*_X Y = \frac{1}{2}(\bar{\mho}_X Y - \bar{\mho}^*_X Y). \tag{11}$$

Let $(\bar{\mathbb{M}}, \bar{\mathfrak{g}})$ be a semi-Riemannian manifold of dimension $(2n+1)$. If $\bar{\mathfrak{g}}$ is a semi-Riemannian metric, ϕ is a $(1,1)$ tensor field, τ is a characteristic vector field and ϖ is a 1-form, such that

$$\bar{\mathfrak{g}}(\phi X, \phi Y) = \bar{\mathfrak{g}}(X,Y) - \varpi(X)\varpi(Y), \qquad \bar{\mathfrak{g}}(\tau,\tau) = 1, \tag{12}$$

$$\phi^2(X) = -X + \varpi(X)\tau, \quad \bar{\mathfrak{g}}(X,\tau) = \varpi(X), \quad \bar{\mathfrak{g}}(\phi X, Y) + \bar{\mathfrak{g}}(X, \phi Y) = 0 \tag{13}$$

which follows that $\phi\tau = 0$ and $\varpi o \phi = 0$ for all $X, Y \in \Gamma(T\mathbb{M})$, then $(\phi, \tau, \bar{\mathfrak{g}})$ is called an almost contact metric structure on $\bar{\mathbb{M}}$.

Definition 3.1. [12] An almost contact metric structure on $\bar{\mathbb{M}}$ is called an indefinite Kenmotsu structure if

$$(\widehat{\bar{\mho}}_X \phi)Y = \bar{\mathfrak{g}}(\phi X, Y)\tau - \varpi(Y)\phi X, \quad \widehat{\bar{\mho}}_X \tau = X - \varpi(X)\tau \tag{14}$$

holds for any $X, Y \in \Gamma(T\mathbb{M})$, where $\widehat{\bar{\mho}}$ is a Levi-Civita Connection.

Definition 3.2. A quadruplet $(\bar{\mho} = \widehat{\bar{\mho}} + K, \bar{\mathfrak{g}}, \phi, \tau)$ is called an indefinite Kenmotsu statistical structure on $\bar{\mathbb{M}}$ if

1. $(\bar{\mathfrak{g}}, \phi, \tau)$ is an indefinite Kenmotsu structure,

2. $(\bar{\mho}, \bar{\mathfrak{g}})$ is a statistical structure on $\bar{\mathbb{M}}$ and

$$K(X, \phi Y) = -\phi K(X, Y) \tag{15}$$

holds for any $X, Y \in \Gamma(T\mathbb{M})$. Then $(\bar{\mathbb{M}}, \bar{\mho}, \bar{\mathfrak{g}}, \phi, \tau)$ is said to be an indefinite Kenmotsu statistical manifold.

If $(\bar{\mathbb{M}}, \bar{\mho}, \bar{\mathfrak{g}}, \phi, \tau)$ is an indefinite Kenmotsu statistical manifold, so is $(\bar{\mathbb{M}}, \bar{\mho}^*, \bar{\mathfrak{g}}, \phi, \tau)$.

Theorem 3.3. *For an almost contact metric structure $(\bar{\mathfrak{g}}, \phi, \tau)$ on an indefinite statistical manifold $(\bar{\mathbb{M}}, \bar{\mho}, \bar{\mho}^*, \bar{\mathfrak{g}})$, $(\bar{\mho}, \bar{\mho}^*, \bar{\mathfrak{g}}, \phi, \tau)$ is said to be an indefinite Kenmotsu statistical struture on $\bar{\mathbb{M}}$ iff:*

$$\bar{\mho}_X \phi Y - \phi \bar{\mho}^*_X Y = \bar{\mathfrak{g}}(\phi X, Y)\tau - \varpi(Y)\phi X \quad and \tag{16}$$

$$\bar{\mho}_X \tau = X - \varpi(X)\tau + \mu(X)\tau \tag{17}$$

hold for all the vector fields X, Y on $\bar{\mathbb{M}}$, where $\mu(X) = \varpi(\bar{\mho}_X \tau) = -\varpi(\bar{\mho}^*_X \tau) = \varpi(K(\tau, \tau))\varpi(X)$.

Proof. For an indefinite Kenmotsu statistical manifold, using (9) and (11), we get

$$\bar{\mho}_X \phi Y - \phi \bar{\mho}^*_X Y = K_X \phi Y + \widehat{\bar{\mho}}_X \phi Y - \phi \widehat{\bar{\mho}}_X Y + \phi K_X Y$$

$$\bar{\mho}_X \phi Y - \phi \bar{\mho}^*_X Y = \varpi(Y)\phi X - \bar{\mathfrak{g}}(\phi X, Y)\tau.$$

Replacing $\bar{\mho}$ by $\bar{\mho}^*$ and Y by τ, we have

$$\phi \bar{\mho}_X \tau = -\phi X + \bar{\mathfrak{g}}(\phi X, \tau)\tau.$$

Therefore, from the concept of almost contact metric structure, we obtain (17).

