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# Deficient Quartic Spline Interpolation

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**Abstract**

In this paper, we have obtained a precise estimate of error bounds for deficient Quartic spline interpolant matching the given function values of the mesh point's and derivative at intermediate point's.

**Keywords:** Deficient Quartic splines, interpolation, Error bound's.

## Introduction

Cubic and higher degree splines are popular for smooth and more approximation of functions (see DeBour [4]). As in the study of linear splines the maximum error between a function and its interpolation can be controlled by mesh spacing but such functions have corner at the joints two linear pieces and therefore the usually require more data than higher order method to get desired accuracy. Starting with the pioneering concept of piecewise polynomial function Schoenberg [8] has studied various aspects of cubic interpolatory spline functions. Later, many authors have extended the study of cubic splines (see Brikoff and deBour [1,2 ], Sharma and Meir [9], Hall and Meyer and Rana [10]). In the direction of higher degree splines. Rana and Dubey [7] have obtained error bounds for quintic interpolatory spline (see also Howell and Verma [6]).

## Existence and Uniqueness

Let a mesh on [0,1] be given by

$$P:\{0 = x_o < x_1 < \cdots < x_n = 1\}$$

with $h = x_i - x_{i-1}$ for $i = 1,2,\ldots,n$. Let $\pi_k$ denote the set of all algebraic polynomials of degree not greater than k. and $s_i$ is the restriction of $s$ over $[x_{i-1}, x_i]$ the class $s(4, P)$ of deficient quartic splines is defined by

$$s(4,P)=\{s: s\in C^2[0,1], s_i\in \pi_4 \text{ for } i=1,2,\dots,n\}$$

Wherein $s^*(4,P)$ denoes the class of all deficient quartic splines $s(4,P)$ which satisfies the boundary conditions

$$s'(x_0)=f'(x_0), s'(x_n)=f'(x_n) \quad (2.1)$$

Writing $v_i=x_{i-1}+\theta h$ with $0<\theta<1$ and following:

**Problem 2.1:** Suppose $f'$ exists over $p$. There under what restriction on $\theta$ does there exists a unique spline interpolation $s\in S^*(4,p)$ of $f$ which satisfies the interpolatory conditions.

$$s(x_i)=f(x_i), i=0,1,\dots,n. \quad (2.2)$$

$$s'(v_i)=f'(v_i), i=1,2,\dots,n. \quad (2.3)$$

where $f'(x_i)'^s$ and $f'(v_i)'^s$ are given functional values and derivatives respectively. and boundary condition (2.1).

In order to investigate problem 2.1 we denote $1-\theta$ by $\theta^*$ and consider a quartic poynomilal $q(z)$ on [0,1] given by

$$q(z)=q(0)P_1(z)+q'(\theta)P_2(z)+q(1)P_3(z)+q'(0)P_4(z)+q'(1)P_5(z) \quad (2.4)$$

where

$$P_1(z)=\left[1-\frac{(1+2\theta+3\theta^2)}{\theta^2}z^2+\frac{2(1+\theta)^2}{\theta^2}z^3-\frac{(1+2\theta)}{\theta^2}z^4\right]$$

$$P_2(z)=[z^2-2z^3+z^4]/\,\theta^2{\theta^*}^2$$

$$P_3(z)=\left[\frac{\theta(3\theta-4)}{{\theta^*}^2}z^2+\frac{4-2\theta^2}{(\theta^*)^2}z^3+\frac{2\theta-3}{(\theta^*)^2}z^4\right]$$

$$P_4(z)=\left[z-\frac{(1+2\theta)}{\theta}z^2+\frac{2+\theta}{\theta}z^3-\frac{z^4}{\theta}\right]$$

$$P_5(z)=\left[\frac{\theta}{\theta^*}z^2-\frac{(1+\theta)}{\theta^*}z^3+\frac{z^4}{\theta}\right]$$

we are now set to answer Problem 2.1 in Theorem.

