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Pre A∗-Algebras and Rings

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Abstract

This paper is a study on algebraic structure of Pre A* - algebra. We prove basic theorems on Pre A* - algebra. We define p-ring. We define Boolean ring and 3-ring. We prove Pre A* - algebra as a Boolean ring and Boolean ring as a Pre A* - algebra. Next we prove Pre A* - algebra as a 3 – ring and also we prove 3– ring as a Pre A* - algebra.

Keywords: Pre A^*-Algebra,p- ring, Boolean ring, 3-ring.

Introduction

Boolean algebras, essentially introduced by Boole in 1850's to codify the laws of thought, have been a popular topic of research since then. A major breakthrough was the duality of Boolean algebras and Boolean spaces as discovered by Stone in 1930's. Stone also proved that Boolean algebras and Boolean rings are essentially the same in the sense that one can convert via terms from one to the other. Since every Boolean algebra can be represented as a field of sets, the class of Boolean algebras is sometimes regarded as being rather uncomplicated. However, when one starts to look at basic questions concerning decidability, rigidity, direct products etc., they are associated with some of the most challenging results.

The study on lattice theory had been made by Birkhoff (1948),and recently Pre A* - algebra had been studied by Chandrasekhara Rao (2007) and Srinivasa Rao(2009). In a draft paper, the equational theory of disjoint alternatives, Manes (1989) introduced the concept of Ada, $(A, \wedge, \vee, (-)', (-)_\pi, 0, 1, 2)$ which however differs from the definition of the Ada by Manes (1993), While the Ada of the earlier draft seems to be based on extending the If –Then –Else concept more on the basis of Boolean algebras, the later concept is based on C- algebra (A, ∧, ∨, $(-)^{\sim}$) introduced by Fernando and Craig (1990).

Koteswara Rao (1994) firstly introduced the concept of A* - algebra (A, ∧, ∨,*,$(-)^{\sim}$, $(-)_{\pi}$, 0, 1, 2) and studied the equivalnence with Ada by Manes (1989), C- algebra by Fernando and Craig (1990) and Ada by Manes (1993)) and its connection with 3-ring, stone type representation and introduced the concept of A* -clone and the If-Then-eise structure over A*-algebra and ideal of A*-algebra. Venkateswara Rao (2000) introduced the concept Pre A* -algebra (A, ∧, ∨, $(-)^{\sim}$) analogous to C-algebra as a reduct ofA*- algebra.

Definition

An algebra (A, ∧,∨, $(-)^{\sim}$) where A is non-empty set, Λ(meet), Λ(join) are binary operations and $(-)^{\sim}$ (tilda) is a unary operation satisfying.

a. $x{\sim}{\sim} = x, \forall x \in A$,
b. $x \wedge x = x, \forall x \in A$
c. $x \wedge y = y \wedge x, \forall x, y \in A$
d. $(x \wedge y){\sim} = x{\sim} \vee y{\sim}, \forall x, y \in A$
e. $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$
f. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$
g. $x \wedge y = x \wedge (x{\sim} \vee y), \forall x, y, z \in A$

is called a Pre A*-algebra.

Example

3 = { 0,1,2 } with operations ∧,∨ $(-)^{\sim}$ defined below is a Pre A* - algebra.

∧	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

∨	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Note

The elements 0,1,2 in the above example satisfy the following laws:

a. $2^{\sim} = 2$
b. $1 \wedge x = x$ for all $x \in 3$

c. $0 \vee x = x, \forall x \in 3$

d. $2 \wedge x = 2 \vee x=2, \forall x \in 3.$

Example

2={0,1} with operations $\wedge,\vee$, $(-)^{\sim}$ defined below is a Pre A*-algebra.

∧	0	1
0	0	0
1	0	1

∨	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

Note

i. $(2, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* - algebra

ii. The identities 1.1 (a) and 1.1 (d) imply that the varieties of Pre A* - algebras satisfies all the dual statements of 1.1 (a) to 1.1 (g).

Note

If (mn) is an axiom in Pre A* - algebra, then $(mn)^{\sim}$ is its dual.