Conversely, replacing Y by ϕY in (16), we derive

$$\phi\{\bar{\mho}_X \phi^2 Y - \phi \bar{\mho}^*_X \phi Y\} = 0, \text{ where } \varpi(\phi Y) = 0 \text{ and } \phi\tau = 0.$$

Further, using (5), (13) and (17), above equation becomes :

$$0 = -\phi \bar{\mho}_X Y + \varpi(Y)\phi \bar{\mho}_X \tau + \bar{\mho}^*_X \phi Y + \bar{\mathfrak{g}}(\phi Y, \bar{\mho}_X \tau)\tau$$

$$= -\phi \bar{\mho}_X Y - \varpi(Y)\phi X + \bar{\mho}^*_X \phi Y + \bar{\mathfrak{g}}(\phi X, Y)\tau$$

which implies

$$\bar{\mho}^*_X \phi Y - \phi \bar{\mho}_X Y = \varpi(Y)\phi X - \bar{\mathfrak{g}}(\phi X, Y)\tau. \tag{18}$$

Hence, using (16) and (18) respectively, we get

$$(\widehat{\bar{\mho}}_X \phi)Y - \varpi(Y)\phi X + \bar{\mathfrak{g}}(\phi X, Y)\tau = K_X \phi Y + \phi K_X Y$$

and

$$(\widehat{\bar{\mho}}_X \phi)Y - \varpi(Y)\phi X + \bar{\mathfrak{g}}(\phi X, Y)\tau = -K_X \phi Y - \phi K_X Y.$$

□

Remark 3.1. *[8] Let $(\bar{\mathfrak{g}}, \phi, \tau)$ be an indefinite Kenmotsu structure on $\bar{\mathbb{M}}$. By setting*

$$K(X,Y) = \mathfrak{g}(X,\tau)\mathfrak{g}(Y,\tau)\tau$$

for any $X, Y \in \Gamma(T\bar{\mathbb{M}})$ such that K satisfies (10) and (15), we obtain an indefinite Kenmotsu statistical structure $(\bar{\mho}^{\lambda} = \hat{\bar{\mho}} + \lambda K, \bar{\mathfrak{g}}, \phi, \tau)$ on $\bar{\mathbb{M}}$ for $\lambda \in C^{\infty}(\bar{\mathbb{M}})$.

4. GEODESIC CONTACT CR-LIGHTLIKE SUBMANIFOLDS

Definition 4.1. [10] Let $(\mathbb{M}, \mathfrak{g}, S(T\mathbb{M}), S(T\mathbb{M}^{\perp}))$ be a lightlike submanifold tangent to the structure vector field τ, immersed in an indefinite Kenmotsu statistical manifold $(\bar{\mathbb{M}}, \bar{\mathfrak{g}})$. Then, $\mathbb{M}$ is a contact CR-lightlike submanifold of $\bar{\mathbb{M}}$ if the following conditions are satisfied:

1. $RadT\mathbb{M}$ is a distribution on $\mathbb{M}$ such that $Rad(T\mathbb{M}) \cap \phi(Rad(T\mathbb{M})) = \{0\}$,
2. there exist vector bundles D_0 and D' over $\mathbb{M}$ such that

$$S(T\mathbb{M}) = \{\phi(Rad(T\mathbb{M})) \oplus D'\} \perp D_o \perp \{\tau\},$$

$$\phi D_o = D_o, \quad \phi(D') = L_1 \perp ltr(T\mathbb{M})$$

where D_o is non-degenerate and L_1 is a vector subbundle of $S(T\mathbb{M}^{\perp})$.

Thus, one has the following decomposition:

$$T\mathbb{M} = D \oplus D' \perp \{\tau\}, \quad D = Rad(T\mathbb{M}) \perp \phi(Rad(T\mathbb{M})) \perp D_o. \tag{19}$$

Also, a contact CR-lightlike submanifold is proper if $D_o \neq \{0\}$ and $L_1 \neq \{0\}$.

Let, the orthogonal complement subbundle to the vector subbundle L_1 in $S(T\mathbb{M}^{\perp})$ be denoted by $L_1^{\perp}$. For a contact CR-lightlike submanifold $\mathbb{M}$, we put

$$\phi X = fX + wX, \quad \forall\, X \in \Gamma(T\mathbb{M}), \tag{20}$$

where $fX \in \Gamma(D)$ and $wX \in \Gamma(L_1 \perp ltr(T\mathbb{M}))$. Similarly, we have

$$\phi W = BW + CW, \quad \forall\, W \in \Gamma(S(T\mathbb{M}^{\perp})), \tag{21}$$

where $BW \in \Gamma(\phi L_1)$ and $CW \in \Gamma(L_1^{\perp})$.

Inspired by [10], the geometry of contact CR-lightlike submanifold of an indefinite Kenmotsu statistical manifold is detailed with an example as follows.

Example 4.2. Consider a semi-Euclidean space $\bar{\mathbb{M}} = (\mathbb{R}^9_2, \bar{\mathfrak{g}})$, with canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$$

where $\bar{\mathfrak{g}}$ is of signature $(-, +, +, +, -, +, +, +, +)$.

The degenerate structure of an indefinite Kenmotsu manifold $(\mathbb{R}^9_2, \phi_o, \tau, \varpi, \bar{\mathfrak{g}})$ for cartesian coordinates $(x^i; y^i; z)$ is given as:

$$\varpi = dz, \qquad \tau = \partial z,$$

$$\bar{\mathfrak{g}} = \varpi\otimes\varpi + e^{2z}(-dx^1\otimes dx^1 - dy^1\otimes dy^1 + dx^3\otimes dx^3 + dy^3\otimes dy^3 + dx^4\otimes dx^4 + dy^4\otimes dy^4),$$

$$\phi_o(\sum_{i=1}^{4}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{4}(Y_i\partial x^i - X_i\partial y_i).$$

If we set $K(X, Y) = \mathfrak{g}(X, \tau)\mathfrak{g}(Y, \tau)\tau$ with $\lambda = 1$, then $(\bar{\mho} = \widehat{\bar{\mho}} + K, \phi_o, \tau, \varpi, \bar{\mathfrak{g}})$ defines an indefinite Kenmotsu statistical structure on $\bar{\mathbb{M}}$ by Remark (3.1).

