

Inversion Theorem of Distributional Mellin–Stieltjes Transform

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Abstract

This work presents an inversion and uniqueness theorem for the Mellin–Stieltjes transform within the framework of generalized functions in distributional sense. Let $f(t, x)$ be a generalized function for which the Mellin–Stieltjes transform exists, and let $F(s, y)$ represent its distributional Mellin–Stieltjes transform. The inversion problem is formulated using the testing-function space $MS_\alpha(\Omega)$, its associated dual space and test functions $\phi(t, x) \in D(\Omega)$.

The proof proceeds through a sequence of four supporting lemmas. The first establishes that the truncated double contour integral arising in the inversion process belongs to the testing-function space and converges with respect to the topology of $MS_\alpha(\Omega)$. The second transforms this contour integral into an equivalent representation involving a Dirichlet-type kernel. The third demonstrates that this kernel acts as an approximation to the identity, converging to the original test function as the contour parameter approaches infinity. Building upon these results, the fourth lemma applies the Cauchy integral formula to prove convergence of the integral representation.

Finally, the inversion theorem is obtained by employing the duality between distributions and test functions together with the analyticity of the transform. The proof further relies on the compact support of the test functions, uniform convergence arguments, integration by parts and the justified interchange of repeated integrals. Consequently, the inversion formula is established in the distributional sense, that is, with convergence in $D'(\Omega)$. This theorem provides a mathematically

rigorous framework for reconstructing a generalized function from its distributional Mellin–Stieltjes transform under the prescribed analytical and convergence assumptions.

Keywords: Mellin Transform, Stieltjes Transform, Mellin- Stieltjes Transform, Generalized function.

1. Introduction:

Integral transforms are powerful analytical techniques widely used in the study of differential equations, integral equations, boundary-value problems and generalized functions. Their main advantage is that they transform a function or distribution into another domain where analytical operations become more manageable. The effectiveness of any transform, however, depends not only on its defining properties but also on the existence of a valid inversion formula, which guarantees the unique recovery of the original function or generalized function.

The Mellin transform is especially useful for problems involving scaling properties and power-law kernels, whereas the Stieltjes transform plays an important role in the study of integral equations, moment problems and complex analysis. The Mellin-Stieltjes transform combines these two ideas by incorporating a Mellin-type factor with complex variable s and a Stieltjes-type kernel involving the variable y . This two-variable formulation provides a natural framework for deriving a double inversion formula.

The present paper studies the inversion of the distributional Mellin-Stieltjes transform in the testing-function space denoted by $MS_\alpha(\Omega)$. For a testing function $\phi(t, x) \in D(\Omega)$, an auxiliary function is introduced in the form

$$\psi(s, y) = \int_0^\infty \int_0^\infty \phi(t, x) t^{-s} (x + y)^{p-1} dt dx.$$

The compact support and smoothness of ϕ ensure that the integral is well defined under the assumptions used in the paper. The analyticity of the Mellin-Stieltjes transform and the entire character of the auxiliary function are then used to justify the contour integrations appearing in the inversion procedure.

The proof of the main theorem is organized through four supporting lemmas. The first lemma shows that a truncated double integral involving the inversion kernel defines a function belonging to the testing-function space $MS_\alpha(\Omega)$.

The second lemma transforms the double contour integral into a more convenient expression involving a sine kernel. This reduction is obtained by substituting the definition of $\psi(s, y)$, changing the order of integration, evaluating the inner Stieltjes-type integral and then calculating the Mellin-type contour integral. The compact support of the test function and continuity of the integrand provide the conditions required for these manipulations.

The third lemma establishes an approximation result based on the classical Dirichlet integral $\int_{-\infty}^{\infty} \frac{\sin(rt)}{t} dt = \pi$.

The proof shows that the resulting expression converges to the test function in the topology of $MS_{\alpha}(\Omega)$ as $r \rightarrow \infty$.

The fourth lemma combines the preceding results. It shows that the complete truncated inversion expression converges in $MS_{\alpha}(\Omega)$ to $\phi(u, v)$ as the contour parameters r and τ tend to infinity. The Mellin-type convergence obtained in the third lemma recovers the first variable, while the Cauchy integral formula is used to recover the second variable.

After establishing these lemmas, the inversion theorem is proved by transferring the inversion operation to the testing function. Let $f(t, x)$ be a Mellin-Stieltjes transformable generalized function and let $F(s, y)$ be its distributional Mellin-Stieltjes transform. The derivative denoted by $F_2(s, y)$, taken with respect to the variable y , appears in the inversion expression. By applying the generalized function to the convergent family of testing functions and using continuity of the dual pairing, the limiting expression is shown to converge to f in $D'(\Omega)$.