**Theorem 2.1.** There exists unique deficient quartic spline $S^*(4,p)$ which satisfies the interpolatory condition (2.2) - (2.3) and boundary conditions (2.1) if

$0<\theta<(\sqrt{33}+1)/16$ or if

$((15-\sqrt{33}/16)<\theta<1.$

**Proof of Theorem 2.1.** Let $t=(x-x_{i-1})/h.\ 0\le t\le 1.$ Then, in view of conditions (2.1) -(2.3).

We now express (2.4) in terms of the restriction $s_i$ of $s$ to $[x_{i-1}, x_i]$ as follows

$$s_i(x)=f(x_{i-1})P_1(t)+f'(v_i)P_2(t)+f(x_i)P_3(t)+h\,s_i'(x_{i-1})P_4(t)+h\,s_i'(x_i)P_5(t) \quad (2.5)$$

Since $s \in C^2[0,1]$, we have form (2.5)

$$J_{i-1}(\theta)s'(x_{i-1}) + J_i(\theta)s'(x_i) + J_{i+1}(\theta)s'(x_{i+1}) = F_i/h, \qquad i = 1,2,\ldots,n. \tag{2.6}$$

$$J_{i-1}(\theta) = -\frac{\theta^*}{\theta}, \quad J_i(\theta) = \frac{(3-2\theta)}{\theta^*} + \frac{(1+2\theta)}{\theta}, \quad J_{i+1}(\theta) = -\frac{\theta}{\theta^*}$$

where

$$F_i = -f(x_i)\left[-\frac{(3\theta^2 - 8\theta + 6)}{\theta^{*3}} + \frac{(1+2\theta+3\theta^2)}{\theta^2}\right] + f(x_{i-1})\left[\frac{\theta^*(3\theta+1)}{\theta^2}\right]$$
$$+ \frac{1}{\theta\theta^*}[f(\theta_{i+1}) - f(\theta_i)] + f(x_{i+1})\left[\frac{\theta(3\theta-4)}{\theta^{*3}}\right].$$

In order to prove theorem 2.1 we shall show that the system of equations (2.6) has a unique set of solution clear $J_{i-1}(\theta)$ is non positive for $0 < \theta \le 1/4$ and non negative for $1/4 \le \theta < 1$ and $J_{i+1}(\theta)$ is non positive for $0 \le \theta \le 3/4$ and non negative for $3/4 \le \theta < 1$.

Further, $J_i(\theta)$ is non negative for $0 < \theta \le 1/2$ and non positive for $1/2 \le \theta < 1$ . Thus, the excess of the absolute value of $J_i(\theta)$ over the sum of the absolute values of $J_{i-1}(\theta)$ and $J_{i+1}(\theta)$ is

$$12\theta(1 - 3\theta + 2\theta^2) = T_1(\theta), \quad say \quad 0 < \theta \le 1/4$$
$$3(1 - 9\theta^2 + 8\theta^3) = T_2(\theta), \quad say \quad 1/4 \le \theta \le 1/2$$
$$-2\theta(6 - 15\theta + 8\theta^2) = T_3(\theta), \quad say \quad 1/2 \le \theta \le 3/4$$
$$-12\theta(1 - 3\theta + 2\theta^2) = T_4(\theta), \quad say \quad 3/4 \le \theta \le 1$$

which turns out to be $> 0$ where $0 < \theta < (\sqrt{33}+1)/16$ and $(15-\sqrt{33})/16 < \theta < 1$. Thus, the coefficient matrix of the system of equations (2.6) is diagonally domain and hence invertible. This completes the proof of Theorem 2.1.

