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Optimal and Uniform Finite Difference Scheme for Singularly Perturbed Riccati Equation

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Abstract

This paper presents a finite difference scheme of order one for singularly perturbed Riccati equation

$$\varepsilon u'(x) = c(x)u^2(x) + d(x)u(x) + e(x), x>0, u(0)=\varphi$$

with a small parameter ε multiplying the first derivative, The scheme is a modified form of the classical Euler's backward scheme of order one. The scheme is both optimal and uniform with respect to the small parameter ε, that is, the solution of the difference scheme satisfies the error estimates of the form:

$$| u (x_i) - u_i | \leq C \min (h , □)$$

where C is independent of i, h and ε. Here h is the step size and x_i is any mesh point. The scheme is implicit in nature and no iteration is needed for the convergence of the solution of the scheme. The scheme presented in this paper is new and it reflects the asymptotic properties of the singularly perturbed Riccati equation. Finally, numerical experiments are presented.

Keywords: singular perturbation problems, exponentially fitted, uniformly convergent, asymptotic expansion, finite difference schemes, Riccati equation,

AMS (MOS) subject classification: 65F05, 65N30, 65N35, 650Y05.

Introdution

Consider the scalar Riccati equation on the interval $\Omega = (0, \infty)$

$$\varepsilon u'(x) = c(x)u^2(x) + d(x)u(x) + e(x), x \in \Omega \qquad (1a)$$

$$u(0) = \varphi \qquad (1b)$$

where ε>0 is a small parameter and c, d and e are smooth functions on Ω . In addition we assume that

$$d^2(x(- 4c(x)e(x) \geq \alpha > 0, x \in \Omega. \quad (2)$$

Equation(1a,b) has wide application in many areas of science such as chemical kinetics[8], mathematical physics[4]. For example in the propagation of auxiliary symmetric waves [7], the input wave impedance to an induction device[5], quadratic periodic optimization, in the design of solar heating system[11], etc.

The equation(1a,b) can be written in the form[2,15]

$$N\, u(x) \equiv \varepsilon\, u'(x) - c(x)[u(x) - a(x)][u(x) - b(x)] = 0\,,\ x \in \Omega\,, \quad (3a)$$

$$u(0) = \varphi \quad (3b)$$

where a(x) and b(x) are the roots of the quadratic equation

$$u^2(x) + [d(x)/c(x)]\, u(x) + [e(x)/c(x)] = 0,\ x \in \Omega\,,\ c(x) \neq 0.$$

The condition(2) is sufficient to guarantee that operator N has a maximum principle and the solution u(x) of expression(3a,b) is unique and bounded[2].

The problem(3a,b) is a singularly perturbed equation with an initial layer at x=0 which is of order ε[2,9,10]. As ε goes to zero, the equation(3a,b) reduces to the form

$$c(x)[u_0(x) - a(x)][u_0(x) - b(x)] = 0\,,\ x \in \Omega\,. \quad (4)$$

The solution of (4) is $u_0(x) = a(x)$ and $u_0(x) = b(x)$. That is, the critical points are clearly a(x) and b(x). We may define the corresponding reduced problem $u_0(x)$ by

$$u_0(x) = a(x),\ x \in \Omega \quad (5)$$

which is the stable critical point., throughout this paper. In general, numerical solution of (3a,b) using the classical Euler's rule will not yield satisfactory result. A modified form of the Euler's rule is presented in [2,9,10] gives satisfactory result for small values of the mesh size but not for small values of ε.

We introduce a uniform mesh of width h on Ω with mesh points $x_i = ih$. We solve problem(3a,b) by finite difference methods of the form:

$$N^h u_i \equiv \varepsilon\, \sigma_{i+1}(\rho)\, D_+ u_i - c_{i+1}[u_{i+1} - a_{i+1}][u_{i+1} - b_{i+1}] = 0\,, \quad (6a)$$

$$u_0 = \Box \quad (6b)$$

where c_{i+1} , , a_{i+1} , b_{i+1} and the fitting factor $\sigma_{i+1}(\rho)$ are specified later. The scheme of this paper is chosen in such a way that it must solve exactly the reduced problem(5) as ε goes to zero, because the scheme which solve exactly the reduced problem(5) are expected to work well for large values of x. If the solution u_i of the scheme (6a,b) satisfies the `reduced equation(5) exactly at the interior points, as ε goes to zero, them we call such finite difference scheme with this property as optimal.