Thus, the principal purpose of this paper is to provide a rigorous distributional justification for the inversion and uniqueness theorem of the Mellin-Stieltjes transform.

2. Testing function space:

2.1 The space MS_{α} :

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I then the space MS_{α} is given by,

$$\begin{aligned} MS_{\alpha} &= \{\phi: \phi \in E_+ / \gamma_{b,c,p,l,q}[\phi(t, x)] \\ &= \sup_{I_1} |\lambda_{b,c}(t) t^{q+1} (1+x)^p D_t^l (xD_x)^q \phi(t, x)| \\ &\leq C_{lq} A^p p^{p\alpha}\} \quad p = 1, 2, 3, \dots \end{aligned}$$

Where the constant A and C_{lq} depends on the testing function $\phi(t, x)$ and

$$\lambda_{b,c} = \begin{cases} t^{-b} & 0 < t \leq 1 \\ t^{-c} & 1 < t < \infty \end{cases}$$

2.2 The space MS_{α}^{β} :

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I then the space MS_{α} is given by,

$$\begin{aligned} MS_{\alpha}^{\beta} &= \{\phi: \phi \in E_+ / \rho_{b,c,p,l,q}[\phi(t, x)] \\ &= \sup_{I_1} |\lambda_{b,c}(t) t^{q+1} (1+x)^p D_t^l (xD_x)^q \phi(t, x)| \\ &\leq C A^p p^{p\alpha} B^l l^{l\beta}\} \quad p, l = 1, 2, 3, \dots \end{aligned}$$

Where the constant A B and C depends on the testing function $\phi(t, x)$ and

3. Distributional Mellin-Stieltjes transform of generalized function in MS_α^*

For $f(t, x) \in MS_\alpha^*$ where MS_α^* is the dual space of MS_α . It contains all distributions of compact support and let $\phi(t, x) \in MS_\alpha$.

Then the distributional Mellin-Stieltjes transform $F(s, y)$ is defined as

$$\begin{aligned} MS\{f(t, x)\} &= F(s, y) \\ &= \left\langle f(t, x), t^{s-1}(x+y)^{-p} \right\rangle \end{aligned}$$

where for each fixed t ($0 < t < \infty$) and x ($0 < x < \infty$), $s > 0, p > 0$ the right-hand side of above equation has a sense as an application of $f(t, x) \in MS_\alpha^*$ to $t^{s-1}(x+y)^{-p} \in MS_\alpha$

4. Inversion Theorem for Mellin-Stieltjes Transform

Lemma 1:

Let $MS\{f(t, x)\} = F(s, y)$, $|\operatorname{Re} s| < a$ and $\rho_1 < \operatorname{Re} y < \rho_2$. For $t, x \in \Omega$ & $\phi(t, x) \in D(\Omega)$ set,

$$\psi(s, y) = \int_0^\infty \int_0^\infty \phi(t, x) t^{-s} (x+y)^{p-1} dt dx \quad \dots$$

(1)

Then for any pair of real numbers r and τ with $y = \rho + iw$ and $0 < \tau < \infty$,

$$\int_{-r}^r \int_0^\tau \left\langle f(u, v), \frac{p u^{s-1}}{(v+y)^{p+1}} \right\rangle \psi(s, y) ds dw = \left\langle f(u, v), \int_{-r}^r \int_0^\tau \frac{p u^{s-1}}{(v+y)^{p+1}} \psi(s, y) ds dw \right\rangle \quad \dots(2)$$

where $y = \rho + iw$ and s & ρ are fixed real numbers such that $s > 0$ and $\rho_1 < \rho < \rho_2$ respectively.

Proof: For $\phi(t, x) = 0$ on Ω , the proof is trivial. Let $\phi(t, x) \neq 0$ on Ω , since $F(s, y)$ is analytic on Ω and $\psi(s, y)$ is entire, the integral on right hand side of equation (2) exists.

First we shall prove that,

$$G(u, v) = \int_{-r}^r \int_0^\tau \frac{p u^{s-1}}{(v+y)^{p+1}} \psi(s, y) ds dw \quad \dots(3)$$

as a function of (u, v) belong to MS_α . For consider,

$$\begin{aligned} \gamma_{b,c,p,l,q} G(u, v) &= \sup_I \left| \lambda_{b,c}(u) u^{q+1} (1+v)^p D_u^l (vD_v)^q G(u, v) \right| \\ &= \sup_I \left| \lambda_{b,c}(u) u^{q+1} (1+v)^p D_u^l (vD_v)^q \int_{-r}^r \int_0^\tau \frac{p u^{s-1}}{(v+y)^{p+1}} \psi(s, y) ds dw \right| \end{aligned}$$

