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A Mathematical Model of Multi-Criteria Transportation Problem with Application of Exponential Membership Function

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Abstract

In this paper, we investigated a multi-criteria transportation problem in fuzzy environment. A simple mathematical model has been proposed to discuss about multi-criteria transportation problem, which is characterized by exponential membership function instead of linear membership function. The model emphasizes on optimal compromise solution and efficient solution of the problem. Two illustrative examples are included to demonstrate this model.

Keywords: Multi-criteria decision making, Fuzzy Transportation problem, Exponential membership function

Introduction

The classical transportation problem (Hitchcock transportation problem) refers to a special class of linear programming problems. Linear programming is one of the most widely used decision making tool for solving real world problems. The transportation model was first developed by Hitchcock (1941). Diaz (1979) developed an algorithm for finding the solution of multi-objective transportation problem. An algorithm for identifying all the non-dominated solutions for a linear multi-objective transportation problem was developed by Isermann(1979). Ringuest and Rinks(1987)developed two interactive algorithm for solving multi-objective transportation problem. The linear interactive, discrete optimization [LINDO](Schrage,1984),general interactive optimizer[GINO](Liebman and Schrage,1981) and TORA packages(Taha,1992) as well as many other commercial and academic packages are useful to find the solution of the transportation problem. The concept of decision making in a fuzzy environment was first proposed by Bellmann and Zadeh(1970).The application of fuzzy optimization technique with the suitable membership function to solve the linear programming problem with several objective functions was applied by Zimmermann(1978). Bit and Biswal(1992) applied the fuzzy programming technique with linear membership function to solve the multi-objective transportation problem. A linear membership function is most commonly used because it is simple and it is defined by fixing two points: the upper and lower levels of acceptability. However, linear membership function is not a suitable representation under many practical situations. Furthermore, if the membership function is interpreted as the fuzzy utility of the decision maker, used for describing levels of indifference, preference towards uncertainty, then a nonlinear membership function gives a better representation than a linear membership function. Li and Lee(1991) used a special type of non-linear(exponential) membership function for the multi-objective linear programming problem. In the multi-objective transportation problem, the objectives alone are considered as fuzzy. Ammar and Youness(2005) investigated the efficient solutions and stability of fuzzy multi-objective transportation problem and proposed an algorithm for the determination of stability set. Liu and Kao(2004) developed a procedure to derive the fuzzy objective value of the fuzzy transportation problem, in that the cost coefficients and the supply and demand quantities are fuzzy numbers basing on extension principle. Wahed(2001) presented a fuzzy programming approach to determine the compromise solution of multi-objective transportation problem(MOTP). In this paper, we are applying the fuzzy programming technique with exponential membership function to solve a vector minimum multi-objective transportation problem

Mathematical model

In a typical transportation problem, a homogeneous product is to be transported from each of m sources to any of n destinations and their capacities are $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_n$ respectively. In addition there is a penalty c_{ij} associated with transporting a unit of the product from source i to destination j. The penalty could represent transportation cost, delivery time, quantity of good delivered, safety of delivery and many others. A variable x_{ij} represents the unknown quantity to be transported from source i to destination j. In the real life, all transportation problems are not single objective. Thus in general the objective will also be controversial. In this paper those transportation problems are considered, which are described by multiple objective functions.

The mathematical model of the multi-objective transportation problem is written as follows:

S1: Minimize $Z_k(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{k} x_{ij}$, $k = 1,2,........, K$ (2.1)

Subject to

$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1,2,....,m$ (2.2)

$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1,2,....,n$ (2.3)

$x_{ij} \geq 0, \quad \forall\, i$ and j (2.4)

Where $Z_k(x) = \{Z_1(x), Z_2(x), ..., Z_k(x)\}$ is a vector of K objective functions, the subscript on $Z_k(x)$ and superscript on c_{ij}^{k} are used to identify the number of objective functions(k=1,2,…, K).Without loss of generality it will be assumed in the whole paper that $a_i > 0 \,\forall\, i, b_j > 0 \,\forall\, j, c_{ij}^{k} \geq 0 \,\forall\, (i,j)$ and $\sum_i a_i = \sum_j b_j$

Definition 2.1: A feasible solution $\bar{x} = \{\overline{x_{ij}}\} \in X$ yields a non-dominated solution to the multi-objective transportation problem if, and only if, there is no other feasible vector $x = \{x_{ij}\} \in X$ such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{k} x_{ij} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{k} \overline{x_{ij}} \quad \text{for all } k$$

and
$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{k} x_{ij} \neq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{k} \overline{x_{ij}} \quad \text{for some } k$$

When this relationship holds $\bar{x}$ is said to be efficient. Clearly, if $\bar{x}$ is not efficient it will yield an inferior or dominated solution (Ringuest & Rinks,1987).