## Error Bounds

In this section, we obtain error bounds of the function $e^{(r)}(x) = f^{(r)}(x) - s^{(r)}(x)$ , $r = 0,1,2$. are estimate for the spline interpolate of theorem 2.1 which are based on the method used by Hall and Meyer [5]. We shall denote by $L_i[f,x]$ the unique deficient quartic agreeing with $f(x_i), f(x_{i+1}), f'(\theta_i), f'(x_i)$ and $f'(x_{i+1})$ and let $f \in C^2[0,1]$. Now consider a twice continuously differentiable quartic spline $s$ of theorem 2.1 we have for $x_{i-1} \le x \le x_i$

$$|f(x) - s(x)| = |f(x) - s_i(x)| \le |f(x) - L_i[f,x]| + |L_i[f,x] - s_i(x)|. \tag{3.1}$$

Thus, it is clear from (3.1) the in order to get the bounds of $e(x)$ we have to estimate point wise bounds of both the terms on the right hand side of (3.1). By a well known theorem of Cauchy [3]. We know that

$$|f(x) - L_i[f,x]| \le \left(\frac{h^5}{5!}\right)(1-t)^2(-t+\theta)\, t^2$$

where

$$t = (x - x_{i-1})/h \text{ and } F = \max_{0 \le x \le 1} |f^{(5)}(x)|$$

We next turn attention to estimate a similar bound for $|L_i[f,x] - s_i(x)]$. Thus, it follows from (2.4) that

$$|L_i[f,x] - s_i(x)| = h|e'(x_{i-1})|\,|P_4(t)| + h|e'(x_i)|\,|P_5(t)| \tag{3.3}$$

For sake of convenience consider $t \le \theta$, we observe that $P_4(t) \ge 0$ for $0 \le \theta \le 1$ and

$P_5(t) \ge 0$ for $1/2 \le \theta \le 1$.

Thus,

$$|P_4(t)| + |P_5(t)| = |P_4(t) + P_5(t)| = K_1(t,\theta), \quad 0 \le \theta \le \tfrac{1}{2}. \tag{3.4}$$

where

$$K_1(t,\theta) = \frac{1}{2\theta\,\theta^*(1-2\theta)}[2\theta\,\theta^*(1-2\theta)t + t^2\{-(1+\theta-8\theta^2)\theta^* + \theta^2(3-4\theta)\}$$
$$+2t^3\{(1-2\theta-2\theta^2)\theta^* - (1-2\theta^2)\theta\} + \{(2-3\theta)\theta - (1-3\theta)\theta^*\}t^4]$$

and

$$|P_4(t)| + |P_5(t)| = |P_4(t) - P_5(t)| = K_2(t,\theta), \quad \tfrac{1}{2} \le \theta \le 1. \tag{3.5}$$

where

$$K_2(t,\theta) = \frac{1}{2\theta\,\theta^*(1-2\theta)}[2\theta\,\theta^*(1-2\theta)t - t^2\{-(1+\theta-8\theta^2)\theta^* + \theta^2(3-4\theta)\}$$
$$+2t^3\{(1-2\theta-2\theta^2)\theta^*\} - t^4\{(1-3\theta)\theta^* - (2-3\theta)\theta\}]$$

Now using (3.4)-(3.5) in (3.3) we have

$$L_i[f,x] - s(x)| \le h\max\{|e'(x_{i-1})|.|e'(x_i)|\}K(t,\theta) \tag{3.6}$$

where $K(t,\theta) = \max\{|K_1(t,\theta)|, |K_2(t,\theta)|\}$

Setting

$$|e'(x_j)| = \max_{i=1,2,\dots,n} |e'(x_i)|,$$

we see that (3.6) may be rewritten as

$$|L_i(f,x) - s(x)| \leq h\,|e'(x_j)|K(t,\theta) \qquad (3.7)$$

It is clear from (3.7) that in order to estimate the bounds of $|e(x)|$ first we have to obtain upper bounds of $|e'(x_j)|$.