Let $\mathbb{M}$ be a submanifold of $\mathbb{R}^9_2$ defined by

$$x^1 = y^4, \quad x^2 = \sqrt{1 - (y^2)^2}, \quad y^2 \neq \pm 1,$$

Thus, the local frame of $T\mathbb{M}$ is as below:

$$l_1 = e^{-z}(\partial x_1 + \partial y_4), \quad l_2 = e^{-z}(\partial x_4 - \partial y_1), \tag{22}$$

$$l_3 = e^{-z}\partial x_3, \quad l_4 = e^{-z}\partial y_3, \quad l_5 = e^{-z}(-\frac{y^2}{x^2}\partial x_2 + \partial y_2), \tag{23}$$

$$l_6 = e^{-z}(\partial x_4 + \partial y_1), \quad l_7 = \tau = \partial z. \tag{24}$$

Using (22), (23) and (24), we have $Rad(T\mathbb{M}) = span\{l_1\}$, $\phi_o(Rad(T\mathbb{M})) = span\{l_2\}$ and $Rad(T\mathbb{M}) \cup \phi_o(Rad(T\mathbb{M})) = \{0\}$. Also, $\phi_o(l_3) = -l_4$ and $\phi_o(l_4) = l_3$, which implies that $\phi_o D_o = D_o$, where $D_o = \{l_3, l_4\}$.

Further, the screen transversal vector bundle and lightlike transversal vector bundle are given by
$S(T\mathbb{M}^\perp) = span\{W = e^{-z}(\partial x_2 + \frac{y^2}{x^2}\partial y_2)$ where $\phi_o(W) = -l_5$ and
$ltr(T\mathbb{M}) = span\{N = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_4)$ where $\phi_o(N) = \frac{1}{2}l_6$, respectively which gives $D^{'} = span\{\phi_o N, \phi_o W\}$.

Therefore, $\mathbb{M}$ becomes a contact CR-lightlike submanifold of the indefinite Kenmotsu statistical manifold $(\mathbb{R}^9_2, \widehat{\bar{\mho}} + K, \phi_o, \tau, \varpi, \bar{\mathfrak{g}})$.

Theorem 4.3. *For a contact CR-lightlike submanifold $\mathbb{M}$ of an indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}$, $D \perp \{\tau\}$ is a totally geodesic foliation iff*

$$\bar{\mathfrak{g}}(h^{*l}(X,\phi Y),\zeta) + \bar{\mathfrak{g}}(h^{l}(X,\phi Y),\zeta) = 0$$

$$\bar{\mathfrak{g}}(h^{*s}(X,Y),\phi W) + \bar{\mathfrak{g}}(h^{s}(X,Y),\phi W) = 0$$

for all $X, Y \in \Gamma(D \perp \{\tau\})$ and $W \in \Gamma(\phi L_1)$.

Proof. $D \perp \{\tau\}$ defines a totally geodesic foliation iff

$$\bar{\mathfrak{g}}(\widehat{\mho}_X Y, \phi\zeta) = \bar{\mathfrak{g}}(\widehat{\mho}_X Y, W) = 0$$

for $X, Y \in \Gamma(D \perp \{\tau\})$ and $W \in \Gamma(\phi L_1)$. Now from the concept of indefinite Kenmotsu statistical manifold and the equations (6) and (13), we have
$\bar{\mathfrak{g}}(\widehat{\mho}_X Y, \phi\zeta) = \bar{\mathfrak{g}}(\widehat{\bar{\mho}}_X Y, \phi\zeta)$
$= \frac{1}{2}[-\bar{\mathfrak{g}}(\phi\bar{\mho}_X Y,\zeta) - \bar{\mathfrak{g}}(\phi\bar{\mho}^*_X Y,\zeta)]$
$= -\frac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}^*_X \phi Y,\zeta) + \bar{\mathfrak{g}}(\bar{\mho}_X \phi Y,\zeta)]$
$= -\frac{1}{2}[\bar{\mathfrak{g}}(h^{*l}(X,\phi Y),\zeta) + \bar{\mathfrak{g}}(h^{l}(X,\phi Y),\zeta)]$

And $\bar{\mathfrak{g}}(\widehat{\mho}_X \phi Y, W) = \frac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}_X \phi Y, W) + \bar{\mathfrak{g}}(\bar{\mho}^*_X \phi Y, W)]$
$= \frac{1}{2}[\bar{\mathfrak{g}}(\phi\bar{\mho}^*_X Y + \varpi(Y)\phi X + \bar{\mathfrak{g}}(\phi X, Y)\tau, W) + \bar{\mathfrak{g}}(\phi\bar{\mho}_X Y + \varpi(Y)\phi X + \bar{\mathfrak{g}}(\phi X, Y)\tau, W)]$
$= -\frac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}^*_X Y, \phi W) + \bar{\mathfrak{g}}(\bar{\mho}_X Y, \phi W)]$
$= -\frac{1}{2}[\bar{\mathfrak{g}}(h^{*s}(X,Y),\phi W) + \bar{\mathfrak{g}}(h^{s}(X,Y),\phi W)]$.

which proves the assertion. □

Theorem 4.4. *If $\mathbb{M}$ is a contact CR-lightlike submanifold of an indefinite Kenmotsu statistical manifold $(\bar{\mathbb{M}}, \bar{\mho}, \bar{\mho}^*, \bar{\mathfrak{g}}, \phi, \tau)$, then D is integrable if and only if*

$$h^{*s}(X,\phi Y) + h^{s}(X,\phi Y) = h^{*s}(Y,\phi X) + h^{s}(Y,\phi X) \quad \forall\, X, Y \in \Gamma(D).$$