Definition 2.2: An ideal solution to the multi objective transportation problem would result in each objective simultaneously realizing its minimum. That is, if

$$Z_k^{*} = \min Z_k = \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^{k} x_{ij}$$

Then the vector $Z^{*} = \left(Z_1^{*}, Z^{*}2, ..., Z_l^{*}\right)$ is an ideal solution. When there is a feasible extreme point x^{*} such that $Z^{*} = \left(Z_1^{*}, Z^{*}2, ..., Z_l^{*}\right) = Z_l\left(x^{*}\right)$ the multi objective transportation problem has an ideal solution. This would mean that for each of the sub problems

$$\min Z_k = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij} \quad k = 1,2,\dots\dots\dots,K$$

Subject to

$$AX \le B \quad X \ge 0 \quad i = 1,2,\dots\dots\dots,K$$

There is at least one identical extreme point which optimizes every Z_k, therefore a compromise solution must be obtained (Ringuest & Rinks,1987).

Definition 2.3: An optimal compromise solution of the multi-objective transportation problem is a solution $\bar{x} = \{\overline{x_{ij}}\} \in X$ which is preferred by the decision maker to all other solutions, taking into consideration all criteria contained in the multi objective functions. Hence, an optimal compromise solution has to be a non dominated solution according to the definition of non dominated solution (Bit et al,1992).

Fuzzy programming Technique for solving multi-objective transportation problem

The MOTP is considered as a vector minimum problem. In this method, we first find the lower bound (minimum value) L_k and the upper bound (maximum value) U_k for the kth objective function z_k, $k = 1, 2, \dots\dots, K$

where

L_k = Aspired level of achievement for objective k

U_k = Highest acceptable level of achievement for objective k

$d_k = U_k - L_k$ the degradation allowance for objective k

On the basis of definitions L_k and U_k we consider the following form of exponential membership function (Li et al,1991) to define the decision maker's level of desirability for the objective function.

$$\mu_k\big(z_k(x)\big) = \exp\left(\frac{\alpha_k\big(z_k(x) - L_k\big)}{L_k - U_k}\right) , \tag{3.1}$$

The exponential membership function has the following properties

$\mu_k\big(z_k(x)\big)$ is a strictly decreasing function for objective function z.

$\mu_k\big(z_k(x)\big) = 1$ if $z_k(x) \le L_k$

$\mu_k\big(z_k(x)\big) = 0$ if $z_k(x) \ge U_k$ and α approaches infinity.

Using the "min" operator to aggregate overall satisfaction and following the Bellman-Zadeh's maximization principle[1] the equivalent non-linear programming problem is

S2: Max λ (3.2)

Subject to

$$\lambda \le \exp\left(\frac{\alpha_k\left(z_k - L_k\right)}{L_k - U_k}\right), \quad k = 1,2,...,K \quad , \tag{3.3}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad i = 1,2,...,m \qquad , \tag{3.4}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1,2,....,n \qquad , \tag{3.5}$$

$$x_{ij} \ge 0 \quad \forall\, i \text{ and } j \tag{3.6}$$

Problem S2 can be transformed into the following linear programming problem by substituting

$$\beta = -\ln\lambda \tag{3.7}$$

Now we have

S3: M*in* β (3.8)

Subject to

$$\beta \ge \left(\frac{\alpha_k\left(L_k - z_k\right)}{L_k - U_k}\right) \quad , \quad k = 1,2,.........,K \tag{3.9}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad i = 1,2,....,m \tag{3.10}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1,2,....,n \tag{3.11}$$

$$x_{ij} \ge 0 \quad \forall\, i \text{ and } j \tag{3.12}$$

Then, the solution procedure of the multi objective transportation problem is briefed in the following steps:

Step 1: Solve the multi-objective transportation problem as a single objective transportation problem K times by taking one of the objectives at a time.

Step 2: Evaluate the kth objective function at the K optimal solutions ($k = 1,2,.........,K$). For each objective function, find its lower and upper bounds (L_k and U_k) corresponding to the set of solutions.

Step 3: Define the membership function as given in equation (3.1)

Step 4: Construct the non-linear programming problem S2and find its equivalent linear programming problem S3

Step 5: Solve S3 by using an integer programming technique to get an optimal solution and evaluate the K objective functions at this optimal compromise solution.

