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Effect of Rotation on Thermal Stability of Superposed Fluid and Porous Layer

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Abstract

In this paper linear stability of a viscous incompressible fluid saturated porous medium under the influence of rotation is investigated. Closed form solutions of velocity, temperature and fluid vorticity in terms of wave number as perturbation parameter have been obtained. The influence of various non-dimensional parameters such as Taylor number, Grashof number, Prandtl number, Darcy number, porosity and wave number on stability characteristics of flow field are discussed numerically.

Keywords: Rotation, convection, porous layer, linear stability analysis.

I. INTRODUCTION

Thermal convection in a rotating porous layer has attracted several researchers due to its importance in astrophysics, geophysics and its applications in many engineering areas. Onset of convection in a rotating porous layer in presence of variable viscosity using both Brinkman and Darcy model has been investigated by Patil and Vaidyanathan [8].

Jou and Liaw [6] analyzed the onset of transient convection in a rotating porous layer by taking into account friction and drag. The problem of hydrodynamic stability of a rotating fluid bounded by a saturated porous medium which is placed at a distance from the axis of rotation subject to centrifugal body force in the absence of gravity has been investigated by Vadasz [11].

The effect of rotation on onset of thermal convection of micropolar fluid in porous medium has been reported by Sharma and Kumar [10]. They concluded that permeability has a stabilizing effect on stationary convection and effect of rotation over stabilizes the system. Vadasz and Olek [12] carried out a study on instability of thermal convection in a rotating porous medium by employing Darcy model with time derivative.

Using Brinkman model, Desaive *et al.* [3] analyzed the effect of Coriolis force on linear stability of free convection in a saturated porous medium which heated from below. Malashetty *et al.* [7] investigated the thermal convection of a rotating anisotropic porous medium using both linear and nonlinear stability analysis.

Allehiany and Abdullah [1] studied the thermal convection of an electrically and thermally conducting viscous incompressible fluid in presence of vertical magnetic field and uniform vertical rotation. Banjar and Abdullah [2] analyzed the effect of Coriolis force on stability of thermal convection in a horizontal fluid layer overlying a porous layer modeled by Brinkman equation and found that marginal convection is bi-modal in nature as convection is dominated by fluid or porous medium depending on depth ratio and effect of rotation.

Saravanan and Brindha [9] investigated the onset of convection in a rotating fluid saturated porous medium of thermal non-equilibrium which is heated at right boundary and cooled at left boundary. Gaikwad and Kamble [4] carried out a theoretical study to analyze the linear stability of double diffusive convection in a rotating anisotropic porous medium in presence of Soret effect.

Hirata *et al.* [5] examined the bimodal nature of onset of convection using linear stability analysis on an incompressible viscous fluid overlying a porous layer. Hence in this paper, the work of Hirata *et al.* [5] has been extended to study the stability of thermal convection of an incompressible viscous fluid in the presence of rotation using method of small oscillations and the analysis is restricted to long wave approximations.

II. MATHEMATICAL FORMULATION

Consider an infinite horizontal rotating incompressible fluid bounded by a saturated porous layer on a fluid layer which is maintained at different constant temperatures T_u and T_l at both the upper wall and lower wall. The rotating frame of reference rotates along the vertical axis with an angular velocity$(0,0,\Omega)$. Boussinesq approximation is employed and variations of density due to temperature is assumed to be $\rho(T) = [1 - \beta_T(T - T_0)]$ where thermal expansion coefficient $\beta_T \geq 0$.

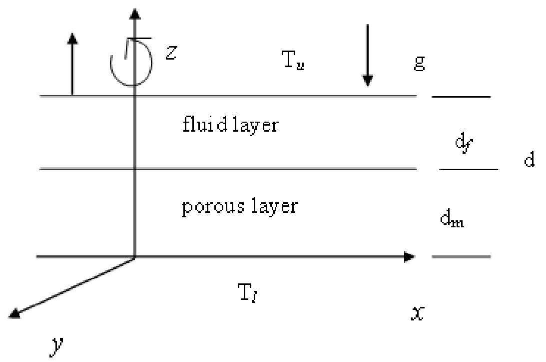

Figure 1. Geometrical Description of the Flow

Under the flow assumptions the governing equations takes the form

$$\nabla.\vec{u} = 0 \tag{1}$$

$$\rho_0\left[\frac{\partial}{\partial t}\left(\frac{\vec{u}}{\phi}\right)+\frac{1}{\phi}\left(\vec{u}.\frac{\nabla\vec{u}}{\phi}\right)+\frac{2}{\phi}(\Omega\times\vec{u})\right] = -\nabla p-\frac{\mu}{K}\vec{u}+\rho g+\frac{\mu_{eff}}{\mu_f}\nabla^2\vec{u} \tag{2}$$

$$\frac{\partial\vec{T}}{\partial t}+\left(\vec{u}.\nabla\vec{T}\right) = \nabla.\left(\alpha\nabla\vec{T}\right) \tag{3}$$

with boundary conditions

$$\vec{u}=\frac{\partial\vec{u}}{\partial z}=\vec{T}=0 \text{ at } z=0 \text{ and } z=d \tag{4}$$

where $\vec{u}$, ρ, p, g ,T, Ω , ρ_0, K, ϕ, μ_{eff}, μ_f, α respectively denote the velocity vector, density, pressure, acceleration due to gravity, temperature, angular velocity, density at reference level, permeability of the medium, porosity, effective viscosity of the porous medium, dynamic viscosity of the fluid and thermal diffusivity.

