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A Computational Procedure for Solving Singular Perturbation Problems Arising in Control System using Shooting Method

K. Selvakumar

Department of Mathematics, Anna University of Technology Tirunelveli,
Tirunelveli-627 007, Tamil Nadu, India.
E-mail: k_selvakumar10@yahoo.com

Abstract

This paper presents a new computational procedure for solving singular perturbation problems arising in control system using shooting method. The convergence of the method is given and the numerical results are presented to illustrate the computational procedure. The computational procedure solves exactly the test problems for small values of the singular perturbation parameter, and it is presented in Tables 1 and 2. No matrix inversion is needed for the computation. Numerical methods of order one are applied to solve the given problem and obtained very good results. Higher order numerical methods can be applied to get higher order convergence. This computational procedure can also be applied to solve partial differential equations.

Keywords: singular perturbation problems, shooting method, exponentially fitted, uniformly convergent, zeroth order, asymptotic expansion, finite difference schemes.

AMS (MOS) subject classification: 65F05, 65N30, 65N35, 65Y05.

Introduction

Consider the following singular perturbation problem arising in control system[1,2,4,5]

$$Lu(t) \equiv \varepsilon u''(t) + a(t)\,u'(t) - b(t)\,u(t) = f(t), \quad 0 < t < 1, \tag{1a}$$

$$B_0\, u(0) \equiv u(0) = \phi_1 \quad B_1\, U(1) \equiv u(1) = \phi_2 \tag{1b}$$

where $\varepsilon > 0$ is a small parameter, a, b and f are smooth functions satisfying $a(t) \geq$

$\alpha > 0$ and $b(t) \geq 0$ with respect to t. The solution of the problem do not converges uniformly to the solution $u_0(t)$ of the reduced problem of (1a,b)

$$a(t)\ u_0'(t) - b(t)\ u_0(t) = f(t),\ 0 < t < 1, \quad (2a)$$

$$u_0(1) = \phi_2 \quad (2b)$$

as ε goes to zero at t = 0. Because of this reason, classical schemes may not produce good approximations to the solution u(t) especially when ε is small. In this paper a good approximation to the problem (1a,b) is presented, The maximum principle and the stability result [2,4] for the solution of the problem (1a,b) are given in the following lemmas, which is needed for error estimation.

Lemma 1
Let v be a smooth function satisfying

$Lv(t) \leq 0,\ 0 \leq t \leq 1,\ B_0\ v(0) \geq 0$ and $B_1\ v(1) \geq 0$. Then

$v(t) \geq 0$ for all $0 \leq t \leq 1$.

Lemma 2
Let v be a smooth function. Then we have the following uniform stability estimate

$$|\ v(t)\ | \leq C\ (|B_0\ v(0)\ | + |B_1\ v(1)\ | + \max |L\ v(t)\ |)$$

for all t in [0,1].

The main aim of the paper is to solve the problem (1a,b) using shooting method. Motivated by the works in [3,7] the given problem (1a,b) can be replaced by an initial value problem(IVP) then the resultant IVP is solved numerically using exponentially fitted finite difference schemes given in [2,4].

Schemes for Initial Value Problems
In this section, two finite difference schemes of order one with variable and constant fitted factors for the IVP

$$u_1'\ t) - u_2(t) = 0\ ,\ u_1(0) = \varphi_1\ , \quad (3a)$$

$$\square\ u_2'(t) + a(t)\ u_2(t) - b(t)\ u_1(t) = f(t)\ .\ u_2(0) = \varphi_3 \quad (3b)$$

where $u_1(t) = u(t)$ and $u_2(t) = u'(t)$ are given.