Pre A*-Algebras and Rings

Basic Theorems in Pre A* - algebra

Theorem 1

De-Morgan laws

Let $(A, \wedge, (-)^{\sim}, 1,)$ be a Pre A* - algebra.

Then,

i. $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$

ii. $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$

Proof

By the definition [1.1(d)] of Pre A* - algebra we have

i. $(a \wedge b) \sim = a\sim \vee b\sim$

ii. By note 1.6, we have

iii. $(a \vee b) \sim = a\sim \wedge b\sim$

Lemma 1: Uniqueness of identity in a Pre A* - algebra:

Let $(A, \wedge, (-)^{\sim}, 1)$ be a Pre A* - algebra and $a \in B(A)$ be an identity for $\wedge$, then $a^{\sim}$ is an identity for $\vee$, a is unique if it exists, denoted by 1 and $a^{\sim}$ by 0 where $B(A)=\{x/x \vee x^{\sim}=1\}$ i.e.,

a. $1 \wedge x = x, \forall\, x \in A$.
b. $0 \vee x = x, \forall\, x \in A$.

Proof: Suppose $a \in B(A)$ is an identity for $\wedge$.

$$\Rightarrow a \wedge x = x, \forall\, x \in A \rightarrow \text{(i)}$$

To prove that $a^{\sim} \in A$ is an identity for $\vee$:

Consider $a^{\sim} \vee x = (a \wedge x\,\sim)\sim = (x\,\sim)\sim$

[Since by (i)] = x

[Since by definition 1.1 (a)]

Therefore $a^{\sim} \vee x = x, \forall\, x \in A$.

Thus $a^{\sim}$ is an identity for $\vee$.

Uniqueness

Suppose a and b are two identities for $\wedge$.

$$\Rightarrow a \wedge x = x, \forall\, x \in A \text{ and}$$

$$b \wedge x = x, \forall\, x \in A$$

Therefore $a \wedge b = b$ and
$b \wedge a = a$

Now $a = b \wedge a$
$= a \wedge b$ [Since by 1.1(c)]
$= b$

Therefore $a = b$

Hence a if it exists is unique.

ie, $1 \wedge x = x, \forall\, x \in A$
$0 \vee x = x, \forall\, x \in A$

ie, 0 is identity for $\vee$
1 is identity for $\wedge$

Lemma 2: Let A be a Pre A* - algebra with 1 and 0 and let $x, y \in A$.

If $x \vee y = 0$, then $x = y = 0$

If $x \vee y = 1$, then $x \vee x^{\sim} = 1$

Proof

(i) Suppose $x \vee y = 0 \rightarrow$ (A)

Consider $x = 0 \vee x = (x \vee y) \vee x$ [By (A)]

$= x \vee (y \vee x)$ [By 1.1 (e)$^{\sim}$]

$= x \vee (x \vee y)$ [By 1.1 (c)$^{\sim}$]

$= (x \vee x) \vee y$ [By 1.1 (e)$^{\sim}$]

$= (x \vee y)$ [By 1.1 (b)$^{\sim}$]

$= 0$ [By (A)]

Therefore $x = 0$

Similarly we can prove that $y = 0$

(ii) Suppose $1 = x \vee y \rightarrow$ (B)

$= x \vee (x^{\sim} \wedge y)$ [By 1.1 (g)$^{\sim}$]

$= (x \vee x^{\sim}) \wedge (x \vee y)$ [By 1.1 (f)$^{\sim}$]

$= (x \vee x^{\sim}) \wedge 1$ [By (B)]

$= x \vee x^{\sim}$ [By Lemma 1]

$x \vee x^{\sim} = 1$

Theorem 2: Let A, be a Pre A* - algebra with 1 and $x, y \in A$.