In quiescent state, basic state flow field are given by $\vec{u}^*=(0,0,0)$, $p^*=p(z)$ and $T^*=T(z)$ and so the temperature field using the boundary conditions becomes

$$T=z+\frac{T_l-T_0}{T_u-T_l} \tag{5}$$

Let the small disturbance in the initial states of velocity, temperature and pressure respectively be denoted by $\vec{u}'(x,z,t)$, $T'(x,z,t)$ and $p'(x,z,t)$. Then the linearized perturbed equations (1) – (3) become

$$\nabla.\vec{u}' = 0 \tag{6}$$

$$\frac{\partial}{\partial t}\left(\frac{\vec{u}'}{\phi}+\frac{2}{\phi}(\Omega\times\vec{u}')\right) = -\frac{1}{\rho_0}\nabla p'-\frac{\nu}{k}\vec{u}'+g\beta_T T'\hat{k}+\frac{1}{\phi}\nabla^2\vec{u}' \tag{7}$$

$$\frac{\partial T'}{\partial t}+(\vec{u}'.\nabla T^*) = \nabla.(\alpha\nabla T') \tag{8}$$

By eliminating pressure term and introducing the non-dimensional variables for length, velocity, time and temperature respectively as follows

$$z=z^*d,\ \ w=w^*\frac{\nu}{d},\ \ t=t^*\frac{d^2}{\nu}, T'=\Delta T\, T^*$$

The linearized system of equations becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{\phi}\nabla^2 w\right)+Ta^{1/2}\frac{\partial\zeta}{\partial z} = -\frac{1}{Da}\nabla^2 w+Gr_T\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{1}{\phi}\nabla^4 w \tag{9}$$

$$\frac{\partial T}{\partial t}+W = \frac{\hat{d}}{P_r}\nabla^2 T \tag{10}$$

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{\phi}\right) = -\frac{1}{Da}\zeta+\frac{1}{\phi}\nabla^2\zeta+Ta^{1/2}\frac{\partial w}{\partial z} \tag{11}$$

where $Da=k/d^2$ (Darcy number), $Gr_T=g\beta_T\rho_0\Delta T d^3/\nu^2$ (Thermal Grashof number)

$P_r=\nu/\alpha_f$ (Prandtl number), $Ta=\left(\frac{2\Omega d^2}{\phi\nu}\right)^2$ (Taylor number)

Also it is assumed that the special variations of α, ϕ and k as null.

Applying normal mode analysis to the dependent variables

$$(\mathrm{w}, \mathrm{T}, \zeta) = \big(\mathrm{W}(\mathrm{z}), \theta(\mathrm{z}), \mathrm{G}(\mathrm{z})\big)e^{(ik_1x+ik_2y+\sigma t)}$$

where $k = \sqrt{k_1^2 + k_2^2}$ is the non-dimensional wave number and σ the growth rate . Substituting the above expression into equations (9) – (11) we get

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\left(\frac{\partial^2}{\partial z^2} - k^2 - \frac{\phi}{Da} - \sigma\right)W = k^2\phi Gr_T\theta + \phi Ta^{1/2}\frac{\partial}{\partial z}G \tag{12}$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 - \sigma\frac{pr}{\hat{d}}\right)\theta = \frac{pr}{\hat{d}}W \tag{13}$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 - \frac{\phi}{Da} - \sigma\right)G = -\phi Ta^{1/2}\frac{\partial}{\partial z}W \tag{14}$$

with the corresponding boundary condition

$$W = \frac{\partial W}{\partial z} = \theta = G = 0 \ \text{ at } \ z = 0 \text{ and } z = 1 \tag{15}$$

III. EIGEN VALUES AND EIGEN FUNCTIONS

Now we expand W, σ, θ and G, in powers of k

$$\left.\begin{aligned} W &= W_0 + k^2W_1 + k^4W_2 + \cdots \\ \sigma &= \sigma_0 + k^2\sigma_1 + k^4\sigma_2 + \cdots \\ \theta &= \theta_0 + k^2\theta_1 + k^4\theta_2 + \cdots \\ G &= G_0 + k^2G_1 + k^4G_2 + \cdots \end{aligned}\right\} \tag{16}$$

Substituting (16) in equations (12) to (14) and collecting the like powers of k we get

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right)W_0 = \phi Ta^{1/2}\frac{\partial}{\partial z}G_0 \tag{17}$$

$$\left(\frac{\partial^2}{\partial z^2} - \sigma_0\frac{Pr}{\hat{d}}\right)\theta_0 = \frac{Pr}{\hat{d}}W_0 \tag{18}$$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right)G_0 = -\phi Ta^{1/2}\frac{\partial}{\partial z}W_0 \tag{19}$$

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right)W_1 = {} & (1+\sigma_1)\frac{\partial^2}{\partial z^2}W_0 + \left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right)W_0 \\ & + \phi Gr_T\theta_0 + \phi Ta^{1/2}\frac{\partial}{\partial z}G_1\end{aligned} \tag{20}$$

$$\left(\frac{\partial^2}{\partial z^2} - \sigma_0\frac{Pr}{\hat{d}}\right)\theta_1 = \frac{Pr}{\hat{d}}W_1 + \left(1 + \sigma_1\frac{Pr}{\hat{d}}\right)\theta_0 \tag{21}$$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right) G_1 = -\phi Ta^{1/2} \frac{\partial}{\partial z} W_1 + (1 + \sigma_1) G_0 \tag{22}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} &W_0 = DW_0 = \theta_0 = G_0 = 0 \ \text{ at } \ z = 0 \text{ and } z = 1 \\ &W_1 = DW_1 = \theta_1 = G_1 = 0 \ \text{ at } \ z = 0 \text{ and } z = 1 \end{aligned}\right\} \tag{23}$$