In this paper the fitting factor will always be chosen so that the difference scheme is uniform with respect to the small parameter ε, that is, if u and u_i are the solutions of (3a,b) and (6a,b) respectively, then at each node x_i there is an error estimate of the form

$$|u(x_i) - u_i| \leq C h^p \qquad (7)$$

where C and p are independent of i i, h and ε.

Uniformly convergent finite difference schemes for the solution of linear and nonlinear singularly perturbed problems have been approached from the point of view of singular perturbations and exponential fitting[2,3,9,10,12,13,15]. Uniform results for nonlinear initial value problems based on exponential fitting appear in[2,9,10,12,15].

Uniformly convergent finite difference schemes of the problem(3a,b) have been proposed by Carroll[2] and O'Reilly[9,10]. In fact, the scheme of Carroll[2] gives uniform error estimate for the case a(x)≠b(x) and not for the case a(x)=b(x). It must be noted that in the fitting factor, Carroll used third order approximation but the scheme is of order one. O'Reilly[9,10] replaced u(x) + d(x)/(2c(x)) by u which becomes a Bernoulli's equation and then framed an explicit finite difference scheme. Both the schemes developed by Carroll and O'Reilly are uniform and they never reflect the asymptotic behavior of the solution for small values of ε

The purpose of the paper is to propose an implicit finite difference scheme for the problem(3a,b) such that the solution of the scheme is uniform using Euler's backward scheme. Also, the scheme must reflect the asymptotic properties of the solution of (3a,b). We derive error estimates of the form:

$$|u(x_i) - u_i| \leq C \min(h, \square) \qquad (8)$$

where C is independent of i, h and ε. Schemes satisfying the inequality (8) are clearly uniform of order one and optimal.

The schemes proposed in [2,9,10] are uniform of order one and satisfies the estimate(7) for p=1. And these schemes are not satisfying the estimate(8) and so they are not optimal. . The scheme presented in [15] is a modified scheme of Carrull[2] which is uniform and optimal but the form of the fitting factor is same as in the scheme of Carroll[2] need a lot of computation.

The scheme presented in this paper is a modified form of the scheme of Euler's backward scheme and it is computationally cheaper than all other explicit schemes available in the literature. The fitting factor of the scheme presented in this paper is computationally cheaper than the fitting factor of the schemes of Carroll[2] and[15] And the scheme works well for moderate and small values of ε and even for large values of the step size h.

Throughout this paper, ρ = h/ε and C will denote a generic constant independent of I, h and ε.

Finite Difference Scheme

In this section an implicit finite difference scheme with a variable fitting factor is presented. The consistency, stability and convergence are discussed. The finite difference scheme for (3a,b) is

$$N^h u_i \equiv \varepsilon\, \sigma_{i+1}(\rho)\, D_+ u_i - c_{i+1}[u_{i+1} - a_{i+1}][u_{i+1} - b_{i+1}] = 0, \qquad (9a)$$

$$u_0 = \square \qquad (9b)$$

where

$$\sigma_{i+1}(\rho) = \sigma(\rho q_0) R_i \qquad (9c)$$

$$\sigma(\rho q_0) = \rho q_0 / [\exp(-\rho q_0) - 1],\ \rho = h/\varepsilon, \qquad (9d)$$

$$q_0 = c_0 (a_0 - b_0), \qquad (9e)$$

$$R_i = [1 - k\exp(i \rho q_0)] / [1 - k\exp((i+1) \rho q_0)], \qquad (9f)$$

$$k = (\square - a_0)/(\square - b_0) \qquad (9g)$$

and

$$c_{i+1} = c(x_{i+1}),\ a_{i+1} = a(x_{i+1}),\ b_{i+1} = b(x_{i+1}). \qquad (9h)$$

The scheme(9a-h) is consistent with the problem(3a,b) in the sense that the discrete problem (9a-h) coincide with the problem(3a,b) when h approaches zero. The scheme(9a-h) satisfies the necessary condition for uniform convergence exactly introduced in[2], that is,

$$\lim \sigma_i(\rho) = (\rho q_0 \exp(\rho q_0) / [1 - \exp(\rho q_0)])\, R_0 \qquad (10a)$$

as limit h goes to zero, where

$$R_0 = [1 - K_0 \exp(i \rho q_0)] / [1 - K_0 \exp((i+1) \rho q_0)] \qquad (10b)$$

$$q_0 = c_0 (a_0 - b_0) \qquad (10c)$$

and

$$K_0 = (\varphi - a_0)/(\varphi - b_0). \qquad (10d)$$

The scheme(9a-h) model the equation(5) exactly as ε goes to zero

$$u_{i1} = u(x_{i+1}). \qquad (11)$$

And so one can expect the scheme(9a-h) to work well for large x. The scheme is exponentially fitted , because the necessary condition(10a-d) gives the minimum requirement on the scheme to model the transient behavior of the problem(3a,b) accurately.