The schemes with variable fitting factor is

$$D_+\ u_{1,i} - u_{2,i} = ((u_{1R}(t_{i+1}) - u_{1R}(t_i))/h) - u_{2R}(t_i)\ , \quad (4a)$$

$$\varepsilon\,\sigma_i\ (-\rho)\ D_+\ u_{2,i} + a(t_i)\ u_{2,i} - b(t_i)\ u_{1,i} = f_i^{\ h_{ii}}\ , \quad (4b)$$

$$u1_{1,0} = \varphi_1\ ,\ u_{2,0} = \varphi_3 \quad (4c)$$

where

$$\sigma_i(-\rho) = \rho\, a(t_i) / [\, 1 - \exp(-\rho\, a(t_i))\,] \;, h = t_{i+1} - t_i \,, \rho = h/\varepsilon, \tag{4d}$$

$$f_i^{\,h} = a(t_i)\; u_{2,R}(t_{i+1}) - b(t_i)\, u_{1R}(t_i)\,, \tag{4e}$$

and $u_{1R}(t)$ and $u_{2R}(t)$ are the solutions of the reduced problem of (3a,b)

$$u_{1R}'(t) - u_{2R}(t) = 0\,, u_{1R}(0) = \varphi_1\,, \tag{5a}$$

$$a(t)\; u_{2R}(t) - b(t)\; u_{1R}(t) = f(t)\,. \tag{5b}$$

To reduce the computational time of the scheme (4a-e), a scheme with constant fitting factor is defined as follows:

$$D_+ u_{1,i} - u_{2,i} = ((u_{1R}(t_{i+1}) - u_{1R}(t_i))/h) - u_{2R}(t_i), \tag{6a}$$

$$\varepsilon\, \sigma(\rho)\; D_+ u_{2,i} + a(t_i)\, u_{2,i+1} - b(t_i)\, u_{1,i} = f_i^{\,h}\;, \tag{6b}$$

$$u_{1,0} = \varphi_1\,, u_{2,0} = \varphi_3\,, \tag{6c}$$

where

$$\sigma(\rho) = \rho a(0) / [\, \exp\{\rho\; a(0)) - 1\,]\;, \rho = h/\varepsilon \tag{6d}$$

and $u_{1R}(t)$, $u_{2R}(t)$ and $f_i^{\,h}$ are defined as in (5a,b) and (4e).

Above schemes are proposed in [4] for the IVP(3a,b). These schemes are consistent, stable and uniformly convergent of order one. And reflect the asymptotic properties of the solution of (3a,b) as ε goes to zero. The application of the schemes (4a-e) and (6a-d) are given in the next section.

Description of the Method

In this section a new computational procedure to solve the problem (1a,b) is presented using shooting method motivated by the works of [3,7]. For the problem (1a,b) we make an initial guess φ_3 for $u'(0)$ from the zeroth order asymptotic expansion for the solution of the problem (1a,b) [2]:

$$u(t, \varepsilon) = u_0(t) + [\, \varphi_1 - u_0(0)\,]\exp(-a(0)\; t/\varepsilon) \tag{7}$$

where $u_0(t)$ is the solution of the problem(2a,b). That is,

$$u'(t_0) = [\, u(t_1, \varepsilon) - u(t_0, \varepsilon)\,] / (t_1 - t_0)$$

where $t_1 \neq t_0$, $t_0 = 0$ and t_1 is a point in the neighborhood of $t = t_0$. Now we denote $u(t, \varphi_3)$ as the solution of the IVP[4]

$$\varepsilon\, u''(t) + a(t)\, u'(t) - b(t)u(t) = f(t), \;\; 0 < t < 1, \tag{8a}$$

$$u(0) = \varphi_1\;, u'(0) = \varphi_3 \tag{8b}$$

such that [3, 7]

$$u(1, \varphi_3) - \varphi_2 = O(\varepsilon) \qquad (9)$$

as t_1 goes to t_0. As ε goes to zero and t_1 goes to $t_0 = 0$, $u(1, \varphi_3) - \varphi_2$ will tend to zero . The IVP (8a,b) can be reduced into a system of IVP for first order ordinary differential equations in which the first equation do not contain the parameter ε and the second equation contain a small parameter ε multiplied at the first derivative,