On solving equations using boundary conditions we get

$$W_0 = A_1 \cosh(rz) + A_2 \sinh(rz) + A_3 \cosh(r_1 z) + A_4 \sinh(r_1 z)$$

$$\theta_0 = A_7 \cosh(r_3 z) + A_8 \sinh(r_3 z) + f_6 \cosh(rz) + f_7 \sinh(rz) + f_8 \cosh(r_1 z) + f_9 \sinh(r_1 z)$$

$$G_0 = A_5 \cosh(r_2 z) + A_6 \sinh(r_2 z) + f_2 \sinh(rz) + f_3 \cosh(rz) + f_4 \sinh(r_1 z) + f_5 \cosh(r_1 z)$$

$$W_1 = A_{13} \cosh(rz) + A_{14} \sinh(rz) + A_{15} \cosh(r_1 z) + A_{16} \sinh(r_1 z) + f_{26} z \cosh(r_2 z) + f_{26} \sinh(r_2 z) + f_{27} \cosh(r_3 z) + f_{28} \sinh(r_3 z)$$

$$\theta_1 = A_{19} \cosh(r_3 z) + A_{20} \sinh(r_3 z) + f_{67} \sinh(rz) + f_{66} \cosh(rz) + f_{69} \sinh(r_1 z) + f_{68} \cosh(r_1 z) + f_{70} z\sinh(r_2 z) + f_{71} \cosh(r_2 z) + f_{72}\, z\cosh(r_2 z) + f_{73} \sinh(r_2 z) + f_{74}\, z\sinh(r_3 z) + f_{75}\, z\cosh(r_3 z)$$

$$G_1 = A_{17} \cosh(r_2 z) + A_{18} \sinh(r_2 z) + f_{46} \sinh(rz) + f_{47} \cosh(rz) + f_{48} \sinh(r_1 z) + f_{49} \cosh(r_1 z) + f_{50} z^2 \cosh(r_2 z) + f_{51} z^2 \sinh(r_2 z) + f_{52} z\cosh(r_2 z) + f_{53} z\sinh(r_2 z) + f_{54} \sinh(r_3 z) + f_{55} \cosh(r_3 z)$$

The zeroth order eigen values are given by the following transcendental equation

$$A_3[\cosh(r_1) - \cosh(r)] + A_4 \left[\sinh(r_1) - \left(\frac{r_1}{r}\right) \sinh(r)\right] = 0$$

$$A_3[r_1 \sinh(r_1) - r\sinh(r)] + A_4[r_1 \cosh(r_1) - r_1 \cosh(r)] = 0$$

The solution of the above expression will not give explicit values of σ_0. Hence the values of σ_0 is obtained using Mathematica 8.0.

The first order approximations of the growth rate is given by

$$\sigma_1 = -\frac{f_{34}}{f_{33}}$$

IV. RESULTS AND DISCUSSION

To get physical insight into the influence of rotation on thermal stability of an viscous fluid bounded by saturated porous medium, the effects of various non-dimensional parameters such as Taylor number Ta, Grashof number Gr, Prandtl number Pr, Darcy number Da and porosity ϕ on temporal growth rate, velocity, temperature and vorticity has been discussed numerically and plotted in figures (2) – (21). We have fixed values of parameter such as $\mathrm{Da} = 0.0001, \mathrm{Ta} = 10.0, \phi = 1.0,\ \mathrm{Pr} = 0.71,\ \mathrm{d} = 0.08,\ \ \mathrm{Gr} = 5.0,\ \mathrm{k} = 0.9$ throughout the entire study of the problem.

Figures (2) – (5) depict the effect of Taylor number Ta and porosity ϕ on growth rate and it is found that increase in Taylor number creates stable mode in the system and porosity induces both stability/ instability in the system.

The influence of Prandtl number Pr and Grashof number Gr on growth rate is shown in figures (6) and (7) and it is observed that increase in Prandtl number and Grashof number tend to stabilize of the system.

Increase in Darcy number tends to increase the instability of the system as depicted in figures (8) and (9). In figures (10) and (11) the effect of wave number k and porosity ϕ with increase in Darcy number on frequency is illustrated and it is found that at Da = 3.0 the instability increases drastically and becomes stable for other values of Darcy number.

Figures (12) and (13) represents the effect of porosity and Taylor number with increase in Grashof number on temporal growth rate and it is inferred that increase in porosity and Taylor number decreases instability of the system.

The effect of Taylor number Ta and porosity ϕ on velocity profile is shown in figures (14) and(15). It is seen that increase in Taylor number and porosity increases the velocity profile.

Increase in Taylor number and porosity decreases the temperature profile as illustrated in figures (16) and (17).

Figures (18) and (19) represents the behavior of fluid vorticity with increase in Taylor number and porosity and it is found that both Taylor number and porosity decreases the vorticity profile.

V. CONCLUSION

We have investigated the linear stability of thermal convection in a rotating viscous fluid which is heated from below and which is confined between infinite horizontal plates using method of small oscillation and the effects of various non-dimensional parameters on characteristics of the flow has been analysed. The following interpretations were made from the findings.

- Taylor number plays a significant role in stabilizing the system and found in agreement with result of Gaikwad and Kamble [4].

- Increase in porosity induces both stability/ instability in the system.
- Increase in Darcy number tends to increase the instability of the system.
- It is found that at Da = 3.0 the instability increases drastically and becomes stable for other values of Darcy number.
- Increase in porosity and Taylor number decreases instability of the system.
- It is seen that increase in Taylor number and porosity enhances the velocity profile.