Following the results of Keller[6], a stability result is given for the solution of the scheme(9a-h) in the form of a Lemma. It is noted that

$$-q_i = \sqrt{d_i^{2} - 4c_i e_i} \geq \sqrt{\alpha} > 0.$$

Lemma 2.1.[6,15]

Assume that $d^2(x) - 4c(x)e(x) \geq \alpha > 0$ for all x Є Ω. Let N^h be the operator defined in (9a-h). If $\{v_i\}$ and $\{w_i\}$ be any two functions then, for all x Є Ω and i≥0

$$|v_i - w_i| \leq |v_0 - w_0| + \max|N^h v_j - N^h w_j|,\ j \geq 0.$$

Following theorem gives the convergence result for the scheme(9a-h). An estimate of the form (8) is obtained in this theorem.

Theorem 2.2

Let u and u_i be the solutions of problem(3a,b) and (9a-h) respectively. Then, at each mesh point x_i, we have the following error estimate,

$$| u (x_i) - u_i | \leq C \min (h , \Box) \quad (12)$$

where C is independent of i, h and ε.

Proof: From the stability result of N^h in the scheme(9a-h), it suffices to prove that

$$| \iota_i | = | N^h u(x_i) - N^h u_i | \leq C \min (h , \Box)$$

where ι_i is the truncation error of the scheme(9a-h) with respect to the problem(3a,b).

For i=0, $\iota_0 = \Box - \Box = 0$.

For I ≥ 1, $\iota_i = N^h u(x_i) - N^h u_i$

$= N^h u(x_i) - 0$

$= N^h u(x_i) - N u(x_{i+1})$

where $N u(x_{i+1}) =0$ since $N u(x) =0$.

Using the asymptotic expansion for the solution of (3a,b) in the interval $[x_i ,x_{i+1}]$

$$u(x) = u_0(x) + v_0(x/\varepsilon) + o(\varepsilon)$$

where

$$u_0(x) = a(x)$$

and

$$v_0(x/\varepsilon) = k [a (0) – b(0)]\exp(q_0 x/\varepsilon) / [1- k \exp(q_0 x-/\varepsilon)] ,$$

$$q_0= c(0) (a(0) – b(0)) , k = [\Box - a(0)]/[\Box - b(0)]$$

where x Є **Ω**, we have

$\iota_1 = N^h u(x_i) - N u(x_{i+1})$

$= \varepsilon \sigma_i (\rho) D_+ u(x_i) - \varepsilon u'(x_{i+1})$

$= \varepsilon [\sigma_i (\rho) – 1] D_+ u_0(x_i) + \varepsilon [D_+ u_0(x_i) - u_0'(x_{i+1})]$

$+ [\varepsilon \sigma_i (\rho) D_+ v_0(x_i /\varepsilon) - \varepsilon v_0'(x_{i+1} /\varepsilon)] + O(\varepsilon)$

$= \varepsilon [\sigma_i (\rho) – 1] D_+ u_0(x_i) + \varepsilon [D_+ u_0(x_i) - u_0'(x_{i+1})] + O(\varepsilon)$

since $\varepsilon v_0'(x_{i+1} /\varepsilon) = \varepsilon \sigma_i (\rho) D_+ v_0(x_i /\varepsilon)$. And

$\varepsilon[\sigma_i (\rho) - 1] = \varepsilon [\sigma (\rho q_0) R_i - 1]$

$= \varepsilon [\sigma (\rho q_0) - 1] + [\varepsilon \sigma (\rho q_0)] [R_i - 1]$

$= \varepsilon [\sigma (\rho q_0) - 1] + \varepsilon \sigma (\rho q_0) [\exp(\rho q_0) - 1] v_0(x_i /\varepsilon)/ [a(0) - b(0)]$