If $x \wedge y = 0$, $x \vee y = 1$, then $y = x^{\sim}$

Proof: If $x \vee y = 1$, then $x \vee x^{\sim} = 1$ [By Lemma (2)]

$\Rightarrow x^{\sim} \wedge x = 0$ (By the duality)

Now $y = 1 \wedge y$

$= (x \vee x^{\sim}) \wedge y$

$= (x \wedge y) \vee (x^{\sim} \wedge y)$ [By 1.1 (f)]

$= 0 \vee (x^{\sim} \wedge y)$

$= (x^{\sim} \wedge x) \vee (x^{\sim} \wedge y)$

$= x^{\sim} \wedge (x \vee y)$ [By 1.1(f)]

$= x^{\sim} \wedge 1$

$= x^{\sim}$

Thus $y = x^{\sim}$

Theorem 3: Let$(A, \wedge, (-)^{\sim}, 1)$ be a Pre A* - algebra.
Then we have the following

i. Involution law:

$(a^{\sim})^{\sim} = a, \forall a \in A$

ii. $0^{\sim} = 1, 1^{\sim} = 0$

Proof: By 1.1 (a) we have (i),
(ii) Since we have

$0 \wedge 1 = 0, 0 \vee 1 = 1$

$1 \wedge 0 = 0, 1 \vee 0 = 1$

and By theorem 2, we have $0^{\sim} = 1, 1^{\sim} = 0$

Pre A* - algebra as a ring

Theorem 4: If $(A, \wedge, (-)^{\sim}, 1)$ is a Pre A* - algebra, then (A, +, ., 1) is a ring where +, . are defined as follows.

(i) $a + b = (a \wedge b^{\sim}) \vee (b \wedge a^{\sim})$, where

$a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$

(ii) $a . b = a \wedge b$

Proof: $(A, \wedge, (-)^{\sim}, 1)$ is a Pre A* - algebra
First we prove (A, +, ., 1) is a ring:
ie, we prove (i) (A, +) is an abelian group
(ii) (A, .) is a semi group
(iii) Distributive laws holds

Since + is a binary operation on A, + is closed.

Associative

Consider $a + (b + c) = a + \{(b \wedge c^{\sim}) \vee (c \wedge b^{\sim})\}$

$= [a \wedge \{(b \wedge c^{\sim}) \vee (c \wedge b^{\sim})\}^{\sim}] \vee$

$[(b \wedge c^{\sim}) \vee (c \wedge b^{\sim}) \wedge a^{\sim}]$

$= [a \wedge \{(b^{\sim} \vee c) \wedge (c^{\sim} \vee b)\}] \vee$

$[\{(b \wedge c^{\sim}) \vee (c \wedge b^{\sim})\} \wedge a^{\sim}]$

$= a \wedge \{[(c^{\sim} \vee b) \wedge b^{\sim}] \vee [(c^{\sim} \vee b) \wedge c]\} \vee$

$\{(b \wedge c^{\sim} \wedge a^{\sim}) \vee (c \wedge b^{\sim} \wedge a^{\sim})\}$

$= a \wedge [\{(b^{\sim} \wedge c^{\sim}) \vee ((b^{\sim} \wedge b)\} \vee$

$\{(c \wedge c^{\sim}) \vee (c \wedge b)\}] \vee$

$\{(b \wedge c^{\sim} \wedge a^{\sim}) \vee (c \wedge b^{\sim} \wedge a^{\sim})\}$

$= a \wedge \{(b^{\sim} \wedge c^{\sim}) \vee (c \wedge b)\} \vee$

$\{(b \wedge c^{\sim} \wedge a^{\sim}) \vee (c \wedge b^{\sim} \wedge a^{\sim})\}$

$= \{(a \wedge b^{\sim} \wedge c^{\sim}) \vee (a \wedge c \wedge b)\} \vee$

$\{(b \wedge c^{\sim} \wedge a^{\sim}) \vee (c \wedge b^{\sim} \wedge a^{\sim})\}$

Similarly we can show that

$(a+b)+c = \{(a \wedge b^{\sim} \wedge c^{\sim}) \vee (a \wedge c \wedge b)\} \vee$

$\{(b \wedge c^{\sim} \wedge a^{\sim}) \vee (c \wedge b^{\sim} \wedge a^{\sim})\}$

Therefore $a+(b+c) = (a+b)+c$, $\forall$ a, b, c $\in$ A

Therefore + is associative.

