

Boundary Layer Flow Analysis of a Class of Shear Thickening Fluids

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Abstract

Inelastic fluids, both shear thickening and shear thinning, are encountered in a number of engineering applications. In such fluids, the relationships between the shear stress and the rate of shear become vitally important in experimental as well as theoretical studies. In this paper, we have considered a two-dimensional steady boundary layer flow of a particular type of shear thickening fluid flowing past a flat plate. Using a specific rheological model for this fluid, we have investigated the combined effect of retaining higher order terms in the constitutive equation as well as perturbation expansions of the physical variables. The boundary layer flow, shown to be governed by a third order non-linear ODE, has been solved by a perturbation method followed by numerical integration. Our focus in this study is to investigate the comparative effects of the various order terms in the perturbation expansions. It is shown that the retention of higher order terms, generally neglected in similar studies, is important to correctly predict the flow features.

Keywords: Inelastic fluid, generalized constitutive equation, engineering applications, stagnation point flow, higher order effects, wall shear stress.

1. INTRODUCTION

The theoretical studies related to non-Newtonian fluids have been a subject of comprehensive investigations. The primary reason for this can apparently be attributed to a vast number of applications covering nearly all areas of engineering and industry including diverse fields such as medicines and biochemical industry. A glance at the huge available literature in this exciting area of research reveals that mathematical analyses of rheological flows gathered momentum with the introduction of empirical (e.g., inelastic fluids) and phenomenological (e.g., viscoelastic fluids) models, particularly during late forties and early fifties. These mathematical models were

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mainly based on nonlinear relations between the stress tensor and the deformation rate tensors.

In the development of inelastic fluid models, in contrast to viscoelastic non-Newtonian fluid models, a number of experimental studies showed that one needs to consider two classes of fluids in real life applications: one, in which the apparent viscosity of the fluid decreases with shear rate, and the other, in which the opposite phenomenon occurs. The fluids belonging to the former class are known in the literature as the shear thinning or pseudo-plastic fluids while the fluids belonging to the second class are classified as shear thickening or dilatant fluids. A number of empirical models have been developed for both classes of fluids, the most common among them being the power law fluid model [*e.g.*, 1 – 5] which could be used for either class.

Research on inelastic fluids has been carried out by both experimentalists and theoreticians due to applications in many applied fields, particularly chemical and food engineering. Some of the specific fields where such fluids arise are industries related to suspensions, polymer solutions, melts, foams, concentrated dispersions (*e.g.*, waxy maize starch dispersions), and polymer industry. Application of shear thickening fluids has also been reported to minimize head and neck injuries. Readers may refer to the related works in literature (see, for instance, [6–13]).

Flows of inelastic fluids, including boundary layer flows over flat surfaces, showing dilatant and pseudo-plastic behavior have been extensively investigated in the literature [14–25]. In the present study, our aim is to revisit a particular facet concerning the flow behavior of a class of dilatant fluids we had investigated before [14, 15, 17, 18, 23]. The non-Newtonian model used to describe the dilatant behavior in these studies was a special model allowing the apparent viscosity of the fluid to be expressed as a power series in I_2 , the second scalar invariant of the rate of strain tensor. In these works, it is assumed that the powers involving I_2 is a polynomial series expansion up to and including either first degree [14, 15, 18, 23] or second degree [17]. The similarity solution analysis of the boundary layer equations led to third order nonlinear ordinary differential equations for introduced similarity functions, together with appropriate number of boundary conditions. The well-defined boundary value problems were solved by a perturbation expansion, in terms of a small non-Newtonian parameter, followed by numerical integration. The perturbation expansion in these analyses was restricted up to 2 or 3 terms over and above the zeroth order representing the corresponding Newtonian fluid flow. In this paper, we have extended an earlier work [17] by assuming the perturbation expansion to have additional term, closely following our recent investigations [23–25]. We have carried out a comprehensive analysis to determine whether the extended perturbation expansion plays significant role in the flow characteristics. It turns out that retention of higher order terms is significant for more accurate description of the flow.