From [3] we have

$\varepsilon| \sigma (\rho q_0) - 1 | \leq C \min (h , \square),$

$\left|\exp(\rho q_0) - 1\right| \leq C \min (1 , \rho),$

$\left| \sigma (\rho q_0)\right| \leq C, \left| v_0(x_i /\varepsilon)\right| \leq C$

and hence

$| \varepsilon [\sigma (\rho q_0) - 1] | \leq C \min (h , \square) + C \min (h , \square)$

$\leq C \min (h , \square).$

Therefore,

$| \iota_i | \leq C \min (h , \square)$ for al $i \geq 0$

since

$\left|D_+ u_0(x_i)\right| \leq C$ and $\left| D_+ u_0(x_i) - u_0'(x_{i+1})\right| \leq C. h$

Using stability result, we have

$| u (x_i) - u_i | \leq | u (0) - u_0 | + \max | N^h u(x_j) - N^h u_j | ,j \geq 0$

$\leq |\square - \square | + \max | \iota_j |$

$\leq \max | \iota_j |$

$\leq C \min (h , \square)$

which is the required estimate.

Hence the explicit scheme (9a-h) is proved to be uniform of order one and optimal.

Numerical Experiment

This section gives numerical results for a singularly perturbed Riccati equation for large values of x. We compare the difference scheme(9a-h) in Table 1 with a number of integration formulae . For the problem 1, we use a common uniform mesh h=1/16 and interval [0,1] respectively and we compute absolute error

$$e_i^{\,h} = \max | u (x_i) - u_i | \text{ for all } i= 0(1)16$$

where $u (x_i)$ and u_i are exact and approximate solutions respectively.

All computations were performed in PASCAL single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, India.

The sample problems which we consider are as follows:

Problem 1[15]

$u'(x) - (3/8) \cos (x)[(1+6 \sin(x))/8]^{-1/2} = \lambda [0.6 \sin(x) +0.1 - 0.8 u^2 (x)] ,$

$u(0) = (1/8)^{1/2}$, $\lambda = 1/\varepsilon$, $\varepsilon \neq 0$.
Numerical results are given in Table 1.

In Tables 1 a comparative study is made. The schemes compared with the scheme(9a-h) presented in this paper are

1. Euler's forward method
2. Euler's backward method
3. Trapezoidal method
4. scheme of Carroll[2]
5. scheme of O'Reilly[9,10]
6. scheme of Selvakumar [15].

The problem 1 with variable coefficients a(x) and b(x) is chosen to show the superiority of the scheme(9a-h). From Table 1 it is observed that the scheme of Carroll[2] and scheme of O'Reilly[9,10] are uniform but not optimal.

It is observed from Table 1 that the scheme of Selvakumar[15] and the scheme(9a-h) are uniform and optimal convergence of O(min (h , □)) . The magnitude of the absolute error produced by the scheme(9a-h) is lesser than that of the scheme of Selvakumar[15].

The fitting factor in the scheme of Carroll[2] and Selvakumar.[15] are same. The scheme(9a-h) is computationally superior than the schemes of Carroll[2] and Selvakumar.[15] in the sense that the fitting factor in the scheme(9a-h) is computationally cheaper than the fitting factor in the scheme of Selvakumar[[15] and Carroll[2].

Table 1

Schemes \ ε	0.01	0.001	0.0001	0.00001
Euler's forward	----	------	------	---
Euler's backward	5.18851E-01	5.75154E-01	5.84423E-01	5.85335E-01
Trapezoidal	-----	----	----	---
scheme od O'Reilly[9,10]	4.31649E-02	5.91179E-02	6.08005E-02	6.09692E-02
scheme of Carroll [2]	4.20973E-02	5.91179E-02	6.08005E-02	6.09691E-02
scheme of Selvakumar[15]	1.36611E-02	1.51849E-03	1.52260E-04	1.52260E-05
scheme(9a-h)	1.17810E-02	1.35896E-03	1.36107E-04	1.36197E=05

Conclusion

In this paper an implicit finite difference scheme of order one is presented for the singularly perturbed Riccati equation . The scheme (9a-h) is a modified form of classical Euler's backward method. The scheme(9a-h) is implicit in nature, exponentially fitted, optimal and uniform of order one. The scheme(9a-h) is applicable even for large values of the step size and small values of the parameter ε.

The scheme(9-h) reflects the asymptotic properties of the solution of the singularly perturbed Riccati equation. The scheme presented in this paper can be applied to solve the second order singularly perturbed boundary value problems

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