Proof. D is integrable if and only if

$$\bar{\mathfrak{g}}([X,Y],\tau) = 0, \quad \bar{\mathfrak{g}}([X,Y],W) = 0 \quad \forall\, X, Y \in \Gamma(D),\ W \in \Gamma(D').$$

Since $\bar{\mathbb{M}}$ is an indefinite Kenmotsu statistical manifold and (5), (6), (12), (19) hold, therefore, we have $\bar{\mathfrak{g}}([X,Y],\tau) = 0$

Also $\bar{\mathfrak{g}}([X,Y],W) = \frac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}_X Y, W) + \bar{\mathfrak{g}}(\bar{\mho}^*_X Y, W) - \bar{\mathfrak{g}}(\bar{\mho}_Y X, W) - \bar{\mathfrak{g}}(\bar{\mho}^*_Y X, W)]$
$= \frac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}^*_X \phi Y + \bar{\mathfrak{g}}(\phi X, Y)\tau, \phi W) + \bar{\mathfrak{g}}(\bar{\mho}_X \phi Y + \bar{\mathfrak{g}}(\phi X, Y)\tau, \phi W)$
$- \bar{\mathfrak{g}}(\bar{\mho}^*_Y \phi X + \bar{\mathfrak{g}}(\phi Y, X)\tau, \phi W) - \bar{\mathfrak{g}}(\bar{\mho}_Y \phi X + \bar{\mathfrak{g}}(\phi Y, X)\tau, \phi W)]$
$= \frac{1}{2}[\bar{\mathfrak{g}}(h^{*s}(X,\phi Y),\phi W) + \bar{\mathfrak{g}}(h^{s}(X,\phi Y),\phi W) - \bar{\mathfrak{g}}(h^{*s}(Y,\phi X),\phi W) - \bar{\mathfrak{g}}(h^{s}(Y,\phi X),\phi W)]$.

Hence, we obtain the desired result using hypothesis.

□

Theorem 4.5. *If $\bar{\mathbb{M}}$ is an indefinite Kenmotsu statistical manifold and $\mathbb{M}$ is a contact CR-lightlike submanifold of $\bar{\mathbb{M}}$, then D' does not define a totally geodesic foliation.*

Proof. D' defines a toatlly geodesic foliation iff

$$\bar{\mathfrak{g}}(\widehat{\mho}_Z W, N) = \bar{\mathfrak{g}}(\widehat{\mho}_Z W, \phi N) = \bar{\mathfrak{g}}(\widehat{\mho}_Z W, X) = \bar{\mathfrak{g}}(\widehat{\mho}_Z W, \tau) = 0 \tag{25}$$

where $Z, W \in \Gamma(D')$, $N \in \Gamma(ltr(T\mathbb{M}))$ and $X \in \Gamma(D_0)$.

Therefore, using (5), (17), we obtain

$\bar{\mathfrak{g}}(\widehat{\mho}_Z W, \tau) = \bar{\mathfrak{g}}(\widehat{\bar{\mho}}_Z W, \tau)$
$= -\frac{1}{2}[\bar{\mathfrak{g}}(W, -Z + \varpi(Z)\tau + \varpi(\bar{\mho}^*_Z \tau)\tau) + \bar{\mathfrak{g}}(W, -Z + \varpi(Z)\tau + \varpi(\bar{\mho}_Z \tau)\tau)]$
$= \bar{\mathfrak{g}}(W, Z)$

And $\bar{\mathfrak{g}}(\widehat{\mho}_Z W, X) = \frac{1}{2}[\bar{\mathfrak{g}}(\phi\bar{\mho}_Z W, \phi X) + \bar{\mathfrak{g}}(\phi\bar{\mho}^*_Z W, \phi X)]$
$= \frac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}^*_Z \phi W - \varpi(W)\varpi(\phi Z) + \bar{\mathfrak{g}}(\phi Z, W)\tau, \phi X) + \bar{\mathfrak{g}}(\bar{\mho}_Z \phi W - \varpi(W)\varpi(\phi Z)$
$+ \bar{\mathfrak{g}}(\phi Z, W), \phi X)]$
$= \frac{1}{2}[\bar{\mathfrak{g}}(-A^*_{\phi W} Z, \phi X) + \bar{\mathfrak{g}}(-A_{\phi W} Z, \phi X)]$

Since $\bar{\mathfrak{g}}(W, Z) \neq 0$ and $\bar{\mathfrak{g}}(-A^*_{\phi W} Z, \phi X) + \bar{\mathfrak{g}}(-A_{\phi W} Z, \phi X) \neq 0$, therefore $D^{'}$ does not define a totally geodesic foliation using (25).

□

Theorem 4.6. *For a contact CR-lightlike submanifold $\mathbb{M}$ of an indefinite Kenmotsu statistical manifold $(\bar{\mathbb{M}}, \bar{\mho}, \bar{\mho}^*, \bar{\mathfrak{g}}, \phi, \tau)$, $D \perp \{\tau\}$ is integrable if and only if*

$$h(X, \phi Y) + h^*(X, \phi Y) = h(Y, \phi X) + h^*(Y, \phi X)$$

for all $X, Y \in D \perp \{\tau\}$.

Proof. Since, $\bar{\mathbb{M}}$ is an indefinite Kenmotsu statistical manifold and (6), (12), (20), (21) hold, therefore, on considering

$$h(X, \phi Y) = \phi\bar{\mho}^*_X Y + \varpi(Y)\phi X - \bar{\mathfrak{g}}(\phi X, Y)\tau - \mho_X \phi Y$$

we obtain, $w(\mho^*_X Y) = h(X, \phi Y) - C(h^{s*}(X, Y)) \quad \forall\, X, Y \in D \perp \{\tau\}$.