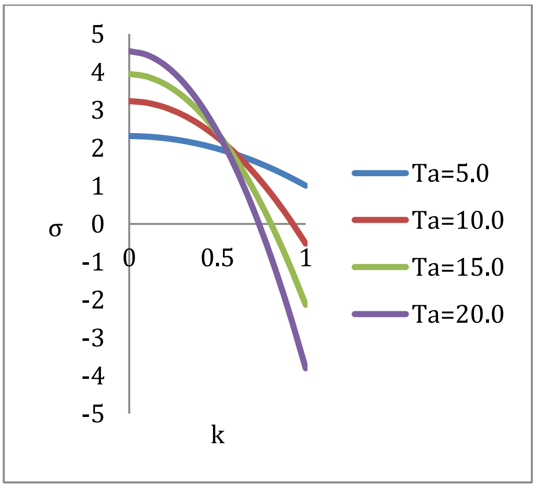

Figure 2. Effect of Taylor number on Growth rate

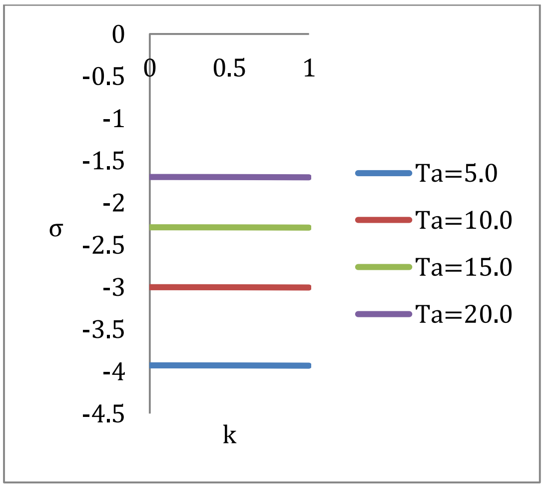

Figure 3. Effect of Taylor number on Growth rate

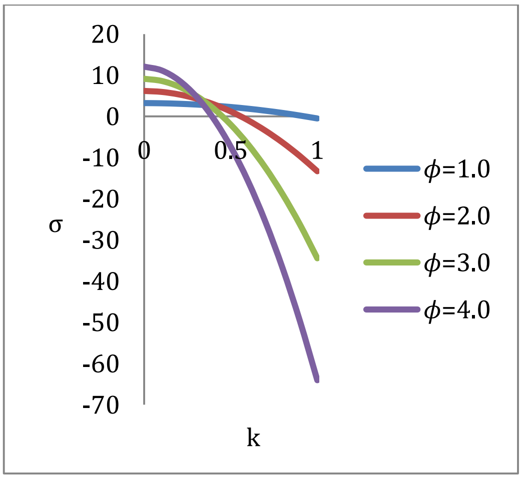

Figure 4. Effect of Porosity on Growth rate

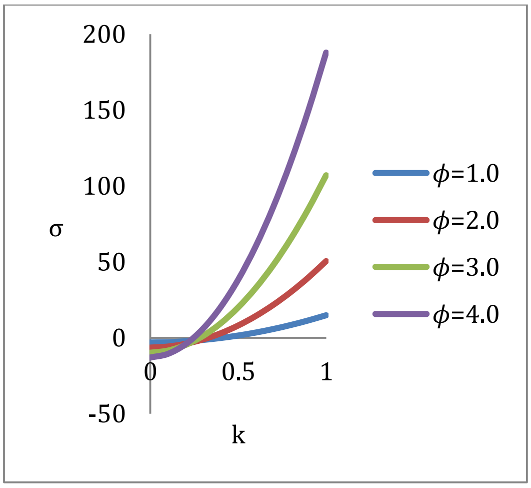

Figure 5. Effect of porosity on Growth rate

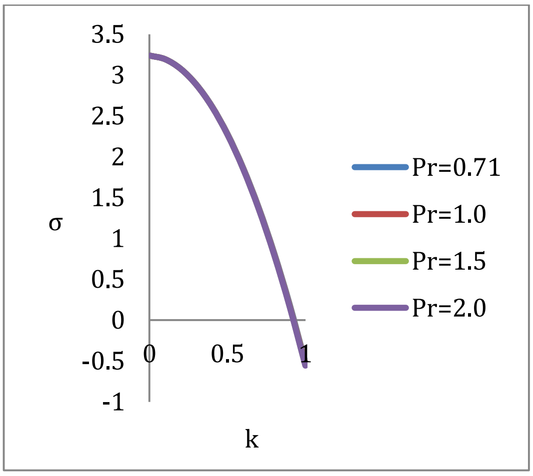

Figure 6. Effect of Prandtl number on Growth rate

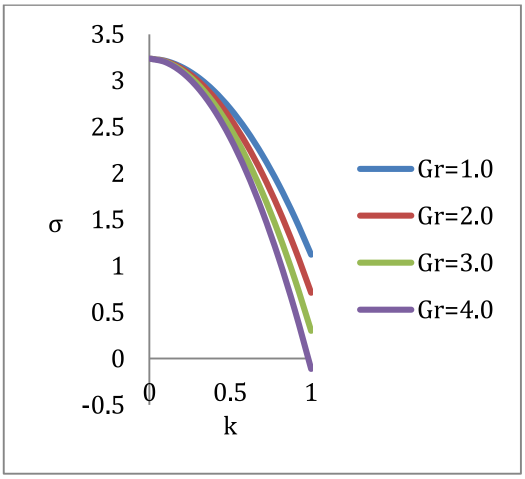

Figure 7. Effect of Grashof number on Growth rate

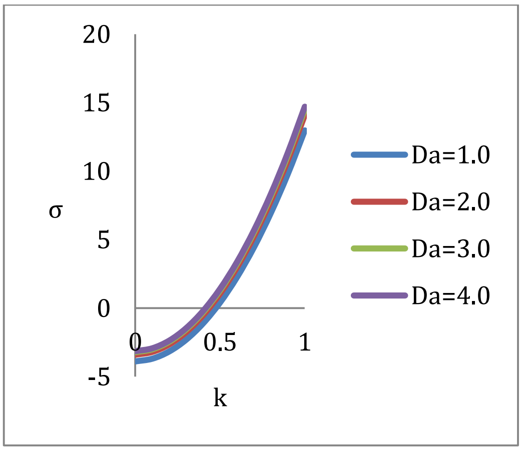

Figure 8. Effect of Darcy number on Growth rate

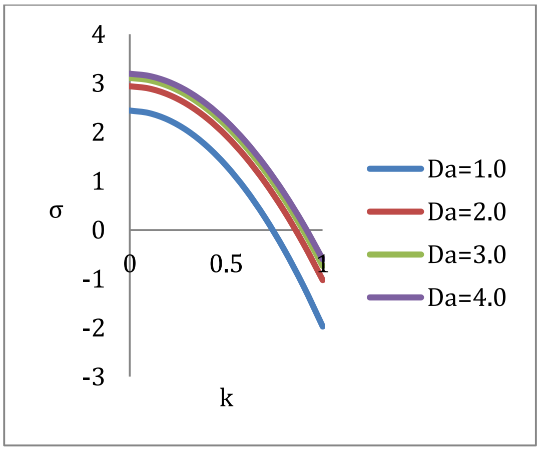

Figure 9. Effect of Darcy number on Growth rate