By the smoothness of the integral we may carry the operator $D_u^l (vD_v)^q$ under the sign of integration. Hence, we get

$$\begin{aligned} \gamma_{b,c,p,l,q} G(u,v) &= \sup_I \left| \lambda_{b,c}(u) u^{q+1} (1+v)^p \int_{-r}^r \int_0^\tau D_u^l (vD_v)^q \frac{p u^{s-1}}{(v+y)^{p+1}} \psi(s,y) ds dw \right| \\ &= \sup_I \left| \int_{-r}^r \int_0^\tau \sum_{j=1}^q \psi(s,y) \lambda_{b,c}(u) u^{q+1} (1+v)^p P(s) u^{s-l-1} v^j (-1)^j P(p) \frac{1}{(v+y)^{p+j+1}} ds dw \right| \end{aligned}$$

Where $P(p)$ and $P(s)$ is polynomial in p and s respectively.

The series on the right-hand side is the series of positive finite terms which is bounded by M_1 (say) for $u > 0$ and $v > 0$. Therefore $\gamma_{b,c,p,l,q} G(u,v) < \infty$, hence $G(u,v)$ belongs to $MS_\alpha(\Omega)$. Therefore, the right-hand side of equation (2) is meaningful.

Partitioning the path of integration on the straight line from $s = a - ir$ to $s = a + ir$ into m intervals, each of length $\frac{2r}{m}$ and $y = \rho - iw$ to $y = \rho + iw$ into $2n$ intervals each of length $\frac{\tau}{n}$. Let $s_i = a + i\sigma$ be any point in the i^{th} interval and $y_j = \rho + iw$ be any point in the j^{th} interval.

Consider,

$$\phi_{m,n}(u,v) = \sum_{i=1}^m \sum_{j=1}^n \psi(s_i, y_j) \frac{p u^{s_i-1}}{(v+y_j)^{p+1}} \frac{2r}{m} \frac{\tau}{n}$$

By applying $f(u,v)$ to above equation term by term, we get

$$\begin{aligned} \langle f(u,v), \phi_{m,n}(u,v) \rangle &= \left\langle f(u,v), \sum_{i=1}^m \sum_{j=1}^n \psi(s_i, y_j) \frac{p u^{s_i-1}}{(v+y_j)^{p+1}} \frac{2r}{m} \frac{\tau}{n} \right\rangle \\ &= \sum_{i=1}^m \sum_{j=1}^n \left\langle f(u,v), \frac{p u^{s_i-1}}{(v+y_j)^{p+1}} \right\rangle \psi(s_i, y_j) \frac{2r}{m} \frac{\tau}{n} \end{aligned}$$

Since $\left\langle f(u,v), \frac{p u^{s-1}}{(v+y)^{p+1}} \right\rangle \psi(s,y)$ is continuous function on $-r < s < r$ and $0 < w < \tau$ the sum on the right-hand side tends to

$$\int_{-r}^r \int_0^\tau \left\langle f(u,v), \frac{p u^{s-1}}{(v+y)^{p+1}} \right\rangle \psi(s,y) ds dw, \quad m, n \rightarrow \infty$$

Next choose a, a', b and b' such that $\sigma_1 < a < \sigma < b < \sigma_2$ and $\rho_1 < a' < \rho < b' < \rho_2$.

Since $f(t, x) \in MS_\alpha^*$ our lemma will be proved when $\phi_{m,n}(u, v)$ converges in MS_α to

$$\int_{-r}^r \int_0^\tau \frac{p u^{s-1}}{(v+y)^{p+1}} \psi(s, y) ds dw \quad \text{as } m, n \rightarrow \infty$$

So, we need merely show that, for each fixed l and q , $|\phi_{m,n}(u, v) - G(u, v)|$ converges uniformly to zero on Ω as $m, n \rightarrow \infty$.

Consider,

$$\begin{aligned} & \gamma_{b,c,p,l,q} [\phi_{m,n}(u, v) - G(u, v)] \\ &= \sup_I \left| \lambda_{b,c}(u) u^{q+1} (1+v)^p D_u^l (v D_v)^q [\phi_{m,n}(u, v) - G(u, v)] \right| \\ &= \sup_I \left| \lambda_{b,c}(u) u^{q+1} (1+v)^p D_u^l (v D_v)^q \left[\sum_{i=1}^m \sum_{j=1}^n \psi(s_i, y_j) \frac{p u^{s_i-1}}{(v+y_j)^{p+1}} \frac{2r \tau}{m n} - \int_{-r}^r \int_0^\tau \frac{p u^{s-1}}{(v+y)^{p+1}} \psi(s, y) ds dw \right] \right| \\ &= \sup_I \left| \lambda_{b,c}(u) u^{q+1} (1+v)^p \sum_{g=1}^q \sum_{i=1}^m \sum_{j=1}^n \psi(s_i, y_j) P(s_i) u^{s_i-l-1} v^g (-1)^g P(p) \frac{1}{(v+y_j)^{p+g+1}} \frac{2r \tau}{m n} - \right. \\ & \quad \left. \lambda_{b,c}(u) u^{q+1} (1+v)^p \int_{-r}^r \int_0^\tau \sum_{g=1}^q \psi(s, y) P(s) u^{s-l-1} v^g (-1)^g \frac{1}{(v+y)^{p+g+1}} ds dw \right| \quad \dots(4) \end{aligned}$$