Similarly, we derive $w(\mho_X Y) = h^*(X, \phi Y) - C(h^s(X, Y))$.

On interchanging X and Y in the above equations, we get

$w(\mho^*_Y X) = h(Y, \phi X) - C(h^{s*}(Y, X))$ and $w(\mho_Y X) = h^*(Y \phi X) - C(h^s(Y, X))$.

Hence, we have $h(X,\phi Y)+h^*(X,\phi Y)-h(Y,\phi X)-h^*(Y,\phi X)=2w([X,Y])$.

Thus, the result follows from the integrability of $D\perp\{\tau\}$.

□

5. GEODESIC CONTACT SCR-LIGHTLIKE SUBMANIFOLDS

Definition 5.1. [10] Let $(\mathbb{M},\mathfrak{g},S(T\mathbb{M}),S(T\mathbb{M}^\perp))$ be a lightlike submanifold tangent to the structure vector field τ, immersed in an indefinite Kenmotsu statistical manifold $(\bar{\mathbb{M}},\bar{\mathfrak{g}})$. Then $\mathbb{M}$ is a contact SCR-lightlike submanifold of $\bar{\mathbb{M}}$ if the following conditions hold:

1. There exist real non-empty distributions D and $D^\perp$ such that
$$S(T\mathbb{M})=D\perp D^\perp\perp\{\tau\},$$
$$\phi(D^\perp)\subset(S(T\mathbb{M}^\perp)),\quad D\cap D^\perp=\{0\}$$
where $D^\perp$ is orthogonally complementary to $D\perp\{\tau\}$ in $S(T\mathbb{M})$.

2. The distributions D and $Rad(T\mathbb{M})$ are invariant with respect to ϕ. It follows that $ltr(T\mathbb{M})$ is also invariant with respect to ϕ. Hence we have the decomposition:
$$T\mathbb{M}=\bar{D}\perp D^\perp\perp\{\tau\},\quad \bar{D}=D\perp Rad(T\mathbb{M}). \tag{26}$$

For any $X\in\Gamma(T\mathbb{M})$ and $W\in\Gamma(S(T\mathbb{M}^\perp))$, we put
$$\phi X=P'X+F'X,\qquad \phi W=B'W+C'W, \tag{27}$$
where $P'X\in\Gamma(\bar{D}),\ F'X\in\Gamma(\phi D^\perp),\ B'W\in\Gamma(D^\perp),\ C'W\in\Gamma(\mu')$ and μ' denotes the orthogonal complement to $\phi(D^\perp)$ in $S(T\mathbb{M}^\perp)$.

Now, an example elaborating the structure of an SCR-lightlike submanifold of an indefinite Kenmotsu statistical manifold following [10] is presented as:

Example 5.2. Consider the indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}=(\mathbb{R}^9_2,\bar{\mho}=\hat{\bar{\mho}}+K,\phi_o,\tau,\varpi,\bar{\mathfrak{g}})$ with the structure defined as in example (4.2).

Now let $\mathbb{M}$ be a submanifold of $\bar{\mathbb{M}}$ defined by
$$x^1=x^2,\qquad y^1=y^2,\qquad x^4=\sqrt{1-(y^4)^2},\qquad y^4\neq\pm1.$$

Then, the local frame of $T\mathbb{M}$ corresponding to the submanifold of indefinite Kenmotsu statistical manifold is constructed as under:
$$l_1=e^{-z}(\partial x_1+\partial x_2),\qquad l_2=e^{-z}(\partial y_1+\partial y_2), \tag{28}$$

$$l_3 = e^{-z}\partial x_3, \qquad l_4 = e^{-z}\partial y_3, \tag{29}$$

$$l_5 = e^{-z}(-y^4\partial x_4 + x^4\partial y_4), \qquad l_6 = \tau = \partial z. \tag{30}$$

Then, $Rad(T\mathbb{M}) = span\{l_1, l_2\}$ as $\phi_o(l_1) = -l_2$ and $D = span\{l_3, l_4\}$ as $\phi_o(l_3) = -l_4$.

Also, $S(T\mathbb{M}^\perp) = span\{W = e^{-z}(x^4\partial x_4 + y^4\partial y_4)\}$ where $\phi_o(W) = -l_5$ and $ltr(T\mathbb{M}) = span\{N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial x_2),\ N_2 = \frac{e^{-z}}{2}(-\partial y_1 + \partial y_2)\}$ where $\phi_o(N_2) = N_1$. This shows that $Rad(T\mathbb{M})$, D and $ltr(T\mathbb{M})$ are invariant with respect to ϕ_o.

Therefore, $\mathbb{M}$ is said to be a contact SCR-lightlike submanifold of the indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}$.

Theorem 5.3. *For a contact SCR-lightlike submanifold $\mathbb{M}$ of an indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}$, the induced connections $\mho$ $(respectively\ \mho^*)$ are metric connections iff ${A^*}'_\zeta X + A'_\zeta X$ and $h^{*s}(X,\zeta) + h^s(X,\zeta)$ have no components in D and $\phi(D^\perp)$ respectively, for all $X \in \Gamma(T\mathbb{M}),\ \zeta \in \Gamma(Rad(T\mathbb{M}))$.*

Proof. Replacing Y by ζ in (16) and simplifying further, we derive $\bar{\mho}_X\phi\zeta = \phi\bar{\mho}^*_X\zeta$ and $\bar{\mho}^*_X\phi\zeta = \phi\bar{\mho}_X\zeta$, for all $X \in \Gamma(T\mathbb{M}),\ \zeta \in \Gamma(Rad(T\mathbb{M}))$.