Let b be the additive identity in A

Then for every a$\in$ A there exists b$\in$A such that a+b=a

Therefore b is the additive identity in A

Let x be the additive inverse in A

Then for every a $\in$A there exists x$\in$A such that a+x=b where b is the additive identity in A

Hence 0 is the additive inversein A.

Now $a + b = (a \wedge b^{\sim}) \vee (b \wedge a^{\sim})$

$= (b \wedge a^{\sim}) \vee (a \wedge b^{\sim})$

$= b + a$

Therefore A, +) is an abelian group.

(ii) Since $a.b = a \wedge b$

$(a.b).c = (a \wedge b).c$

$= (a \wedge b) \wedge c$

$= a \wedge (b \wedge c)$ (Since A is a Pre A* - algebra By 1.1 (e))

$= a.(b.c)$

Therefore . is associative

Hence (A, .) is a semi-group.

(iii) Since A is a Pre A* - algebra,by 1.1(f) we have

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$

Thus distributive laws holds in A.

Therefore we have (A, +, . , 1) is a ring.

Definition: p is an integer. A ring (R, +,., 0) is called a p-ring if

$x^p = x, \forall x \in R,$

$px = 0, \forall x \in R$

Example: If p = 3 then (R, +, ., 0) is called 3 – ring.

Boolean ring: A ring (R, +, .) is said to be a Boolean ring if it satisfies the idempotent law ie, $x^2 = x, \forall x \in R$

Pre A* - algebra as a Boolean ring

Theorem 5: If $(A, \wedge, (-)^{\sim}, 1)$ is a Pre A* - algebra, then (A, +, ., 1) is a Boolean ring where +, . are defined as follows:

(i) $a + b = (a \wedge b^{\sim}) \vee (b \wedge a^{\sim})^{\sim}$, where

$a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$

(ii) $a . b = (a \wedge b)$

Proof: If $(A, \wedge, (-)^{\sim}, 1)$ is a Pre A* - algebra then (A, +, ., 1) is a ring

(Since By theorem 4)

Now $x^2 = x . x$

$= x \wedge x$

$= x$ (Since A is a Pre A* - algebra)

Therefore (A, +, ., 1) is a Boolean ring.

Theorem 6: If (A, +, ., 1) is a Boolean ring, then (A, $\wedge$, $(-)^{\sim}$, 1) is a Pre A* - algebra, where

$a^{\sim} = 1 - a$

$a \wedge b = [1-(1-a)][1-(1-b)]$

Proof

(A, +, ., 1) is a Boolean ring.

To Prove (A, $\wedge$, $(-)^{\sim}$, 1) is a Pre A* - algebra :

(a) $(x^{\sim})^{\sim} = 1 - x^{\sim}$

$= 1 - (1 - x)$

$= x, \forall x \in A$

(b) $x \wedge x = x . x$

$= x^2$

$= x$ (Since A is a Boolean ring)

Therefore $x \wedge x = x, \forall x \in A$

(c) $x \wedge y = [1-(1-x)][1-(1-y)]$

$= [1-(1-y)][1-(1-x)]$

$= y \wedge x$

Therefore $x \wedge y = y \wedge x, \forall x, y \in A$

(d) $(x \wedge y)^{\sim} = 1 - (x \wedge y)$

$= (1 - x) \wedge (1 - y)$

$= x^{\sim} \vee y^{\sim}$

Therefore $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall x, y \in A$

(e) Since A is a ring, we have

$x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in A$

(f) Since A is a ring, we have

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$

(g) $x \wedge (x^{\sim} \vee y) = x \wedge (x \wedge y^{\sim})^{\sim}$

$= x \wedge \{1 - (x \wedge y^{\sim})\}$

$= x \wedge \{1 - [x \wedge (1 - y)]\}$

$= x \wedge \{(1 - x) \vee [1 - (1 - y)]\}$

$= x \wedge \{(1 - x) \vee y\}$

$= [x \wedge (1 - x)] \vee (x \wedge y)$

$= (x \wedge x^{\sim}) \vee (x \wedge y)$

$= 0 \vee (x \wedge y)$ (Since $(x \wedge x^{\sim})=0, \forall x \in B(A)$)