Illustrative examples

To illustrate the fuzzy approach, we consider the following two examples of multi objective transportation problem (taken from [9])

Example 1

$$\min Z_1 = x_{11} + 2x_{12} + 7x_{13} + 7x_{14} + x_{21} + 9x_{22} + 3x_{23} + 4x_{24} + 8x_{31} + 9x_{32} + 4x_{33} + 6x_{34}$$
$$\min Z_2 = 4x_{11} + 4x_{12} + 3x_{13} + 3x_{14} + 5x_{21} + 8x_{22} + 9x_{23} + 10x_{24} + 6x_{31} + 2x_{32} + 5x_{33} + x_{34}$$

s/t
$$x_{11} + x_{12} + x_{13} + x_{14} = 8$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 19$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 17$$
$$x_{11} + x_{21} + x_{31} = 11$$
$$x_{12} + x_{22} + x_{32} = 3$$
$$x_{13} + x_{23} + x_{33} = 14$$
$$x_{14} + x_{24} + x_{34} = 16 \qquad x_{ij} > 0,\ i = 1,2,3;\ j = 1,2,3,4$$

The optimal solution of each single objective transportation problem is

$$X_1 = (5,3,0,0,6,0,0,13,0,0,14,3)$$
$$X_2 = (0,0,8,0,11,2,6,0,0,1,0,16)$$

The set of non-dominated solutions is

$$Z(X_1) = (143,265)$$
$$Z(X_2) = (208,167)$$

Thus the upper and lower bound of each objective are

$$143 \le Z_1 \le 208 \text{ and } 167 \le Z_2 \le 265$$

The membership function of both $Z_1(x)$and $Z_2(x)$ with parameter α=2 are

$$\mu_1(z_1(x)) = \exp\left(\frac{2(z_1(x) - 143)}{143 - 208}\right)$$

and

$$\mu_2(z_2(x)) = \exp\left(\frac{2(z_2(x) - 167)}{167 - 265}\right)$$

Now problem S3 is written as follows

$$\min \beta$$

subject to

$$2x_{11} + 4x_{12} + 14x_{13} + 14x_{14} + 2x_{21} + 18x_{22} + 6x_{23} + 8x_{24} + 16x_{31} + 18x_{32} + 8x_{33} + 12x_{34} - 65\beta \le 286$$

$$8x_{11} + 8x_{12} + 6x_{13} + 6x_{14} + 10x_{21} + 16x_{22} + 18x_{23} + 20x_{24} + 12x_{31} + 4x_{32} + 10x_{33} + 2x_{34} - 98\beta \le 334$$

$$x_{11} + x_{12} + x_{13} + x_{14} = 8$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 19$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 17$$
$$x_{11} + x_{21} + x_{31} = 11$$
$$x_{12} + x_{22} + x_{32} = 3$$
$$x_{13} + x_{23} + x_{33} = 14$$
$$x_{14} + x_{24} + x_{34} = 16$$
$$\lambda' \ge 0 \qquad x_{ij} \ge 0 \ \forall i, j$$

The problem is solved by the TORA package the optimal compromise solution is presented as follows:

$$X^* = (x_{11} = 4, x_{12} = 3, x_{13} = 1, x_{21} = 7, x_{23} = 12, x_{33} = 1, x_{34} = 16),$$

Therefore $Z_1^* = 160$, $Z_2^* = 195$ with $\beta = .57$ and $\lambda^* = .56$.

Example-2

$$\text{Min } z_1 = 9x_{11} + 12x_{12} + 9x_{13} + 6x_{14} + 9x_{15} + 7x_{21} + 3x_{22} + 7x_{23} + 7x_{24} + 5x_{25} + 6x_{31} + 5x_{32}$$
$$+ 9x_{33} + 11x_{34} + 3x_{35} + 6x_{41} + 8x_{42} + 11x_{43} + 2x_{44} + 2x_{45}$$

$$\text{Min } z_2 = 2x_{11} + 9x_{12} + 8x_{13} + x_{14} + 4x_{15} + x_{21} + 9x_{22} + 9x_{23} + 5x_{24} + 2x_{25} + 8x_{31} + x_{32}$$
$$+ 8x_{33} + 4x_{34} + 5x_{35} + 2x_{41} + 8x_{42} + 6x_{43} + 9x_{44} + 8x_{45}$$

$$\text{Min } z_3 = 2x_{11} + 4x_{12} + 6x_{13} + 3x_{14} + 6x_{15} + 4x_{21} + 8x_{22} + 4x_{23} + 9x_{24} + 2x_{25} + 5x_{31} + 3x_{32}$$
$$+ 5x_{33} + 3x_{34} + 6x_{35} + 6x_{41} + 9x_{42} + 6x_{43} + 3x_{44} + x_{45}$$

s/t

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 5$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 4$$