$$\text{Now } \left| \sum_{g=1}^q \lambda_{b,c}(u) u^{q+1} (1+v)^p u^{s-l-1} v^g \frac{1}{(v+y)^{p+g+1}} \right| \rightarrow 0 \text{ as}$$

$|u| \rightarrow \infty$ and $|v| \rightarrow \infty$ because $s > 0$. So for given $\varepsilon > 0$ we can select $T > 0$ and $X > 0$ as large that for all $|u| > T$ and $|v| > X$

$$\left| \sum_{g=1}^q \lambda_{b,c}(u) u^{q+1} (1+v)^p u^{s-l-1} v^g \frac{1}{(v+y)^{p+g+1}} \right| < \frac{\varepsilon}{3} \left[\int_{-r}^r \int_0^\tau \left| \sum_{g=1}^q \psi(s, y) P(s) (-1)^g P(p) \right| ds dw \right]^{-1}$$

Since $\phi(t, x) \neq 0$, the right-hand side is finite. Now for all $|u| > T$ and $|v| > X$, the magnitude of the second term on the right side of equation (4) is bounded by $\frac{\varepsilon}{3}$.

Moreover, again for $|u| > T$ and $|v| > X$, the magnitude of the first term on right side of equation (4) is bounded by,

$$\frac{\varepsilon}{3} \left[\int_{-r}^r \int_0^\tau \left| \sum_{g=1}^q \psi(s, y) P(s) (-1)^g P(p) \right| ds dw \right]^{-1} \sum_{g=1}^q \sum_{i=1}^m \sum_{j=1}^n |\psi(s_i, y_j) P(s_i) (-1)^g P(p)| \frac{2r \tau}{m n}$$

We can choose m_0, n_0 so large that for all $m > m_0$ and $n > n_0$ the last expression is less than $\frac{2\varepsilon}{3}$. Therefore, for all $|u| > T$ and $|v| > X$ and all $m > m_0$ and $n > n_0$

$$\gamma_{b,c,p,l,q} [\phi_{m,n}(u, v) - G(u, v)] < \varepsilon$$

Finally,

$$\lambda_{b,c}(u) u^{q+1} (1+v)^p \sum_{g=1}^q \psi(s, y) P(s) P(p) u^{s-l-1} v^g (-1)^g \frac{1}{(v+y)^{p+g+1}}$$

is a uniformly continuous function of (t, x, s, w) on the domain $-T \leq u \leq T, 0 \leq v \leq X$, $-r < s \leq r$ and $0 \leq w \leq \tau$. Then in the view of equation (4) there exist m_1 and n_1 such that $m > m_1$ and $n > n_1$,

$$\gamma_{b,c,p,l,q} [\phi_{m,n}(u, v) - G(u, v)] < \varepsilon \text{ on } -T \leq u \leq T, 0 \leq v \leq X \text{ as well.}$$

Thus when $m > \max(m_0, m_1)$ and $n > \max(n_0, n_1)$, $\gamma_{b,c,p,l,q} [\phi_{m,n}(u, v) - G(u, v)] < \varepsilon$ on $0 < u < \infty$ and $0 < v < \infty$ which gives the result.

Lemma 2:

For $\phi \in D(\Omega)$, set $\psi(s, y)$ as in lemma 1 then

$$\frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \psi(s, y) \frac{p u^{s-1}}{(v+y)^{p+1}} dw ds = \frac{-i}{2\pi^2} \int_0^\infty \int_{-\infty}^\infty \phi(ue^{-t}, x) \frac{1}{(v-x)} e^{(a-1)t} \frac{\sin rt}{t} dt dx$$

Proof: We shall prove the result by justifying the steps in the following manipulations and by considering compact support of $\phi(t, x) \in D(\Omega)$