Replacing $s'(x_j)$ by $e'(x_j)$ in (2.6) it follows that

$$(1-\theta)^2(1-4\theta)e'(x_{j-1}) - (1+6\theta-24\theta^2+16\theta^3)e'(x_j)$$
$$+\theta^2(3-4\theta)e'(x_{j+1}) = E(f) \qquad (3.8)$$

where

$$E(f) = \frac{F_i}{h} + (1-\theta)^2(1-4\theta)f'(x_{j-1}) - (1+6\theta-24\theta^2+16\theta^3)f'(x_j)$$
$$+\theta^2(3-4\theta)f'(x_{j+1}) \qquad (3.9)$$

Observing that $E(f)$ is a linear functional which vanishes for polynomials of degree 4 or less, we have on an application of the peano theorem [3].

$$E(f) = \int_{x_{j-1}}^{x_{j+1}} \frac{f^{(5)}(y)}{14}\, E[(x-y)_+^4]\,dy \qquad (3.10)$$

From (3.9) it follows that

$$|E(f)| \leq \frac{1}{14}F\int_{x_{j-1}}^{x_{j+1}} |E[(x-y)_+^4]|dy \qquad (3.11)$$

Next, we observe that from (3.9) that for $x_{j-1} \leq y \leq x_{j+1}$

$$E[(x-y)_+^4] = 4(1+6\theta-24\theta^2+16\theta^3)\,(x_j-y)_+^3 + 4\theta^2(3-4\theta)(x_{j+1}-y)_+^3$$
$$-4[3\theta^*\theta^2\{(x_{j+1}-y)^4 - (x_j-y)_+^4\} - (\theta_{j+1}-y)_+^3 - 3\theta\theta^*(x_j-y)_+^4]\,/h$$
$$-\frac{(x_i-y)^4}{h}\left[-\frac{(3\theta^2-8\theta+6)}{\theta^{*2}} + \frac{(1+2\theta+3\theta^2)}{\theta^2}\right] + [(\theta_{i+i}-y)^4 - (\theta_i-y)^4]$$
$$+\frac{(x_{i+j}-y)^4}{h}\frac{(\theta(3\theta-y))}{\theta^{*2}}.$$

In order to evaluate the integral of the right hand side of (3.11) we rewrite the expression (3.12) in the symmetric form. Thus,

$$E[(x-y)_+^4 = \frac{4}{h}[3\theta\theta^{*2}(x_j-y)^4 \;-\theta^{*2}(1+8\theta)h(x_j-y)^3$$
$$+3h(1-(3-2\theta)\theta^2h)(x_j-y)^3 \;-3h^2(\theta^2h+(1-2\theta))(x_j-y)$$

$$+h^3((1-3\theta+3\theta^2)-h\theta^3)] \qquad x_{j-1}\le y\le \theta_j$$

$$=\frac{4}{h}[3\theta\theta^{*3}(x_j-y)^4-(1-\theta^{*3}(1+8\theta)h)(x_j-y)^3$$

$$+3\theta h(1+(2\theta-3)\theta\ \ h)(x_j-y)^2-3\theta^2h^2(1-h)(x_j-y)$$

$$+(1-h)\theta^3h^3] \qquad \theta_j\le y\le x_j$$

$$=\frac{4}{h}[-3\theta^*\theta^2(x_j-y)^4+(1+(8\theta-9)\theta^2h)(x_j-y)^3$$

$$+3\theta h(1+(2\theta-3)\theta\ \ h)(x_j-y)^2+3\theta^2h^2(1-h)(x_j-y)$$

$$+(1-h)\theta^3h^3] \qquad x_j\le y\le \theta_{j+1}$$

$$=-\frac{4}{h}\theta^2\left[\theta h+3\theta^*(x_j-y)\ \ \right](x_j-y+h)^3 \qquad \theta_{j+1}\le y\le x_{j+1}$$