$x_{31}+x_{32}+x_{33}+x_{34}+x_{35}=2$

$x_{41}+x_{42}+x_{43}+x_{44}+x_{45}=9$

$x_{11}+x_{21}+x_{31}+x_{41}=4$

$x_{12}+x_{22}+x_{32}+x_{42}=4$

$x_{13}+x_{23}+x_{33}+x_{43}=6$

$x_{14}+x_{24}+x_{34}+x_{44}=2$

$x_{15}+x_{25}+x_{35}+x_{45}=4$

$x_{ij}\geq 0,\quad i=1,2,3,4,\quad j=1,2,3,4,5.$

The optimal solution of each single objective transportation problem is

$X_1=(0,0,5,0,0,0,4,0,0,0,1,0,1,0,0,3,0,0,2,4)$

$X_2=(3,0,0,2,0,0,0,0,0,4,0,2,0,0,0,1,2,6,0,0)$

$X_3=(3,2,0,0,0,0,0,4,0,0,0,2,0,0,0,1,0,2,2,4)$

The set of non-dominated solutions is

$Z(X_1)=(102,148,100)$

$Z(X_2)=(157,72,86)$

$Z(X_3)=(129,126,64)$

Thus the upper and lower bound for each objective are $102\leq Z_1\leq 157,\ 72\leq Z_2\leq 148, 64\leq Z_3\leq 100$

Problem S3 is developed and solved by using the TORA package. the optimal compromise solution is presented as follows:

$$X^*=(x_{11}=3,x_{14}=2,x_{22}=2,x_{23}=2,x_{32}=2,x_{41}=1,x_{43}=4,x_{45}=4),$$

therefore $Z_1^*=127, Z_2^*=104, Z_3^*=76$ with $\beta=.59$ and $\lambda^*=.55$.

For the multi objective transportation problem with K objective functions the fuzzy programming method gives K non-dominated solutions and an optimal compromise solution. Above two examples are solved by the interactive approach (Ringuest and Rinks,1987). The interactive approach gives more than K non-dominated and dominated solutions. Using each non-dominated solution, the decision-maker has to look out for the best compromise solution out of set of non-dominated solutions. This search is continuously repeated until a satisfactory solution is achieved.

In example-1 Ringuest and Rinks(1987) obtained three non-dominated solutions and the point(156,200) as the most optimal value of the objective function. In fuzzy programming method we got two non-dominated solutions (143,265) and (208,167) and the point (160,195) as the most optimum value of the objective function. The point (160,195) obtained by fuzzy programming method is very close to the ideal solution (143,167) in comparison to the point (156,200) obtained by interactive approach. Thereby we achieved the best optimal compromise solution by fuzzy linear programming.

In example -2 Ringuest et al.(1987) have obtained six non-dominated and six dominated solutions. The optimal compromise solution obtained by Ringuest interactive approach is point (127,104,76). In fuzzy programming method we get three non-dominated solutions (102,148,100),(157,72,86),(129,126,64) and the point (127,104,76) as the most optimal value of the objective function. Thus, in fuzzy programming method the optimum values of objective functions are similar to the Ringuest interactive approach. The solution with two approaches are illustrated in the below table1:

Table 1

	Ideal solution	Fuzzy approach results	Interactive approach results
Example-1 Z_1 Z_2	143 167	160 195	156 200
Example-2 Z_1 Z_2 Z_3	102 72 64	127 104 76	127 104 76

Conclusions

Fuzzy programming algorithm is a more convenient and feasible method for finding an optimal compromise solution for the multi-objective transportation problem. In this paper, a fuzzy programming approach with exponential membership is used to find an optimal compromise solution for the multi-objective transportation problem. The fuzzy approach features are described as:

1. It is easy and simple to use for the decision maker and can be easily implemented to solve linear multi objective programming problems.
2. The fuzzy approach gives a better optimal compromise solution of large problem(such as increasing the number of objectives and constraints). This feature shows that the fuzzy approach is better than interactive approach in solving multi-objective transportation problem.
3. Fuzzy approach is applicable to all types of multi-objective transportation problem, the vector minimum problem and the vector maximum problem.
4. Fuzzy programming approach with exponential membership function is a suitable representation in many realistic situations. This feature makes this approach more realistic than the approach using linear membership function in solving multi-objective transportation problem.

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