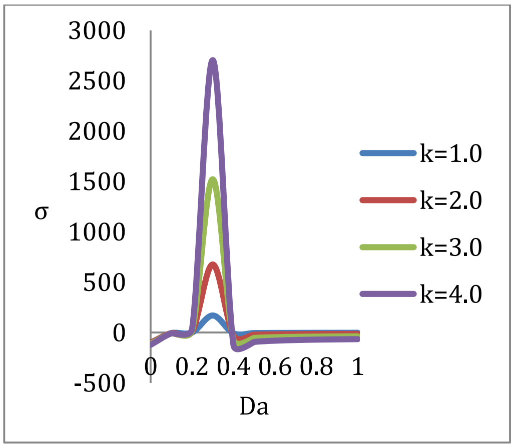

Figure 10. Effect of wave number on Growth rate

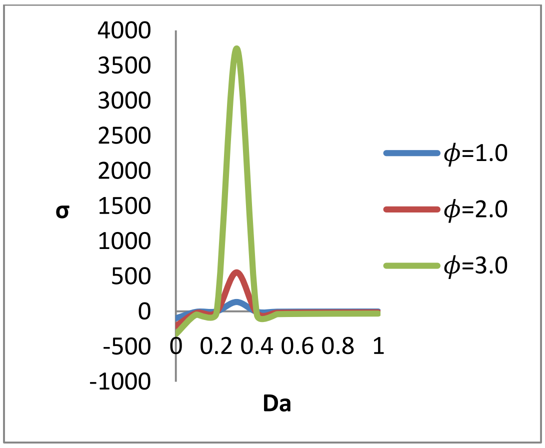

Figure 11. Effect of porosity on Growth rate

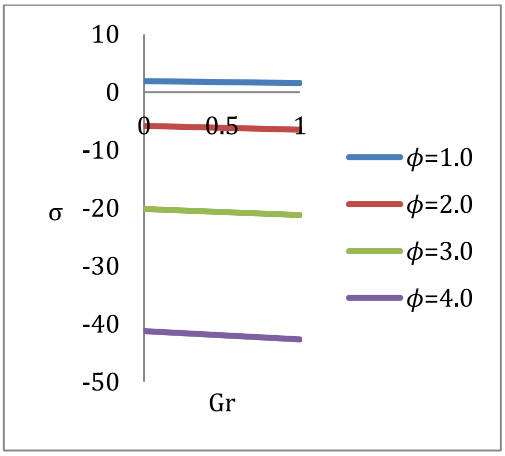

Figure 12. Effect of porosity on Growth rate

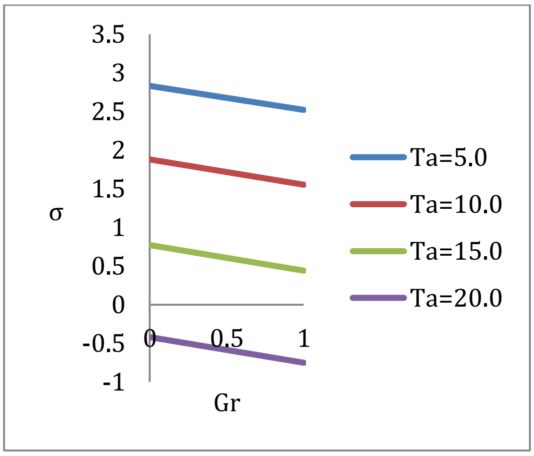

Figure 13. Effect of Taylor number on Growth rate

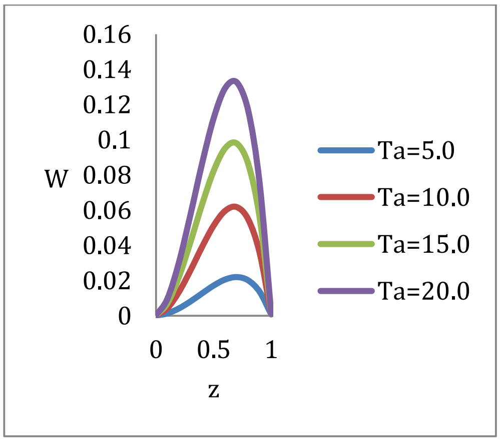

Figure 14. Effect of Taylor number on Velocity profile

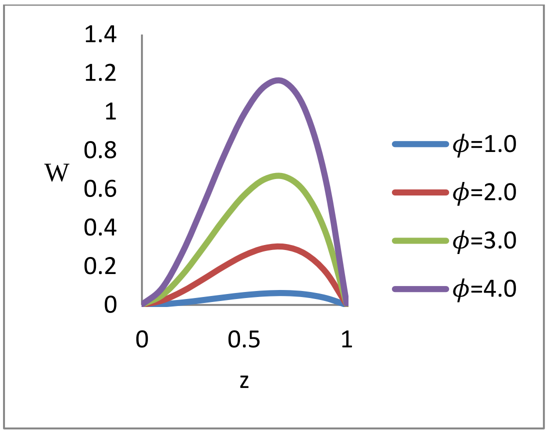

Figure 15. Effect of porosity on Velocity profile

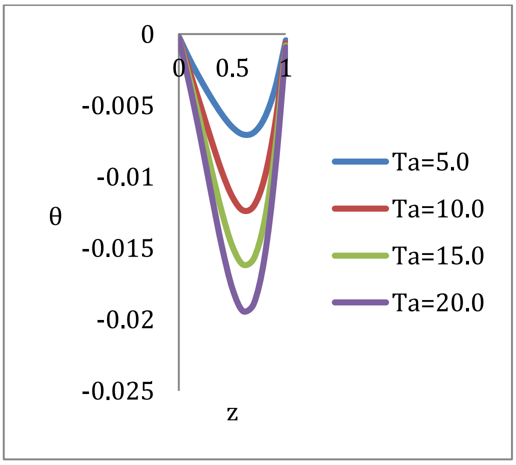

Figure 16. Effect of Taylor number on temperature profile

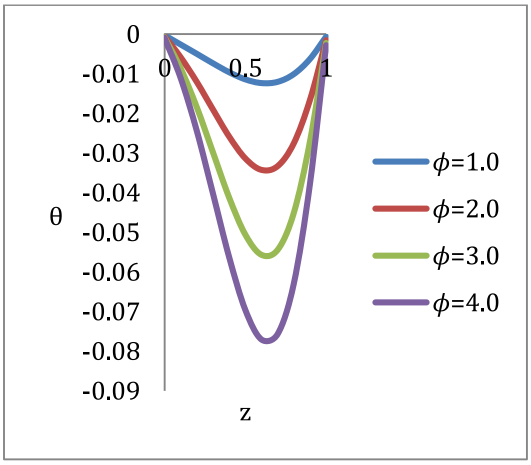