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \psi(s, y) \frac{p u^{s-1}}{(v+y)^{p+1}} dw ds \\ &= \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \frac{p u^{s-1}}{(v+y)^{p+1}} \left[\int_0^\infty \int_0^\infty \phi(t, x) t^{-s} (x+y)^{p-1} dt dx \right] dw ds \\ &= \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\infty \int_0^\infty \phi(t, x) t^{-s} p u^{s-1} \left[\int_0^\tau \frac{(x+y)^{p-1}}{(v+y)^{p+1}} dy \right] dt dx ds \end{aligned}$$

By using the proper substitution and as $v \rightarrow \infty$ the bracket from the right-hand side

tends to $\frac{1}{p(v-x)}$. Hence,

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \psi(s, y) \frac{p u^{s-1}}{(v+y)^{p+1}} dw ds \\ &= \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\infty \int_0^\infty \phi(t, x) \frac{1}{u} \left(\frac{u}{t}\right)^s \frac{p}{p(v-x)} dt dx ds \\ &= \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty \phi(t, x) \frac{1}{(v-x)} \left[\frac{1}{u} \int_{a-ir}^{a+ir} \left(\frac{u}{t}\right)^s ds \right] dt dx \end{aligned}$$

Now,

$$\left[\frac{1}{u} \int_{a-ir}^{a+ir} \left(\frac{u}{t}\right)^s ds \right] = 2i \left(\frac{u}{t}\right)^a \frac{\sin(r \log u / t)}{u \log(u / t)}$$

Hence,

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \psi(s, y) \frac{p u^{s-1}}{(v+y)^{p+1}} dw ds \\ &= \frac{i}{2\pi^2} \int_0^\infty \int_0^\infty \phi(t, x) \frac{1}{(v-x)} \left(\frac{u}{t}\right)^{a-1} \frac{\sin(r \log u / t)}{t \log(u / t)} dt dx \\ &= \frac{i}{2\pi^2} \int_0^\infty \int_{-\infty}^\infty \phi(ue^{-t}, x) \frac{1}{(v-x)} e^{(a-1)t} \frac{\sin rt}{t} dt dx \end{aligned}$$

Lemma 3:

Let r be real number with $a < s < b$. Also let $\phi(t, x) \in D(\Omega)$, then

$$\frac{1}{\pi} \int_{-\infty}^\infty \phi(ue^{-t}, x) e^{(a-1)t} \frac{\sin rt}{t} dt \text{ converges in } MS_\alpha \text{ to } \phi(u, x) \text{ as } r \rightarrow \infty$$

Proof: In the following assume that $r > 0$. It is known that,

$$\int_{-\infty}^\infty \frac{\sin rt}{t} dt = \pi \quad \dots(5)$$

Thus, our objective is to prove that for each $k=0,1,2,3,\dots$ $B_r(u, x)$ converges uniformly to zero on $-\infty < t < \infty$ and $r \rightarrow \infty$, where,

$$\begin{aligned} B_r(u, x) &= \frac{1}{\pi} \lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q \int_{-\infty}^{\infty} [\phi(ue^{-t}, x) - \phi(t, x)] e^{(a-1)t} \frac{\sin rt}{t} dt \\ &= \frac{1}{\pi} \left\{ \lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q \int_{-\infty}^{\infty} \phi(ue^{-t}, x) e^{(a-1)t} \frac{\sin rt}{t} dt \right. \\ &\quad \left. - \lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q \int_{-\infty}^{\infty} \phi(t, x) e^{(a-1)t} \frac{\sin rt}{t} dt \right\} \\ &= \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} H(t, u, x) \sin rt dt \right\} \end{aligned}$$

Where

$$H(t, u, x) = \frac{\lambda_{b,c}(u) u^{q+1} (1+x)^p e^{(a-1)t}}{t} \left[D_u^l(xD_x)^q \phi(ue^{-t}, x) - D_u^l(xD_x)^q \phi(t, x) \right] \quad \dots(6)$$

Therefore,

$$\begin{aligned} B_r(u, x) &= \frac{1}{\pi} \left\{ \int_{-\infty}^{-\delta} H(t, u, x) \sin rt dt + \int_{-\delta}^{\delta} H(t, u, x) \sin rt dt + \int_{\delta}^{\infty} H(t, u, x) \sin rt dt \right\} \\ &= \frac{1}{\pi} \int_{-\infty}^{-\delta} H(t, u, x) \sin rt dt + \frac{1}{\pi} \int_{-\delta}^{\delta} H(t, u, x) \sin rt dt + \frac{1}{\pi} \int_{\delta}^{\infty} H(t, u, x) \sin rt dt \\ &= I_{1,r}(u, x) + I_{2,r}(u, x) + I_{3,r}(u, x) \end{aligned}$$

where $I_{1,r}(u, x)$, $I_{2,r}(u, x)$ and $I_{3,r}(u, x)$ denotes the quantities obtained by integrating over $-\infty < t < -\delta$, $-\delta < t < \delta$ and $\delta < t < \infty$ respectively where $\delta > 0$