$$u_1'(t) - u_2(t) = 0, \qquad (10a)$$

$$\varepsilon u_2'(t) + a(t)\, u_2(t) - b(t)\, u_1(t) = f(t),\ 0,< t < 1 , \qquad (10b)$$

$$u_1(0) = \varphi_1 ,\ u_2(0) = \varphi_3 \qquad (10c)$$

where $u_1(t) = u(t)$ and $u_2(t) = u'(t)$. Here the non –uniformity occurs only at the derivative but not in the solution. Now the IVP (10a-c) can be solved numerically using the scheme (4a-e). The finite difference scheme for (10a-c) is

$$D_+ u_{1,i} - u_{2,i} = ((u_{1R}(t_{i+1}) - u_{1R}(t_i))/h) - u_{2R}(t_i) , \qquad (11a)$$

$$\varepsilon\sigma_i(-\rho)\, D_+ u_{2,i} + a(t_i)\, u_{2,i} - b(t_i)\, u_{1,i} = f_i^{\,h} , \qquad (11b)$$

where $\sigma_i(-\rho)$, $u_{1R}(t)$, $u_{2R}(t)$ and $f_i^{\,h}$ are defined as in (4d), (5a-b) and (4e). Hence the given problem (1a,b) can be solved numerically using shooting method with the help of one-step methods without matrix inversion.

In the following the convergence of the solution $u(t, \varphi_3)$ of the IVP (8a,b) to the solution $u(t)$ of the problem (1a,b) is given.

Theorem 1

Let $u(t)$ and $u(t, \varphi_3)$ are the solutions of the problems (1a,b) and (8a,b) respectively. Then, for all t in [0,1] we have

$$| u(t) - u(t, \varphi_3) | \leq C\varepsilon . \qquad (12)$$

Proof. The proof of the theorem follows from the stability result (1.2.Lemma 2). From (1a, b) and (8a, b), we have,

$$L[u(t) - u(t, \varphi_3)] = Lu(t) - Lu(t, \varphi_3) = f(t) - f(t) = 0,$$

$$B_0[u(0) - u(0, \varphi_3)] = B_0 u(0) - B_0 u(0,\varphi_3) = \varphi_1 - u(0, \varphi_3) = \varphi_1 - \varphi_1 = 0$$

and

$$B_1[u(1) - u(1, \varphi_3)] = B_1 u(1) - B_1 u(1, \varphi_3) = \varphi_2 - u(1, \varphi_3).$$

From (9) we have

$$| B_1[u(1) - u(1, \varphi_3)] | \leq | \varphi_2 - u(1, \varphi_3) | \leq C\varepsilon.$$

Now, using stability result, we have

$$| u(t) - u(t, \varphi_3) | \le C\, \varepsilon .$$

Note.
When ε goes to zero, the solution $u(t, \varphi_3)$ of (8a,b) converges to the solution $u(t)$ of (1a,b).

The following theorem gives the main results of this section, that is, the convergence of the computational method.

Theorem 2
Let $u(t)$ and u_i be the solutions of (1a,b) and (11a-b) respectively. Then we have,

$$| u(t_i) - u_i | \le C(\varepsilon + h).$$

Proof. From [4] we have the estimate for the solution of the scheme (11a-b) as

$$| u(t_i, \varphi_3) - u_i | \le C\, h.$$

From Theorem 1 we have

$$| u(t_i) - u(t_i, \varphi_3) | \le C\, \varepsilon .$$

Therefore

$$| u(t_i) - u_i | \le | u(t_i) - u(t_i, \varphi_3) | + | u(t_i, \varphi_3) - u_i |$$
$$\le C\varepsilon + C h \le C(\varepsilon + h).$$