Figure 17. Effect of porosity on temperature profile

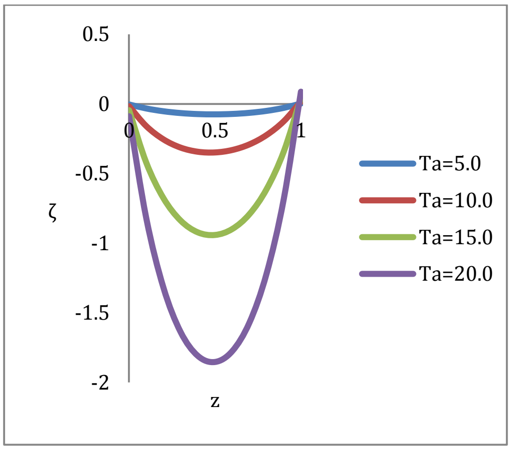

Figure 18. Effect of Taylor number on fluid vorticity

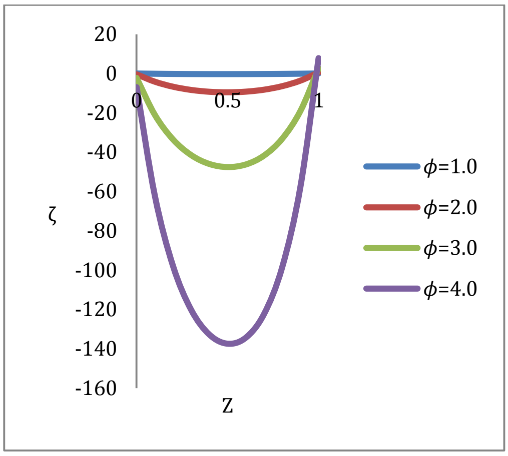

Figure 19. Effect of porosity on fluid vorticity

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APPENDIX

$r = (\frac{\phi}{Da} + \sigma_0) + i\phi\sqrt{Ta}$; $r_1 = \sqrt{r}$; $A_1 = 1$; $A_3 = -1$; $A_2 = \frac{-r_1}{r} A_4$; $A_4 = \frac{\cosh(r_1) - \cosh(r)}{\sinh(r_1) - \frac{r_1}{r}\sinh(r)}$;

$r_2 = \sqrt{\sigma_0 + \frac{\phi}{Da}}$; $f_1 = -Ta^{1/2}\phi$; $A_5 = -(f_3 + f_5)$;

$A_6 = -\frac{1}{\sinh(r_2)}\{A_5 \cosh(r_2) + f_2 \sinh(r) + f_3 \cosh(r) + f_4 \sinh(r_1) + f_5 \cosh(r_1)\}$;