First consider $I_{2,r}(u, x)$: The function $H(t, u, x)$ is a continuous function of (t, u, x) for all u and $t \neq 0$. Moreover since ϕ is smooth, equation (6) tends to,

$$\lambda_{b,c}(u) u^{q+1} (1+x)^p D_t \left[\frac{e^{(a-1)t}}{t} D_u^l(xD_x)^q \phi(ue^{-t}, x) \right]_{t=0} \quad \dots(7)$$

as $t \rightarrow 0$. Upon assigning the value from above equation (7) to $H(0, u, x)$ we obtain a function $H(t, u, x)$ which is continuous everywhere. Since ϕ is of bounded support,

$H(t, u, x)$ is bounded on the domain $\{(t, u, x) : -\delta < t < \delta, -\infty < u < \infty, 0 < x < \infty\}$ by the constant M (say). Thus, for given $\varepsilon > 0$, we can choose δ so small that,

$$|I_{2,r}(u, x)| = \frac{1}{\pi} \left| \int_{-\delta}^{\delta} H(t, u, x) \sin rt \, dt \right| \leq \frac{2M\delta}{\pi} > \varepsilon,$$

for $-\infty < u < \infty$ and $0 < x < \infty$. Fix δ in this way.

Next consider $I_{1,r}(u, x)$: set

$$I_{1,r}(u, x) = J_{1,r}(u, x) - J_{2,r}(u, x)$$

where,

$$J_{1,r}(u, x) = \frac{1}{\pi} \int_{-\infty}^{-\delta} \lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q \phi(ue^{-t}, x) e^{(a-1)t} \frac{\sin rt}{t} \, dt$$

$$J_{2,r}(u, x) = \frac{1}{\pi} \lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q \phi(u, x) \int_{-\infty}^{-r\delta} e^{(a-1)z} \frac{\sin z}{z} \, dz$$

Since $\lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q \phi(u, x)$ is continuous and bounded support, it is bounded on $-\infty < u < \infty$. By the convergence of the improper integral $\int_{-\infty}^0 \frac{\sin z}{z} \, dz$ it follows that $J_{2,r}(u, x)$ tends uniformly to zero $-\infty < u < \infty$, $0 < x < \infty$ and $r \rightarrow \infty$.

To show that $J_{1,r}(u, x)$ does the same, first integrate by parts and use the fact that $\phi(u, x)$ is of bounded support to obtain

$$J_{1,r}(u, x) = \frac{1}{\pi} \lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q e^{-(a-1)\delta} \phi(ue^{-\delta}, x) \frac{\cos r\delta}{r\delta} \\ + \frac{1}{\pi r} \int_{-\infty}^{-\delta} \cos(rt) \cdot D_t \left[\frac{1}{t} \lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q \phi(ue^{-t}, x) e^{(a-1)t} \right] dt$$

The first term on the right-hand side tends uniformly to zero on $-\infty < u < \infty$, $0 < x < \infty$ and $r \rightarrow \infty$ because δ is fixed and

$\lambda_{b,c}(u) u^{q+1} (1+x)^p D_u^l(xD_x)^q e^{-(a-1)\delta} \phi(ue^{-\delta}, x)$ is a bounded function of u . Moreover, by using the product rule of differentiation with respect to t for equation (7)

$$\begin{aligned}
\lambda_{b,c}(u)u^{q+1}(1+x)^p D_t \left[\frac{1}{t} D_u^l (xD_x)^q \phi(ue^{-t}, x) e^{(a-1)t} \right] &= \lambda_{b,c}(u)u^{q+1}(1+x)^p D_u^l (xD_x)^q \phi(ue^{-t}, x) e^{(a-1)t} \left(\frac{-1}{t^2} \right) \\
&\quad - \frac{1}{t} \lambda_{b,c}(u)u^{q+1}(1+x)^p (xD_x)^q u D_u^{l+1} \phi(ue^{-t}, x) e^{(a-1)t} \\
&\quad + \frac{1}{t} \lambda_{b,c}(u)u^{q+1}(1+x)^p D_u^l (xD_x)^q \phi(ue^{-t}, x) (a-1) e^{(a-1)t} \\
\lambda_{b,c}(u)u^{q+1}(1+x)^p D_t \left[\frac{1}{t} D_u^l (xD_x)^q \phi(ue^{-t}, x) e^{(a-1)t} \right] &= \lambda_{b,c}(u)u^{q+1}(1+x)^p D_u^l (xD_x)^q \phi(ue^{-t}, x) e^{(a-1)t} \left(\frac{-1}{t^2} \right) \\
&\quad + \frac{e^{(a-1)t}}{t} \lambda_{b,c}(u)u^{q+1}(1+x)^p D_u^l (xD_x)^q [(a-1)\phi - u D_u \phi] \\
&\quad \dots(8)
\end{aligned}$$