Note
When the scheme (6a-d) is applied to the IVP (10a-c) the computational time will be reduced at each nodal points. From [4] we have

$$| u(t_i, \varphi_3) - u_i | \le C h$$

where $u(t_i, \varphi_3)$ is the solution of the IVP (8a,b) and u_i is the solution of the scheme (6a-d). Therefore, using the estimate (12)

$$| u(t_i) - u_i | \le | u(t_i) - u(t_i, \varphi_3) | + | u(t_i, \varphi_3) - u_i |$$
$$\le C\varepsilon + C h \le C(\varepsilon + h).$$

Test Examples and Numerical Results

Example 1. Consider the following homogenous problem which arises in control system

$$\varepsilon u''(t) + u'(t) = 0, \; 0 < t < 1, \qquad (13a)$$

$$u(0) = \square_1 = 1, u(1) = \square_2 = 2 . \quad (13b)$$

The initial guess φ_3 for u'(0) is given by

$$\varphi_3 = u'(0) = (\varphi_2 - \varphi_1) [1 - \exp(-t_1 / \varepsilon)] / t_1$$

where t_1 is a point in the neighborhood of $t_0 = 0$ from the zeroth order asymptotic expansion for the solution of (13a,b)

$$u(t,\varepsilon) = \varphi_2 + (\varphi_1 - \varphi_2) \exp(-t / \varepsilon).$$

The corresponding IVP is

$$\varepsilon\, u''(t) + u'(t) = 0, 0 < t < 1, \quad (14a)$$

$$u(0) = \varphi_1, u'(0) = \varphi_3 \quad (14b)$$

whose solution is

$$u(t, \varphi_3) = \varphi_1 + \varepsilon \varphi_3 [1 - \exp(-t / \varepsilon)]$$

such that

$$u(1, \varphi_3) - \varphi_2 = (\varphi_2 - \varphi_1) (((1/ \sigma(-t_1 /\varepsilon))[1 - \exp(-1/\varepsilon)]) - 1)$$

where

$$\sigma(-t_1 / \varepsilon) = (t_1 /\varepsilon)/ [1-\exp(-t_1 / \varepsilon)].$$

As t_1 goes to zero,

$\sigma(-t_1 / \varepsilon)$ goes to one and so

$$u(1, \varphi_3) - \varphi_2 = (\varphi_1 - \varphi_2) \exp(-1/\varepsilon).$$

Again as ε goes to zero , we have,

$$u(1, \varphi_3) - \varphi_2 = 0.$$

The IVP (14a, b) can be reduced into a system of the form

$$u_1'(t) - u_2(t) = 0, u_1(0) = \varphi_1, \quad (15a)$$

$$\varepsilon u_2'(t) + u_2(t) = 0, u_2(0) = \varphi_3, \quad (15b)$$

where $u_1(t) = u(t)$ and $u_2(t) = u'(t)$. The IVP (15a , b) can be solved numerically using the schemes given in section 2. It is observed that the scheme (4a-e) reduces to the scheme (6a-d) when applied to the IVP (13a, b).

Numerical results are given in Table 1. It is observed that the computational procedure solves exactly the sample problem (13a,b) even for large values of h =

1/16 and small value of $\varepsilon = 10^{-4}$. The shooting error , $u(1, \varphi_3) - \varphi_2 = 0$, for large step size h = 1/16 and small value of $\varepsilon = 10^{-4}$. Similarly for $\varepsilon = 10^{-5}$ the shooting error is zero for large step size h = 1/16.

Example 2.. Consider the following homogeneous problem with variable coefficient which arises in control system

$$\varepsilon u''(t) + (1+t) u't) = 0 , 0 < t < 1,$$
$$u(0) = 1, u(1) = 2.$$

The numerical results are given in Table 2 using the scheme (4a-e). It is observed that the computational procedure solves exactly the sample problem even for large values of h = 1/16 and small value of $\varepsilon = 10^{-4}$. The shooting error , $u(1, \varphi_3) - \varphi_2 = 0$ for large step size h = 1/16 and small value of $\varepsilon = 10^{-4}$. Similarly for $\varepsilon = 10^{-5}$ the shooting error is zero for large step size h = 1/16. Same numerical result is got on using the scheme (6a-d) but it take less time for computation.