$= (x \wedge y)$

Therefore $x \wedge y) = x \wedge (x^{\sim} \vee y), \forall x, y \in A$

Therefore $(A, \wedge, (-)^{\sim}, 1)$ is a Pre A* - algebra

Pre A*-algebra as 3-ring

Theorem 7: If $(A, \wedge, (-)^{\sim}, 1)$ is a Pre A* - algebra then (A, +, ., 1) is a 3-ring where +, . are defined as follows.

(i) $a + b = (a \wedge b^{\sim}) \vee (b \wedge a^{\sim})$, where

$a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$

(ii) $a . b = a \wedge b$

Proof

By theorem 4, we have (A, +, ., 1) is a ring.

To prove (A, +, . , 1) is a 3-ring,

We prove $x^3 = x, \forall x \in A$ & $3x = 0, \forall x \in A$

Now $x^3 = x^2 . x$

$= x^2 \wedge x$

$= x . x \wedge x$

$= (x \wedge x) \wedge x$

$= x \wedge x$ [Since By 1.1 (b)]

$= x$ [Since By 1.1 (b)]

Therefore $x^3=x$

Now $3x = x + x + x$

$= 2x + x$

$= 0+x$ (Since A is a Boolean ring)

$= (0 \wedge x^{\sim}) \vee (x \wedge 0)$

$= 0 \vee 0$

= 0

Therefore 3x = 0, ∀ x∈A

Hence A, +, ., 1) is a 3-ring.

Theorem 8: If (A, +, ., 1) is a 3 – ring, then (A, ∧, (-)˜, 1) is a Pre A* - algebra, where (i) a˜ = 1 – a

(ii) a ∧ b =-[1-(1-a)][1-(1-b)]

Proof: (A, +, ., 1) is a 3 – ring (given)

To Prove (A, ∧, (-)˜, 1) is a Pre A* - algebra :

a) (x˜)˜ = 1 - x˜ [by (i)]

= 1 – (1 – x)

= x

Thus (x˜)˜ = x, ∀ x ∈ A

b) x ∧ x = x.x

= x2

= x (Since A is aBoolean ring)

Therefore x ∧ x = x, ∀ x ∈ A

(c) x ∧ y = [1-(1-x)][1-(1-y)]

= [1-(1-y)][1-(1-x)]

=y ∧ x (By (ii))

Therefore x ∧ y = y ∧ x, ∀ x, y ∈ A

(d) (x ∧ y)˜ = 1 – (x ∧ y)

= (1 – x) ∨ (1 – y)

= x ˜ ∨ y ˜

Therefore (x ∧ y)˜ = x ˜ ∨ y ˜, ∀ x, y ∈ A

(e) Since A is a ring, x ∧ (y ∧ z) = (x ∧ y) ∧ z, ∀ x, y, z ∈ A

(f) Since A is a ring, we have

x ∧ (y ∨ z) = (x ∧ y) ∨ (x ∧ z), ∀ x, y, z ∈ A

(g) x ∧ (x˜ ∨ y) = x ∧ (x ∧ y˜)˜

$= x \wedge \{1 - (x \wedge y^{\sim})\}$

$= x \wedge \{1 - (x \wedge (1 - y)\}$

$= x \wedge \{(1 - x) \vee 1 -(1 - y)\}$

$= x \wedge \{(1 - x) \vee y\}$

$= [x \wedge (1 - x)] \vee (x \wedge y)$

$= (x \wedge x^{\sim}) \vee (x \wedge y)$

$= 0 \vee (x \wedge y)$ (Since $x \wedge x^{\sim} = 0, \forall x \in B(A)$)

$= x \wedge y$

Therefore $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x, y \in A$

Therefore $(A, \wedge, (-)^{\sim}, 1)$ is a Pre A* - algebra.

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