But for every q $\lambda_{b,c}(u)u^{q+1}(1+x)^p D_u^l (xD_x)^q \phi(ue^{-t}, x) e^{(a-1)t}$ is bounded. This is because $D_u^l (xD_x)^q \phi(ue^{-t}, x)$ is bounded and has its support contained in the strip $\{(t, u, x) : |t, u, x| < A\}$ where A is sufficiently large number whereas $\lambda_{b,c}(u)u^{q+1}(1+x)^p e^{(a-1)t}$ is a bounded on the strip. Thus equation (8) is bounded on the domain. $\{(t, u, x) : -\infty < t < -\delta, -\infty < u < \infty, 0 < x < \infty\}$ by $N(\text{say})$. This result and assumption that the support of $\phi(u, x)$ is contained in the interval $-A < u < A$ implies that the second term on the right-hand side of equation (8) is bounded by $\frac{2NA}{\pi}$, which tends to zero as $r \rightarrow \infty$. Therefore, $I_{1,r}(u, x)$ tends uniformly to zero on $-\infty < u < \infty, 0 < x < \infty$ as $r \rightarrow \infty$. A similar argument shows that, $I_{3,r}(u, x)$ also tends uniformly to zero on $-\infty < u < \infty, 0 < x < \infty$ as $r \rightarrow \infty$. Thus, we have established that $\lim_{r \rightarrow \infty} |B_r(u, x)| \leq \varepsilon$ and since $\varepsilon > 0$ is arbitrary, implying that $B_r(u, x)$ converges uniformly to zero, for every $k=0,1,2,3 \dots \dots \dots -\infty < u < \infty, 0 < x < \infty$ by using equation (5).

Lemma 4:

For $\phi(t, x) \in D(\Omega)$, set $\psi(s, y)$ as in lemma 1, then

$$\frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \psi(s, y) \frac{p u^{s-1}}{(v+y)^{p+1}} dw ds \quad \dots(9)$$

Converges in $MS_\alpha(\Omega)$ to $\phi(u, v)$ as $r, \tau \rightarrow \infty$

Proof: Set $G(u, v)$ as in lemma 1. In lemma 1 we have proved that $\gamma_{b,c,p,l,q} G(u, v) < \infty$, hence $G(u, v)$ converges in $MS_\alpha(\Omega)$. Therefore, $\gamma_{b,c,p,l,q} |(4\pi^2)^{-1} G(u, v)| < \infty$ and $(4\pi^2)^{-1} G(u, v)$ converges in $MS_\alpha(\Omega)$.

Hence to prove the lemma we have to prove that equation (9) converges to

$\phi(u, v)$. Equation (9) can be written by using equation (1) as

$$\frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \frac{p u^{s-1}}{(v+y)^{p+1}} \left[\int_0^\infty \int_0^\infty \phi(t, x) t^{-s} (x+y)^{p-1} dt dx \right] dw ds$$

The order of integration for the repeated integral herein may be changed because $\phi(t, x)$

is of bounded support and the integrand is continuous function of (t, x, s, w) and then by using lemma 2 the integral reduces to,

$$\frac{i}{2\pi^2} \int_0^\infty \int_{-\infty}^\infty \phi(ue^{-t}, x) \frac{1}{(v-x)} e^{(a-1)t} \frac{\sin rt}{t} dt dx$$

By changing the order of integration, we get

$$\frac{-1}{2\pi i} \int_0^\infty \left[\pi^{-1} \int_{-\infty}^\infty \phi(ue^{-t}, x) e^{(a-1)t} \frac{\sin rt}{t} dt \right] \frac{1}{(v-x)} dx$$

In view of lemma 3, the term in the bracket converges in $MS_\alpha(\Omega)$ to $\phi(u, x)$ as $r \rightarrow \infty$. Therefore, above equation tends to

$$\frac{-1}{2\pi i} \int_0^\infty \phi(u, x) \frac{1}{(v-x)} dx$$

Now, by using well known Cauchy integral formula, the last expression tends to $\phi(u, v)$ as $v \rightarrow \infty$.

We are now ready to prove inversion theorem.