2 GOVERNING EQUATIONS FOR STEADY FLOW

For the two-dimensional incompressible flow considered here, we assume the generalized constitutive equation of an inelastic fluid in the form [17]

$$\tau_{ij} = \mu(I_2)e_{ij} \quad (1)$$

$$\mu(I_2) = \mu_0 + \mu_1 I_2 + \mu_2 I_2^2 + \mu_3 I_2^3 + \dots \quad (2)$$

where τ_{ij} is the shear stress tensor, e_{ij} is the rate of strain tensor, I_2 is the second scalar invariant of the rate of strain tensor, and $\mu_0, \mu_1, \mu_2, \dots$ are the material parameters of the fluid. In this study, we consider the fundamental equations of the steady flow corresponding to the approximation of $\mu(I_2)$ up to and including the second degree terms in I_2 . Such higher degree approximations are known to exhibit varying extents of a type of shear thickening effect in the rheological fluid. Thus, for our present study, we assume

$$\mu(I_2) = \mu_0 + \mu_1 I_2 + \mu_2 I_2^2 \quad (3)$$

Using Eq (3) in the momentum equations of fluid motion, and standard boundary layer approximations corresponding to the flow configuration considered here, it can be shown that the x and y components of the momentum equation reduce, respectively, to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \left[\nu_0 + 3\nu_1 \left(\frac{\partial u}{\partial y} \right)^2 + 5\nu_2 \left(\frac{\partial u}{\partial y} \right)^4 \right] \frac{\partial^2 u}{\partial y^2} \quad (4)$$

$$\frac{\partial p}{\partial y} = 0 \quad (5)$$

where u and v are, respectively, the x and y components of velocity, p is the pressure and ρ is the density, $\nu_0 = \mu_0/\rho$, $\nu_1 = \mu_1/\rho$ and $\nu_2 = \mu_2/\rho$. The equation of continuity is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6)$$

3. FLOW NEAR A TWO-DIMENSIONAL STAGNATION POINT

The stagnation point flow corresponds to the flow of a fluid near the stagnation region of a solid boundary. Such flows have been widely investigated in literature due to their applications in a number of engineering and industrial problems. For the stagnation point flow of the dilatant fluid considered here, the governing equations, as obtained in the previous section, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (7)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \left[\nu_0 + 3\nu_1 \left(\frac{\partial u}{\partial y} \right)^2 + 5\nu_2 \left(\frac{\partial u}{\partial y} \right)^4 \right] \frac{\partial^2 u}{\partial y^2} \quad (8)$$

$$\frac{\partial p}{\partial y} = 0 \quad (9)$$

where U is the mainstream velocity. The boundary conditions for the velocity field are

$$u = 0, v = 0 \text{ at } y = 0, u \rightarrow U \text{ as } y \rightarrow \infty \quad (10)$$

In order to solve Eqs (7)–(9) subject to the conditions (10), we let

$$U = U_1 x, \quad \eta = \sqrt{\frac{U_1}{\nu}} y, \quad \psi = \sqrt{\nu U_1} x f(\eta), \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (11)$$

It can be verified that the continuity equation is automatically satisfied by ψ .

The velocity components u and v now become

$$u = U f'(\eta), \quad v = -\sqrt{\nu U_1} f(\eta) \quad (12)$$

Using the expressions in Eqs (11) and (12) into Eq (8), we eventually obtain an ode in the form

$$f''' + f f'' - (f')^2 + 1 + \alpha c (f'')^2 f''' + \beta c^2 (f'')^4 f''' = 0 \quad (13)$$

where $\alpha = (3\nu_1 L^2 U_1^3)/\nu_0^2$, $\beta = (5\nu_2 L^4 U_1^6)/\nu_0^3$, $c = (x/L)^2$ and L is a characteristic length scale. In Eq (13), the primes denote differentiation with respect to η .

The parameters α and β play a vital role in the study of dilatant fluids described by our model, namely, the truncated Eq (3). They characterize the ratios of rheological effects and the Newtonian viscous effects of successive higher orders. In this work, one of our interests is to assess the relative effects of α and β . To this end, and further to make our analysis amenable to analytical treatment, we assume $\beta = \epsilon \alpha$, ($0 < \epsilon < 1$). Thus, our analysis will be dominated by two key rheological parameters α and ϵ besides the non-dimensional parameter representing the longitudinal coordinate from the stagnation point. Now, the transformed boundary conditions become

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1 \quad (14)$$

The boundary layer flow problem thus reduces to the solution of the boundary value problem (bvp) given by Eqs (13) and (14) whose solution can be sought either numerically or using a perturbation expansion.

4. SOLUTION OF THE BOUNDARY VALUE PROBLEM

One may first of all note that Eq (13) subject to the conditions (14) describes a well-posed boundary value problem. This bvp may be contrasted with some other similar studies for viscoelastic fluids [26–28] in which the corresponding velocity functions have been shown to be governed by equations whose orders do not match the number of physical boundary conditions. However, the authors of such studies overcame this difficulty by resorting to a perturbation technique and thereby reducing the governing non-linear equations into systems of equations in each of which the order of equation matched the number of boundary conditions. When $\alpha = 0$, Eq (13) reduces to the corresponding well-known equation for viscous fluids. The nonlinear terms here — $\alpha c (f'')^2 f'''$ and $\beta c^2 (f'')^4 f'''$ — are consequences of the non-Newtonian fluid model considered in our present work. Of special note is the presence of the longitudinal coordinate represented by the parameter c in these terms. This indicates that it is natural to consider solution of Eq (13) at cross-sections near the stagnation point.