Example 3.. Consider the following non homogeneous problem with variable coefficient which arises in fluid dynamics

$$\varepsilon u''(t) + u'(t) = 1+2t, 0 < t < 1,$$
$$u(0) = 0, u(1) = 2.$$

The numerical results are given in Table 3. It is observed that the scheme (4a-e) reduces to the scheme (6a-d) when applied to example 3. The shooting error , $u(1, \varphi_3) - \varphi_2 = 3.906499818E-03$ for large step size h = 1/16 and small value of $\varepsilon = 10^{-4}$. Similarly for $\varepsilon = 10^{-5}$ the shooting error is 3.96499818E-03 for large step size h = 1/16. It is also observed that for small values of ε the shooting error is constant.

Example 4.. Consider the following homogeneous problem which arises in control system

$$\varepsilon u''(t) + u'(t) - u(t) = 0 , 0 < t < 1,$$
$$u(0) = 1, u(1) = 2.$$

The numerical results are given in Table 4. It is observed that the scheme (4a-e) reduces to the scheme (6a-d) when applied to example 4. The shooting error , $u(1, \varphi_3) - \varphi_2 = 8.7998364932E-02$ for large step size h = 1/16 and small value of $\varepsilon = 10^{-4}$. Similarly for $\varepsilon = 10^{-5}$ the shooting error is 8.7998364932E-02 for large step size h = 1/16. It is also observed that for small values of ε the shooting error is constant.

Note.

When the exact solution of the initial value problem (8a,b) is not known , using the numerical solution of (8a,b), we can check the condition (9). In Tables 1-4 , the condition (9) is checked and given for the respective Test problems to illustrate the computational procedure

Table 1

$h = 1/16,\ \varepsilon = 1.0000000000E\text{-}04,\ \varphi_3 = 1.6000000000E{+}01$

shooting error: $u(1, \varphi_3) - \varphi_2 = 0.0000000000E{+}00$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
6.25000E-02	2.00000E+00	2.00000E+00	0.00000E+00
1.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
2.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
4.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.00000E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.62500E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
7.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
9.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.00000E+00	2.00000E+00	2.00000E+00	0.00000E+00

$h = 1/16,\ \varepsilon = 1.0000000000E\text{-}05,\ \varphi_3 = 1.6000000000E{+}01$

shooting error: $u(1, \varphi_3) - \varphi_2 = 0.0000000000E{+}00$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
6.25000E-02	2.00000E+00	2.00000E+00	0.00000E+00
1.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
2.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
4.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.00000E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.62500E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
7.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
9.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.00000E+00	2.00000E+00	2.00000E+00	0.00000E+00

Table 2

$h = 1/16,\ \varepsilon = 1.0000000000E\text{-}05,\ \varphi_3 = 1.6000000000E{+}01$

shooting error: $u(1, \varphi_3) - \varphi_2 = 0.0000000000E{+}00$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
6.25000E-02	2.00000E+00	2.00000E+00	0.00000E+00
1.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
2.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
4.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.00000E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.62500E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
7.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
9.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.00000E+00	2.00000E+00	2.00000E+00	0.00000E+00

$h = 1/16,\ \varepsilon = 1.0000000000E\text{-}05,\ \varphi_3 = 1.6000000000E{+}01$

shooting error: $u(1, \varphi_3) - \varphi_2 = 0.0000000000E{+}00$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
6.25000E-02	2.00000E+00	2.00000E+00	0.00000E+00
1.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
2.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
3.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
4.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.00000E-01	2.00000E+00	2.00000E+00	0.00000E+00
5.62500E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.25000E-01	2.00000E+00	2.00000E+00	0.00000E+00
6.87500E-01	2.00000E+00	2.00000E+00	0.00000E+00
7.50000E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.12500E-01	2.00000E+00	2.00000E+00	0.00000E+00
8.75000E-01	2.00000E+00	2.00000E+00	0.00000E+00
9.37500E-01	2.00000E+00	2.00000E+00	0.00000E+00
1.00000E+00	2.00000E+00	2.00000E+00	0.00000E+00