Inversion Theorem:

Let $f(t, x)$ be a Mellin-Stieltjes transformable generalized function and $F(s, y)$ is the distributional Mellin-Stieltjes transform of $f(t, x)$ then in sense of convergence in

$$D'(\Omega), f(t, x) = \lim_{r, \tau \rightarrow \infty} \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^{\rho+iw} F_2(s, y) t^{-s} (x+y)^{p-1} dy ds$$

where $y = \rho + iw$, s and ρ are any fixed real numbers with $-r < s < r$ and $\rho_1 < \rho < \rho_2$ and F_2 is the partial derivative of $F(s, y)$ with respect to y .

Proof: The idea off this proof is to transfer the inversion formula onto transform of $\phi(t, x) \in D(\Omega)$ and to use the fact that the resulting expression converges to ϕ with respect to the topology of the testing function space MS_α .

Let $\phi(t, x) \in D(\Omega)$ and we have to show that,

$$\lim_{r, \tau \rightarrow \infty} \left\langle \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^{\rho+iw} F_2(s, y) t^{-s} (x+y)^{p-1} dy ds, \phi(t, x) \right\rangle = \langle f, \phi \rangle$$

From the analyticity of $F(s, y)$ on Ω , the above integral is continuous function of (t, x)

Therefore, the left-hand side without the notation can be written as,

$$\int_0^\infty \int_0^\infty \phi(t, x) \left[\frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^{\rho+iw} F_2(s, y) t^{-s} (x+y)^{p-1} dy ds \right] dt dx$$

Considering $y = \rho + iw$

$$\frac{1}{4\pi^2} \int_0^\infty \int_0^\infty \phi(t, x) \int_{a-ir}^{a+ir} \int_0^\tau F_2(s, y) t^{-s} (x+y)^{p-1} dw ds dt dx$$

Since $\phi(t, x)$ has a compact support and integrand is a continuous function of (t, x, s, w) , the order of integration may be changed to yield,

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau F_2(s, y) \int_0^\infty \int_0^\infty \phi(t, x) t^{-s} (x+y)^{p-1} dw ds dt dx \\ &= \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \left\langle f(u, v), \frac{p u^{s-1}}{(v+y)^{p+1}} \right\rangle \int_0^\infty \int_0^\infty \phi(t, x) t^{-s} (x+y)^{p-1} dw ds dt dx \\ &= \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \left\langle f(u, v), \frac{p u^{s-1}}{(v+y)^{p+1}} \right\rangle \psi(s, y) dw ds && \dots \text{by using equation (1)} \\ &= \left\langle f(u, v), \frac{1}{4\pi^2} \int_{a-ir}^{a+ir} \int_0^\tau \psi(s, y) \frac{p u^{s-1}}{(v+y)^{p+1}} dw ds \right\rangle && \dots \text{by lemma 1} \\ &= \langle f(u, v), \phi(u, v) \rangle && \dots \text{by lemma 4} \end{aligned}$$

As $r, \tau \rightarrow \infty$ which completes the proof of the inversion theorem.

5. Uniqueness Theorem

In the theory of integral transforms, the uniqueness theorem is of special importance because of this theorem the question of recovery of a function is possible from its transform. In the next theorem we have shown that if two functions have the same Mellin-Stieltjes transform then they coincide almost everywhere.

Theorem:

If $MS\{f(t, x)\} = F(s, y)$ for $t, x \in \Omega_f$ and $MS\{g(t, x)\} = G(s, y)$ for $t, x \in \Omega_g$

$|\operatorname{Res}| < a$ and $\rho_1 < \operatorname{Re} y < \rho_2$. If $\Omega_f \cap \Omega_g$ is not empty and if $F(s, y) = G(s, y)$, for

s and $y \in \Omega_f \cap \Omega_g$ then $f = g$ in the sense of equality in $D^*(\Omega)$.

Proof: f and g must assign the same value to each $\phi \in D$. By inversion theorem and equating $F(s, y)$ and $G(s, y)$ in

$$\langle f(t, x) - g(t, x), \phi(t, x) \rangle = \lim_{\substack{r \rightarrow \infty \\ t \rightarrow \infty}} \langle \frac{1}{4\pi^2} \int_0^r \int_0^\tau \{F(s, y) - G(s, y)\} t^{-s} (x + y)^{p-1} dy ds, \phi(t, x) \rangle = 0$$

Thus $f = g$ in $D^*(\Omega)$.

6. Conclusions

This paper establishes an inversion and uniqueness theorem for the distributional Mellin-Stieltjes transform in the framework of generalized functions. Using the testing-function space $MS_\alpha(\Omega)$ and its dual, the original generalized function is recovered in the distributional sense. The proof relies on contour integration, the Dirichlet integral and the Cauchy integral formula to justify the inversion process. The result provides a rigorous theoretical basis for the inversion and uniqueness of the Mellin-Stieltjes transform in generalized-function spaces.

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