Now integrating $|E[(x-y)_+^4]|$ over $[x_{j-1},x_{j+1}]$, we have

$$\int_{x_{j-1}}^{x_{j+1}}|E[(x-y)_+^4]|\,dy=\frac{h^5}{5}K'(\theta,h) \tag{3.13}$$

where

$$K'(\theta,h)=[\theta^{*3}h(12\theta(1-\theta^{*5})-5(1-\theta^{*4})+20(1+(2\theta-3)\theta^2h)(1-\theta^{*3})$$

$$-30(\theta^2h+(1-2\theta))(1-\theta^{*2})+20\theta(1-3\theta+3\theta^2-\theta^3h)$$

$$+h\theta^{*4}\left(12\theta\theta^{*3}+5(1-\theta^{*2}(1+8\theta)h)\right)+20\theta\theta^{*3}(1+(2\theta-3)\theta h$$

$$+10\theta\theta^*(1-h)(3\theta\theta^*+2\theta^2)+12\theta^7\theta^*-5\theta^4(1+(8\theta-9)\theta^2h)$$

$$-20\theta^4(1+(2\theta-3)\theta h)-10\theta^4(1-h)+12\theta^2\theta^{*6}h.]$$

Now (3.11) – (3.13) give us

$$|E(f)|\le\frac{Fh^5}{5!}|K'(\theta,h)|. \tag{3.14}$$

Thus, from (3.8) and (3.12), it follows that

$$|e'(x_j)|\le\frac{Fh^5}{5!}\frac{|K'(\theta,h)|}{12\theta(-\theta+\theta^*)\theta^*}$$

$$|e'(x_j)|\le\frac{Fh^5}{6!}K^*(\theta,h),$$

where

$$K^*(\theta,h)=\frac{|K'(\theta,h)|}{12\theta(-\theta+\theta^*)\theta^*}.$$

Now using (3.2), (3.7) along with (3.13) in (3.1), we have

$$|e(x)| \leq \frac{h^5}{5!}|t^2(\theta - t)(1-t)^2|F + F\frac{h^4}{5!}K^*(\theta, h)K(t,\theta)$$

$$\leq F\frac{h^5}{5!}|C(t)| \tag{3.15}$$

where

$$C(t) = \left[t^2(\theta - t)(1-t)^2 + \frac{K^*(\theta, h)K(t,\theta)}{6h}\right]$$

Thus, we prove the following

**Theorem 3.1:** Suppose $s(x)$ is the Quartic spline interpolation of theorem 2.1 interpolating a function $f(x)$ and $f \in C^5[0,1]$, thus

$$|e(x)| \leq F\frac{h^5}{5!}\max|f^5(x)| \tag{3.16}$$

where

$$K = \max_{0 \leq x \leq 1}|C(t)|.$$

Also, we have

$$|e'(x_i)| \leq F\frac{h^5}{6!}K^*(\theta, h) \tag{3.17}$$

where

$$K^*(\theta, h) = \frac{|K'(\theta, h)|}{2\theta(-\theta + \theta^*)\theta^*},$$

and $K'(\theta, h)$ obtained in (3.13).

Also, we have

$$|e'(x_i)| \leq K_1\frac{h^5}{6!}|f^5(x)|K^*(\theta, h) \tag{3.18}$$

where $K_1$ is positive constant. Equation (3.15) Proves (3.16). Inequality (3.17) is a direct consequence of (3.14).

Now, we shall show that inequality (3.16) is best possible in the limit Considering $f(x) = \frac{x^5}{5!}$ and using the Cauchy formula (Davis) [3], we have

$$\frac{x^5}{5!} - L_t\left[\frac{t^5}{5!}, x\right] = \frac{h^5}{5!}(1-t)^2 t^2(\theta - t) \tag{3.19}$$

Moreover, for the function under consideration (3.8) gives the following for equally spaces knot's

$$\left(\frac{h^5}{5!}\right) = (1-\theta)^2(1-4\theta)e'(x_{j-1}) + (-1-6\theta+24\theta^2-16\theta^3)e'(x_j) + \theta^2(3-4\theta)$$

$$= \frac{h^5}{5!}K'(\theta,h). \tag{3.20}$$

Consider for a moment.

$$e'(x_{j-1}) = e'(x_j) = e'(x_{j+1}) = \frac{h^5}{6!}\frac{K'(\theta,h)}{2\theta\theta^*(1-2\theta)} = \frac{h^5}{6!}K^*(\theta,h) \quad \text{(say)} \tag{3.21}$$