Now using (6), (8) and (27), we obtain

$$\mho_X\phi\zeta + h^l(X,\phi\zeta) + h^s(X,\phi\zeta) = \phi(-{A^*}'_\zeta X + {\mho^*}'^{\perp}_X\zeta) + \phi(h^{*l}(X,\zeta)) + \phi(h^{*s}(X,\zeta)).$$

Therefore, taking the tangential parts, we have

$$\mho_X\phi\zeta = -P'({A^*}'_\zeta X) + \phi({\mho^*}'^{\perp}_X\zeta) + B'(h^{*s}(X,\zeta)). \tag{31}$$

Similarly, we derive

$$\mho^*_X\phi\zeta = -P'(A'_\zeta X) + \phi({\mho'}^{\perp}_X\zeta) + B'(h^s(X,\zeta)). \tag{32}$$

Since the induced connection is a metric connection iff $Rad(T\mathbb{M})$ is parallel with respect to $\mho$ (respectively $\mho^*$), so by letting $Rad(T\mathbb{M})$ to be parallel, we have $\mho_X\phi\zeta = 0$ and $\mho^*_X\phi\zeta = 0$.

From (31) and (32), we get

$\bar{\mathfrak{g}}(P'({A^*}'_\zeta X + A'_\zeta X), \phi Z) = 0$ and $\bar{\mathfrak{g}}(B'(h^{*s}(X,\zeta) + h^s(X,\zeta)), \phi Z) = 0$

where $P'(A^{*\prime}_{\zeta}X), P'(A'_{\zeta}X) \in \Gamma(\bar{D})$, $B'(h^{*s}(X,\zeta)), B'(h^{s}(X,\zeta)) \in \Gamma(D^{\perp})$ and $\phi Z \in \Gamma(D^{\perp})$.

This shows that $A^{*\prime}_{\zeta}X + A'_{\zeta}X$ has no components in D and $h^{*s}(X,\zeta) + h^{s}(X,\zeta)$ has no components in $\phi(D^{\perp})$.

Conversely, from (31) and (32), we have

$$\mho_X\phi\zeta,\ \mho^*_X\phi\zeta \in \Gamma(Rad(T\mathbb{M})).$$

Since $Rad(T\mathbb{M})$ is parallel, therefore the induced connections $\mho$ (respectively $\mho^*$) are metric connections.

□

Definition 5.4. Let $\bar{\mathbb{M}}$ be an indefinite Kenmotsu statistical manifold and $\mathbb{M}$ be a contact SCR-lightlike submanifold of $\bar{\mathbb{M}}$. Then, $\mathbb{M}$ is said to be $D^{\perp}$-totally geodesic contact SCR-lightlike submanifold with respect to $\bar{\mho}$ (respectively $\bar{\mho}^*$) if $h(X,Y) = 0$ (respectively $h^*(X,Y) = 0$) $\forall\ X,Y \in \Gamma(D^{\perp})$.

Theorem 5.5. *For a contact SCR-lightlike submanifold $\mathbb{M}$ of an indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}$, $\mathbb{M}$ is $D^{\perp}$-totally geodesic if and only if*

$$\bar{\mathfrak{g}}(D^{*l}(X,F'Y),\phi\zeta) + \bar{\mathfrak{g}}(D^{l}(X,F'Y),\phi\zeta) = 0$$

$$\bar{\mathfrak{g}}(\mho^{*s}{}_X\phi Y, C'W) + \bar{\mathfrak{g}}(\mho^{s}_X\phi Y, C'W) = \bar{\mathfrak{g}}(A^*_{\phi Y}X, B'W) + \bar{\mathfrak{g}}(A_{\phi Y}X, B'W)$$

for all $X,Y \in \Gamma(D^{\perp})$.

Proof. Using (12) and (27) for a contact SCR-lightlike submanifold $\mathbb{M}$ of the indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}$, we have

$$\begin{aligned}&\bar{\mathfrak{g}}(h^{l}(X,Y),\zeta) + \bar{\mathfrak{g}}(h^{*l}(X,Y),\zeta) = \bar{\mathfrak{g}}(\bar{\mho}_XY,\zeta) + \bar{\mathfrak{g}}(\bar{\mho}^*_XY,\zeta)\\ &= \bar{\mathfrak{g}}(\bar{\mho}^*_X\phi Y - \varpi(Y)\phi X + \bar{\mathfrak{g}}(\phi X,Y)\tau,\phi\zeta) + \bar{\mathfrak{g}}(\bar{\mho}_X\phi Y - \varpi(Y)\phi X + \bar{\mathfrak{g}}(\phi X,Y)\tau,\phi\zeta)\end{aligned}$$

which gives

$$\bar{\mathfrak{g}}(h^{l}(X,Y),\zeta) + \bar{\mathfrak{g}}(h^{*l}(X,Y),\zeta) = \bar{\mathfrak{g}}(D^{*l}(X,F'Y),\phi\zeta) + \bar{\mathfrak{g}}(D^{l}(X,F'Y),\phi\zeta) \quad (33)$$

And $\bar{\mathfrak{g}}(h^{l}(X,Y),W) + \bar{\mathfrak{g}}(h^{*l}(X,Y),W) = \bar{\mathfrak{g}}(\bar{\mho}_XY,W) + \bar{\mathfrak{g}}(\bar{\mho}^*_XY,W)$
$= \bar{\mathfrak{g}}(\bar{\mho}^*_X\phi Y,\phi W) + \bar{\mathfrak{g}}(\bar{\mho}_X\phi Y,\phi W)$