$r_3 = \sqrt{\frac{\sigma_0 Pr}{d}}$; $f_6 = \frac{\frac{Pr}{d}A_1}{r^2 - r_3{}^2}$; $f_7 = \frac{\frac{Pr}{d}A_2}{r^2 - r_3{}^2}$; $f_8 = \frac{\frac{Pr}{d}A_3}{r_1{}^2 - r_3{}^2}$; $f_6 = \frac{\frac{Pr}{d}A_4}{r_1{}^2 - r_3{}^2}$;

$A_7 = -(f_6 + f_8)$; $A_8 = -\frac{1}{\sinh(r_3)}\{A_7 \cosh(r_3) + f_6 \cosh(r) + f_7 \sinh(r) + f_8 \cosh(r_1) + f_9 \sinh(r_1)\}$; $f_{10} = Gr\phi$; $f_{11} = \phi Ta^{1/2}$; $f_{12} = -Gr\phi r_2{}^2$;

$f_{13} = r^4 A_1 - 2r_2{}^2 r^2 A_1 + r_2{}^4 A_1 + f_{10} r^2 f_6 + f_{12} f_6$; $f_{14} = r^4 A_1 - r_2{}^2 r^2 A_1 + f_{11} r f_2$;

$f_{15} = r^4 A_2 - 2r_2{}^2 r^2 A_2 + r_2{}^4 A_2 + f_{10} r^2 f_7 + f_{12} f_7$; $f_{16} = r^4 A_2 - r_2{}^2 r^2 A_2 + f_{11} r f_3$;

$f_{17} = r_1{}^4 A_3 - 2r_2{}^2 r_1{}^2 A_3 + r_2{}^4 A_3 + f_{10} r_1{}^2 f_7 + f_{12} f_8$; $f_{18} = r_1{}^4 A_3 - r_2{}^2 r_1{}^2 A_3 + f_{11} r_1 f_4$;

$f_{19} = r_1{}^4 A_4 - 2r_2{}^2 r_1{}^2 A_4 + r_2{}^4 A_4 + f_{10} r_1{}^2 f_9 + f_{12} f_9$; $f_{20} = r_1{}^4 A_4 - r_2{}^2 r_1{}^2 A_4 + f_{11} r_1 f_5$;

$f_{21} = f_{12} A_7 + f_{10} r_3{}^2 A_7$; $f_{22} = f_{12} A_8 + f_{10} r_3{}^2 A_8$; $f_{23} = r^2(r^2 - r_2{}^2)$;

$f_{24} = r_1{}^2(r_1{}^2 - r_2{}^2)$; $f_{25} = \frac{f_{11} A_5}{2r_2{}^2}$; $f_{26} = \frac{f_{11} A_6}{2r_2{}^2}$; $f_{27} = \frac{f_{21}}{r_3{}^2(r_3{}^2 - r_2{}^2)}$; $f_{28} = \frac{f_{22}}{r_3{}^2(r_3{}^2 - r_2{}^2)}$

$A_{13} = (f_{13} + (1 + \sigma_1) f_{14}) f_{23}$; $A_{14} = (f_{15} + (1 + \sigma_1) f_{16}) f_{23}$; $A_{15} = (f_{17} + (1 + \sigma_1) f_{18}) f_{24}$;

$A_{16} = (f_{19} + (1 + \sigma_1) f_{20}) f_{24}$;

$f_{29} = \left\{f_{18} f_{24} \cosh(r) + \frac{1}{r}(f_{20} f_{24} r_1)\sinh(r) - f_{18} f_{24} \cosh(r_1) - f_{20} f_{24} \sinh(r_1)\right\}$;

$f_{30} = \Big\{[(f_{17} + f_{18}) f_{24} + f_{27}] \cosh(r) + \frac{1}{2}[(f_{19} + f_{20}) f_{24} r_1 + f_{26} + r_3 f_{28}] \sinh(r) - -(f_{17} + f_{18}) f_{24} \cosh(r_1) - (f_{19} + f_{20}) f_{24} \sinh(r_1) - f_{25} \sinh(r_2) + f_{26} \cosh(r_2) + f_{27} \cosh(r_3) + f_{28} \sinh(r_3)\Big\}$;

$f_{33} = [f_{29}(r_1 \sinh(r_1) - r\sinh(r)) - f_{31}(\cosh(r_1) - \cosh(r))];$

$f_{34} = [f_{30}(r_1 \sinh(r_1) - r\sinh(r)) - f_{32}(\cosh(r_1) - \cosh(r))];$

$A_9 = -[A_{11} + (f_{13} + (1+\sigma_1)f_{14})f_{23} + (f_{17} + (1+\sigma_1)f_{18})f_{24} + f_{27}];$

$A_{10} = -\frac{1}{r}[r_1 A_{12} + (f_{15} + (1+\sigma_1)f_{16})f_{23} r + (f_{19} + (1+\sigma_1)f_{20})f_{24} r_1 + f_{26} + r_3 f_{28}\ ;$

$A_{11} = 1.0; \quad A_{12} = \frac{\sigma_1 f_{29} + f_{30} - \cos h(r_1) + \cosh(r)}{\sin h(r_1) - \frac{r_1}{r}\sinh(r)};$

$f_{35} = (1+\sigma_1)\ ;\ f_{36} = (f_1 r A_{13} + f_2 f_{35})\ ;\ f_{37} = (f_1 r A_{14} + f_3 f_{35});\ f_{38} = (f_1 r_1 A_{15} + f_4 f_{35});$

$f_{39} = (f_1 r_1 A_{16} + f_5 f_{35})\ ;\ f_{40} = f_1 r_2 f_{25}\ ;\ f_{41} = f_1 r_2 f_{26};\ f_{42} = f_1 f_{25} + A_6 f_{35};$