We have from (3.4)

$$L_i[f,x] - s(x) = \frac{h^4}{6!}K^*(\theta,h) \tag{3.22}$$

Combining (3.19) with (3.22), we get for $x_i \le x \le x_{i+1}$.

$$f(x) - s(x) = \frac{h^4}{6!}K^*(\theta,h) + \frac{h^5}{5!}(1-t)^2t^2(\theta-t)$$

$$= \frac{h^5}{5!}\left[\frac{K'(\theta,h)}{6h} + (1-t)^2t^2(\theta-t)\right] \tag{3.23}$$

Form (3.23), it is clearly observed that (3.15) is the best possible provided we could prove that

$$e'(x_{i-1}) = e'(x_{i+1}) = e'(x_i) = \frac{h^5}{6!}K^*(\theta,h) \tag{3.24}$$

In fact (3.24) is attained only in the limit the difficulty will take place in the case of boundary condition i.e. $e'(x_0) = e'(x_n) = 0$, However it can be shown that as we move many subintervals away from the boundaries $e(x_i) \to \frac{h^5}{6!}, K'(\theta,h)$. For that we shall apply (3.20) inductively move away from the end condition $e'(x_0) = e'(x_n) = 0$. The first step in this direction is to show that $e'(x_i) \ge 0$, for $i = 0,\dots,n,$ which can be proved by contradictory assumption.

Let $e(x_i) < 0$, for some $i$, $i = 1,2,\dots,n-1$, Now making a use of (3.17), we get

$$\frac{h^5}{6!}K^*(\theta,h) \ge \max|e(x_j)|$$

$$\ge \theta^{*2}(1-4\theta)e'(x_{j-1}) + \theta^2(3-4\theta)e'(x_{j+1})$$

$$\ge \theta^*(1-4\theta)e'(x_{j-1}) + \theta(3-4\theta)e'(x_{j+1}) + (1+6\theta-24\theta^2+16\theta^3)e'(x_j)$$

$$> \frac{h^3}{5!}\, K'(\theta, h)$$

$$\Rightarrow K^*(\theta, h) > 6K'(\theta, h)$$

This is a contradiction
so $e'(x_j) \geq 0$ for $i = 0, \ldots, n$.

From (3.20), we can write

$$(1 + 6\theta - 24\theta^2 + 16\theta^3) e'(x_j) = \frac{h^3}{5!} K'(\theta, h) + \theta^{2}(1 - 4\theta) e'(x_{j-1}) + \theta^2(3 - 4\theta) e'(x_{j+1})$$

$$\Rightarrow e'(x_j) \leq \frac{h^3}{6!} \frac{K'(\theta, h)}{K(\theta)} \qquad (3.25)$$

where $K(\theta) = 1 + 6\theta - 24\theta^2 + 16\theta^3$.
Hence again using (3.25) in (3.20), we have

$$e'(x_j) \leq \frac{h^3}{6!} \frac{K'(\theta, h)}{K(\theta)} \left[1 + \frac{K_1(\theta)}{K(\theta)} + \frac{K_1^2(\theta)}{K^2(\theta)}\right]$$

Repeated use of (3.25), we get

$$e'(x_j) \leq \frac{h^3}{6!} \frac{K^*(\theta, h)}{K(\theta)} \left[1 + \frac{K_1(\theta)}{K(\theta)} + \left(\frac{K_1(\theta)}{K(\theta)}\right)^2 + \cdots\right] \qquad (3.26)$$

Now, it can be seen easily that the r.h.s. of (3.26) $\rightarrow \frac{h^3}{6!} K^*(\theta, h)$ and hence limiting case

$$e'(x_i) \leq \frac{h^3}{6!} K^*(\theta, h)$$

which verify proof of (3.17).
Hence in the limiting case when $f(x) = \frac{x^5}{5}$ from (3.26) $e'(x_i) \rightarrow \frac{h^3}{3!} K^*(\theta, h)$.
This proves Theorem 3.1.

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