Then

$$\begin{aligned}\bar{\mathfrak{g}}(h^{l}(X,Y),W) + \bar{\mathfrak{g}}(h^{*l}(X,Y),W) &= \bar{\mathfrak{g}}(-A^*_{\phi Y}X, B'W) + \bar{\mathfrak{g}}(\mho^{*s}{}_X\phi Y, C'W)\\ &\quad + \bar{\mathfrak{g}}(-A_{\phi Y}X, B'W) + \bar{\mathfrak{g}}(\mho^{s}_X\phi Y, C'W).\end{aligned} \quad (34)$$

If $\mathbb{M}$ is $D^{\perp}$-totally geodesic, then

$$\bar{\mathfrak{g}}(h^l(X,Y),\zeta)=0, \qquad \bar{\mathfrak{g}}(h^{*l}(X,Y),\zeta)=0$$

$$\bar{\mathfrak{g}}(h^s(X,Y),W)=0 \qquad \bar{\mathfrak{g}}(h^{*s}(X,Y),W)=0$$

for all $X,Y\in\Gamma(D^{\perp})$.

Therefore, the hypothesis alongwith (33) and (34) leads to the desired result.

Conversely, using (33) and (34) for all $X,Y\in\Gamma(D^{\perp})$, $\mathbb{M}$ becomes $D^{\perp}$-totally geodesic.

□

Theorem 5.6. *Let* $(\bar{\mathbb{M}},\bar{\mho},\bar{\mho}^*,\bar{\mathfrak{g}},\phi,\tau)$ *be an indefinite Kenmotsu statistical manifold and* $\mathbb{M}$ *be a contact* SCR*-lightlike submanifold of* $\bar{\mathbb{M}}$*. Then* $\bar{D}\perp\{\tau\}$ *defines a totally geodesic foliation in* $\mathbb{M}$ *iff*

$$\bar{\mathfrak{g}}(h^{*s}(X,\phi Y)+h^s(X,\phi Y),\phi Z)=0$$

for all $X,Y\in\Gamma(\bar{D}\perp\{\tau\})$.

Proof. As per the concept of indefinite Kenmotsu statistical manifold and from (12), for all $X,Y\in\Gamma(\bar{D}\perp\{\tau\})$, we have

$$\begin{aligned}&\bar{\mathfrak{g}}(\widehat{\mho}_XY,Z)=\bar{\mathfrak{g}}(\widehat{\bar{\mho}}_XY,Z)\\&=\tfrac{1}{2}[\bar{\mathfrak{g}}(\phi\bar{\mho}_XY,\phi Z)+\bar{\mathfrak{g}}(\phi\bar{\mho}^*_XY,\phi Z)]\\&=\tfrac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}^*_X\phi Y-\varpi(Y)\phi X+\bar{\mathfrak{g}}(\phi X,Y)\tau,\phi Z)+\bar{\mathfrak{g}}(\bar{\mho}_X\phi Y-\varpi(Y)\phi X+\bar{\mathfrak{g}}(\phi X,Y)\tau,\phi Z)]\\&=\tfrac{1}{2}[\bar{\mathfrak{g}}(h^{*s}(X,\phi Y),\phi Z)+\bar{\mathfrak{g}}(h^s(X,\phi Y),\phi Z)]\end{aligned}$$

Since, $\bar{D}\perp\{\tau\}$ defines a totally geodesic foliation in $\mathbb{M}$ iff

$$\bar{\mathfrak{g}}(\widehat{\mho}_XY,Z)=0 \quad \forall\ X,Y\in\Gamma(\bar{D}\perp\{\tau\})$$

Hence the proof.

□

Theorem 5.7. *Let* $\mathbb{M}$ *be a contact* SCR*-lightlike submanifold of an indefinite Kenmotsu statistical manifold* $\bar{\mathbb{M}}$*. Then* $D^{\perp}$ *and* $\bar{D}$ *do not define a geodesic foliation on* $\mathbb{M}$.

Proof. $D^{\perp}$ defines a totally geodesic foliation in a contact SCR-lightlike submanifold $\mathbb{M}$ iff

$$\bar{\mathfrak{g}}(\widehat{\mho}_XY,Z)=\bar{\mathfrak{g}}(\widehat{\mho}_XY,\tau)=\bar{\mathfrak{g}}(\widehat{\mho}_XY,N)=0$$

$\forall\ X,Y\in\Gamma(D^{\perp}),\ Z\in\Gamma(\bar{D}),\ N\in\Gamma(ltr(T\mathbb{M}))$.

Now using (5) and (17),

$$\bar{\mathfrak{g}}(\widehat{\mho}_X Y, \tau) = \frac{1}{2}[-\bar{\mathfrak{g}}(Y, -X + (\varpi(X) + \mu(X))\tau) - \bar{\mathfrak{g}}(Y, -X + (\varpi(X) + \mu(X))\tau)]$$
$$= \bar{\mathfrak{g}}(X, Y)$$

For the choice of non-null vectors $X, Y \in \Gamma(D^{\perp})$, we have $\bar{\mathfrak{g}}(X, Y) \neq 0$. Hence, $D^{\perp}$ does not define a totally geodesic foliation on $\mathbb{M}$.