$f_{43} = f_1 f_{26} + A_5 f_{35};\ f_{44} = f_1 r_3 f_{27};\ f_{45} = f_1 r_3 f_{28};\ f_{46} = \frac{f_{36}}{r^2 - {r_2}^2};\ f_{47} = \frac{f_{37}}{r^2 - {r_2}^2};$

$f_{48} = \frac{f_{38}}{{r_1}^2 - {r_2}^2};\ f_{49} = \frac{f_{39}}{{r_1}^2 - {r_2}^2};\ f_{50} = \frac{f_{41}}{4r_2};\ f_{51} = \frac{f_{40}}{4r_2};\ f_{52} = \frac{f_{42}}{2r_2};\ f_{53} = \frac{f_{43}}{2r_2};$

$f_{54} = \frac{f_{44}}{{r_3}^2 - {r_2}^2};\ f_{55} = \frac{f_{45}}{{r_3}^2 - {r_2}^2};\ A_{17} = -(f_{47} + f_{49} + f_{55})\ ;$

$A_{18} = -\frac{1}{\sinh(r_2)}\{A_{17}\cosh(r_2) + f_{46}\sinh(r) + f_{47}\cosh(r) + f_{48}\sinh(r_1) + f_{49}\cosh(r_1) +$
$f_{50}\cosh(r_2) + f_{51}\sinh(r_2) + f_{52}\cosh(r_2) + f_{53}\sinh(r_2) + f_{54}\sinh(r_3) + f_{55}\cosh(r_3)\};$

$f_{56} = \frac{Pr}{d}; \quad f_{57} = (1 + \frac{Pr}{d}\sigma_1)\ ; \quad f_{58} = f_{56}A_{13} + f_{57}f_6\ ; \quad f_{59} = f_{56}A_{14} + f_{57}f_7;$

$f_{60} = f_{56}A_{15} + f_{57}f_8\ ;\ f_{61} = f_{56}A_{16} + f_{57}f_9\ ;\ f_{62} = f_{56}f_{25}\ ;\ f_{63} = f_{56}f_{26}\ ;$

$f_{64} = f_{57}A_7 + f_{56}f_{27}\ ;\ f_{65} = f_{57}A_8 + f_{56}f_{28}\ ;\ f_{66} = \frac{f_{58}}{r^2 - {r_3}^2};\ f_{67} = \frac{f_{59}}{r^2 - {r_3}^2}\ ;$

$f_{68} = \frac{f_{60}}{{r_1}^2 - {r_3}^2};\quad f_{69} = \frac{f_{61}}{{r_1}^2 - {r_3}^2}\ ;\quad f_{70} = \frac{f_{62}}{{r_2}^2 - {r_3}^2}\ ;\ f_{71} = \frac{-2r_2 f_{62}}{({r_2}^2 - {r_3}^2)^2};$

$f_{72} = \frac{f_{63}}{{r_2}^2 - {r_3}^2};\quad f_{73} = \frac{-2r_2 f_{63}}{({r_2}^2 - {r_3}^2)^2}\ ;\ f_{74} = \frac{f_{64}}{2r_3};\quad f_{75} = \frac{f_{65}}{2r_3};$

$A_{19} = -(f_{66} + f_{68} + f_{71})\ ;$

$A_{20} = -\frac{1}{\sinh(r_3)}\{A_{19}\cosh(r_3) + f_{66}\cosh(r) + f_{67}\sinh(r) + f_{68}\cosh(r_1) + f_{69}\sinh(r_1) +$
$f_{70}\sinh(r_2) + f_{71}\cosh(r_2) + f_{72}\cosh(r_2) + f_{73}\sinh(r_2) + f_{74}\sinh(r_3) + f_{75}\cosh(r_3)\};$

$\mathrm{f}_{31} = \{\mathrm{r}\mathrm{f}_{18}\mathrm{f}_{24}\sinh(\mathrm{r}) + \mathrm{f}_{20}\mathrm{f}_{24}\cosh(\mathrm{r}) - \mathrm{f}_{18}\mathrm{f}_{24}\,\mathrm{r}_1\sinh(\mathrm{r}_1) - \mathrm{f}_{20}\mathrm{f}_{24}\mathrm{r}_1\cosh(\mathrm{r}_1)\}$;

$\mathrm{f}_{32} = \{\mathrm{r}[(\mathrm{f}_{17} + \mathrm{f}_{18})\mathrm{f}_{24} + \mathrm{f}_{27}]\sinh(\mathrm{r}) + [(\mathrm{f}_{19} + \mathrm{f}_{20})\mathrm{f}_{24}\mathrm{r}_1 + \mathrm{f}_{26} + \mathrm{r}_3\mathrm{f}_{28}]\cosh(\mathrm{r}) -$
$(\mathrm{f}_{17} + \mathrm{f}_{18})\mathrm{f}_{24}\,\mathrm{r}_1\sinh(\mathrm{r}) - (\mathrm{f}_{19} + \mathrm{f}_{20})\mathrm{f}_{24}\mathrm{r}_1\cosh(\mathrm{r}_1) - [\mathrm{f}_{25}(\mathrm{r}_2\cosh(\mathrm{r}_2) + \sinh(\mathrm{r}_2))] -$
$[\mathrm{f}_{26}(\mathrm{r}_2\sinh(\mathrm{r}_2) + \cosh(\mathrm{r}_2))] - \mathrm{r}_3\mathrm{f}_{27}\sinh(\mathrm{r}_3) - \mathrm{r}_3\mathrm{f}_{28}\cosh(\mathrm{r}_3)\}$;