Table 3

$h = 1/16$, $\varepsilon = 1.0000000000E-04$, $\varphi_3 = 1.0625000000E+00$

shooting error: $u(1, \varphi_3) - \varphi_2 = 3.9062499818E-03$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
6.25000E-02	7.03125E-02	6.65937E-02	3.71875E-03
1.25000E-01	1.44531E-01	1.40800E-01	3.73125E-03
1.87500E-01	2.26562E-01	2.22819E-01	3.74375E-03
2.50000E-01	3.16406E-01	3.12650E-01	3.75625E-03
3.12500E-01	4.14062E-01	4.10294E-01	3.76875E-03
3.75000E-01	5.19531E-01	5.15750E-01	3.78125E-03
4.37500E-01	6.32812E-01	6.29019E-01	3.79375E-03
5.00000E-01	7.53906E-01	7.50100E-01	3.80625E-03
5.62500E-01	8.82812E-01	8.78994E-01	3.81875E-03
6.25000E-01	1.01953E+00	1.01570E+00	3.83125E-03
6.87500E-01	1.16406E+00	1.16022E+00	3.84375E-03
7.50000E-01	1.31641E+00	1.31255E+00	3.85625E-03
8.12500E-01	1.47656E+00	1.47269E+00	3.86875E-03
8.75000E-01	1.64453E+00	1.64065E+00	3.88125E-03
9.37500E-01	1.82031E+00	1.81642E+00	3.89375E-03
1.00000E+00	2.00391E+00	2.00000E+00	3.90625E-03

$h = 1/16$, $\varepsilon = 1.0000000000E-05$, $\varphi_3 = 1.0625000000E+00$

shooting error: $u(1, \varphi_3) - \varphi_2 = 3.9062500000E-03$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
6.25000E-02	7.03125E-02	6.64250E-02	3.88750E-03
1.25000E-01	1.44531E-01	1.40643E-01	3.88875E-03
1.87500E-01	2.26563E-01	2.22673E-01	3.89000E-03
2.50000E-01	3.16406E-01	3.12515E-01	3.89125E-03
3.12500E-01	4.14063E-01	4.10170E-01	3.89250E-03
3.75000E-01	5.19531E-01	5.15638E-01	3.89375E-03
4.37500E-01	6.32813E-01	6.28918E-01	3.89500E-03
5.00000E-01	7.53906E-01	7.50010E-01	3.89625E-03
5.62500E-01	8.82813E-01	8.78915E-01	3.89750E-03
6.25000E-01	1.01953E+00	1.01563E+00	3.89875E-03
6.87500E-01	1.16406E+00	1.16016E+00	3.90000E-03
7.50000E-01	1.31641E+00	1.31250E+00	3.90125E-03
8.12500E-01	1.47656E+00	1.47266E+00	3.90250E-03
8.75000E-01	1.64453E+00	1.64063E+00	3.90375E-03
9.37500E-01	1.82031E+00	1.81641E+00	3.90500E-03
1.00000E+00	2.00391E+00	2.00000E+00	3.90625E-03