Also, using (12), (16) and (26), we have

$$\bar{\mathfrak{g}}(\widehat{\mho}_X Y, Z) = \frac{1}{2}[\bar{\mathfrak{g}}(\bar{\mho}_X Y, Z) + \bar{\mathfrak{g}}(\bar{\mho}^*_X Y, Z)]$$
$$= \frac{1}{2}[-\bar{\mathfrak{g}}(\phi Y, \phi\bar{\mho}^*_X Z) - \bar{\mathfrak{g}}(\phi Y, \phi\bar{\mho}_X Z)]$$
$$= -\frac{1}{2}[\bar{\mathfrak{g}}(\phi Y, \bar{\mho}_X \phi Z - \varpi(Z)\phi X + \bar{\mathfrak{g}}(\phi X, Z)\tau) + \bar{\mathfrak{g}}(\phi Y, \bar{\mho}^*_X \phi Z - \varpi(Z)\phi X + \bar{\mathfrak{g}}(\phi X, Z)\tau)]$$
$$= -\frac{1}{2}[\bar{\mathfrak{g}}(\phi Y, \mho_X \phi Z) + \bar{\mathfrak{g}}(\phi Y, \mho^*_X \phi Z)]$$
$$\neq 0 \qquad \forall\, X, Y \in \Gamma(\bar{D}),\ Z \in \Gamma(D^{\perp}).$$

which implies that $\bar{D}$ does not define a totally geodesic foliation on $\mathbb{M}$.

□

Definition 5.8. A contact SCR-lightlike submanifold of indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}$ is said to be mixed totally geodesic contact SCR-lightlike submanifold with respect to $\bar{\mho}$ (respectively $\bar{\mho}^*$) if $h(X, Y) = 0$ (respectively $h^*(X, Y) = 0$) $\forall\, X \in \Gamma(\bar{D} \perp \{\tau\})$ and $\forall\, Y \in \Gamma(D^{\perp})$.

Theorem 5.9. *For an indefinite Kenmotsu statistical manifold $\bar{\mathbb{M}}$, let $\mathbb{M}$ be a contact SCR-lightlike submanifold. Then, $\mathbb{M}$ is mixed totally geodesic iff*

$$\bar{\mathfrak{g}}(D^{*l}(X, F'Y), \phi\zeta) + \bar{\mathfrak{g}}(D^l(X, F'Y), \phi\zeta) = 0$$

$$\bar{\mathfrak{g}}({\mho^*}^s_X \phi Y, C'W) + \bar{\mathfrak{g}}(\mho^s_X \phi Y, C'W) = \bar{\mathfrak{g}}(A^*_{\phi Y} X, B'W) + \bar{\mathfrak{g}}(A_{\phi Y} X, B'W)$$

for all $X \in \Gamma(\bar{D} \perp \{\tau\})$ and $\forall\, Y \in \Gamma(D^{\perp})$.

Proof. Let $\mathbb{M}$ be mixed totally geodesic contact SCR-lightlike submanifold of $\bar{\mathbb{M}}$. Then, we have

$$\begin{aligned} \bar{\mathfrak{g}}(h^l(X, Y), \zeta) &= 0, \qquad \bar{\mathfrak{g}}(h^{*l}(X, Y), \zeta) = 0 \\ \bar{\mathfrak{g}}(h^s(X, Y), W) &= 0 \qquad \bar{\mathfrak{g}}(h^{*s}(X, Y), W) = 0 \end{aligned} \tag{35}$$

for all $X \in \Gamma(\bar{D} \perp \{\tau\})$ and $\forall\, Y \in \Gamma(D^{\perp})$.

Since, $\bar{\mathbb{M}}$ is the indefinite Kenmotsu statistical manifold, therefore using (12) and (27) for all $X \in \Gamma(\bar{D} \perp \{\tau\})$ and $\forall\, Y \in \Gamma(D^{\perp})$, we have

$$\bar{\mathfrak{g}}(h^l(X,Y),\zeta)+\bar{\mathfrak{g}}(h^{*l}(X,Y),\zeta)=\bar{\mathfrak{g}}(\phi\bar{\mho}_XY,\phi\zeta)+\bar{\mathfrak{g}}(\phi\bar{\mho}^*_XY,\phi\zeta)+\varpi(\bar{\mho}_XY)\varpi(\zeta)$$
$$+\varpi(\bar{\mho}^*_XY)\varpi(\zeta)=\bar{\mathfrak{g}}(\bar{\mho}^*_X\phi Y,\phi\zeta)+\bar{\mathfrak{g}}(\bar{\mho}_X\phi Y,\phi\zeta)$$

that is, $\bar{\mathfrak{g}}(h^l(X,Y),\zeta)+\bar{\mathfrak{g}}(h^{*l}(X,Y),\zeta)=\bar{\mathfrak{g}}(D^{*l}(X,F'Y),\phi\zeta)+\bar{\mathfrak{g}}(D^l(X,F'Y),\phi\zeta)$

and $\bar{\mathfrak{g}}(h^l(X,Y),W)+\bar{\mathfrak{g}}(h^{*l}(X,Y),W)=\bar{\mathfrak{g}}(\phi\bar{\mho}_XY,\phi W)+\bar{\mathfrak{g}}(\phi\bar{\mho}^*_XY,\phi W)$
$=\bar{\mathfrak{g}}(\bar{\mho}^*_X\phi Y-\varpi(Y)\phi X+\bar{\mathfrak{g}}(\phi X,Y)\tau,\phi W)+\bar{\mathfrak{g}}(\bar{\mho}_X\phi Y-\varpi(Y)\phi X+\bar{\mathfrak{g}}(\phi X,Y)\tau,\phi W)$

that is, $\bar{\mathfrak{g}}(h^l(X,Y),W)+\bar{\mathfrak{g}}(h^{*l}(X,Y),W)=\bar{\mathfrak{g}}(-A^*_{\phi Y}X,B'W)+\bar{\mathfrak{g}}({\mho^*}^s_X\phi Y,C'W)$

$$+\bar{\mathfrak{g}}(-A_{\phi Y}X,B'W)+\bar{\mathfrak{g}}(\mho^s_X\phi Y,C'W).$$

So, the required result is obtained following (35).

□

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