Table 4

$h = 1/16$, $\varepsilon = 1.0000000000E-04$, $\varphi_3 = -3.4335782862E+00$

shooting error: $u(1, \varphi_3) - \varphi_2 = 8.7998364932E-02$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
6.25000E-02	7.87396E-01	7.86161E-01	1.23512E-03
1.25000E-01	8.56050E-01	8.36858E-01	1.91913E-02
1.87500E-01	9.11811E-01	8.90824E-01	2.09868E-02
2.50000E-01	9.72281E-01	9.48267E-01	2.40142E-02
3.12500E-01	1.03667E+00	1.00941E+00	2.72690E-02
3.75000E-01	1.10530E+00	1.07447E+00	3.08329E-02
4.37500E-01	1.17843E+00	1.14371E+00	3.47243E-02
5.00000E-01	1.25633E+00	1.21737E+00	3.89653E-02
5.62500E-01	1.33929E+00	1.29571E+00	4.35779E-02
6.25000E-01	1.42761E+00	1.37902E+00	4.85838E-02
6.87500E-01	1.52159E+00	1.46759E+00	5.40043E-02
7.50000E-01	1.62156E+00	1.56170E+00	5.98598E-02
8.12500E-01	1.72784E+00	1.66167E+00	6.61697E-02
8.75000E-01	1.84077E+00	1.76782E+00	7.29521E-02
9.37500E-01	1.96070E+00	1.88048E+00	8.02235E-02
1.00000E+00	2.08800E+00	2.00000E+00	8.79984E-02

$h = 1/16$, $\varepsilon = 1.0000000000E-05$, $\varphi_3 = -3.4335782862E+00$

shooting error: $u(1, \varphi_3) - \varphi_2 = 8.7998364932E-02$

t_i	u_i	$u(t_i)$	$u(t_i) - u_i$
0.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
6.25000E-02	7.87396E-01	7.86095E-01	1.30039E-03
1.25000E-01	8.56050E-01	8.36794E-01	1.92560E-02
1.87500E-01	9.11811E-01	8.90760E-01	2.10507E-02
2.50000E-01	9.72281E-01	9.48204E-01	2.40769E-02
3.12500E-01	1.03667E+00	1.00934E+00	2.73300E-02
3.75000E-01	1.10530E+00	1.07441E+00	3.08919E-02
4.37500E-01	1.17843E+00	1.14365E+00	3.47807E-02
5.00000E-01	1.25633E+00	1.21731E+00	3.90185E-02
5.62500E-01	1.33929E+00	1.29566E+00	4.36273E-02
6.25000E-01	1.42761E+00	1.37898E+00	4.86287E-02
6.87500E-01	1.52159E+00	1.46755E+00	5.40440E-02
7.50000E-01	1.62156E+00	1.56166E+00	5.98934E-02
8.12500E-01	1.72784E+00	1.66164E+00	6.61964E-02
8.75000E-01	1.84077E+00	1.76780E+00	7.29710E-02
9.37500E-01	1.96070E+00	1.88047E+00	8.02335E-02
1.00000E+00	2.08800E+00	2.00000E+00	8.79984E-02

Conclusions

From the above discussion, we conclude that the given singular perturbation problem arising in control system can be solved by shooting method. The given problem is replaced by an initial value problem with an initial guess from the zeroth order asymptotic expansion for the solution. Then the resultant initial value problem is solved using exponentially fitted finite difference schemes The dissimilarity between the given problem and the transformed initial value problem is , the non uniformity occurs in the solution of the given problem and the non uniformity occurs in the derivative of the solution of the transformed initial value problem respectively at t = 0.

The computational procedure presented in this paper solves exactly the Test problem 1 and problem 2 for small values of $\varepsilon = 10^{-4}$. The advantage of this procedure is , there is no need for matrix inversion as in tri-diagonal difference schemes. A one step method is used in the computational procedure. To study the local behavior of the solution in the neighborhood of ε, one can apply the computational procedure presented in [5].

Using a scheme of order one, in this paper a good approximation to the solution of the given problem is obtained. To get higher order convergence one can apply the higher order numerical methods presented in [1,2,3,4,6]. The computational procedure can also be applied to solve the partial differential equations

All computations were performed in Pascal single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, Tamil Nadu, India.

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