We now turn our attention to analyzing the influence of the non-Newtonian parameter α on the velocity profiles in the boundary layer and associated wall stress. We shall thus showcase the higher order non-Newtonian effects vis-à-vis the basic Newtonian flow. For this, we shall adopt a perturbation expansion of the governing similarity function $f(\eta)$ and obtain solutions for various orders for the governing function $f(\eta)$ and its derivative. We write

$$f(\eta) = f_0(\eta) + \alpha f_1(\eta) + \alpha^2 f_2(\eta) + \alpha^3 f_3(\eta) + \alpha^4 f_4(\eta) + \dots \quad (15)$$

Using Eq (15) in Eqs (13) and (14) and equating coefficients of various powers of α , we obtain sets of boundary value problems corresponding to the various order terms. Restricting ourselves up to terms of order three — zeroth, first, second and third order — the system of equations and the corresponding boundary conditions can be obtained as

$$f_0''' + f_0 f_0'' - (f_0')^2 + 1 = 0 \quad (16)$$

$$f_0(0) = 0, \quad f_0'(0) = 0, \quad f_0'(\infty) = 1$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = -c(f_0'')^2 f_0''' - \epsilon c^2 (f_0'')^4 f_0''' \quad (17)$$

$$f_1(0) = 0, \quad f_1'(0) = 0, \quad f_1'(\infty) = 0$$

$$\begin{aligned} f_2''' + f_0 f_2'' - 2f_0' f_2' + f_0'' f_2 &= (f_1')^2 - f_1 f_1'' - c \left[(f_0'')^2 f_1''' + 2f_0'' f_0''' f_1'' \right] \\ &\quad - \epsilon c^2 \left[(f_0'')^4 f_1''' + 4(f_0'')^3 f_0''' f_1'' \right] \end{aligned} \quad (18)$$

$$f_2(0) = 0, \quad f_2'(0) = 0, \quad f_2'(\infty) = 0$$

$$\begin{aligned} f_3''' + f_0 f_3'' - 2f_0' f_3' + f_0'' f_3 &= 2f_1' f_2' - f_1 f_2'' - f_1'' f_2 \\ &\quad - c \left[(f_0'')^2 f_2''' + 2f_0'' f_1'' f_1''' \right. \\ &\quad \left. + f_0''' (f_1'')^2 + 2f_0'' f_0''' f_2'' \right] \\ &\quad - \epsilon c^2 \left[(f_0'')^4 f_2''' + 4(f_0'')^3 f_0''' f_2'' \right. \\ &\quad \left. + 4(f_0'')^3 f_1'' f_1''' + 6(f_0'')^2 f_0''' (f_1'')^2 \right] \end{aligned} \quad (19)$$

$$f_3(0) = 0, \quad f_3'(0) = 0, \quad f_3'(\infty) = 0$$

In applications, the prediction of the effect of the non-Newtonian parameter on the local wall shear stress is of great importance. For the model considered here, the non-dimensional skin friction coefficient τ at the bounding wall $y = 0$, is given by

$$\tau = \sqrt{c} \left[f''(0) + \frac{1}{3}\alpha c \{f''(0)\}^3 + \frac{1}{5}\epsilon \alpha c^2 \{f''(0)\}^5 \right] \quad (20)$$

In order to obtain the longitudinal and the transverse velocity profiles, we need to compute $f(\eta)$ and $f'(\eta)$ by integrating numerically the boundary value problems given by Eqs (16)–(19) using a suitable numerical method. In these equations, it may be noted that the equations describing the Newtonian boundary layer flow, Eq (16), can be treated independently from the remaining coupled equations. We have thus solved first the Newtonian flow equation using a shooting method. The other equations are then solved in succession by the same method.

5. RESULTS AND DISCUSSION

We now proceed to discuss the effect of the governing non-dimensional parameters on the flow, with a clear focus on the relative importance of inclusion of various order terms in the perturbation expansion. Of the three parameters, viz., c , α and ϵ , we shall in fact endeavor to showcase the effect of the key rheological parameter α in our analysis. This has been done by assessing its impact on (i) velocity components in the boundary layer and (ii) percentage increases in the computed values of the similarity functions, representing longitudinal and transverse velocity components, through inclusion of various order terms. For the sake of completeness, we shall also show the effects of the variations in the parameters c and ϵ on f and f' . These parameters represent, respectively, the extent of the deviation from the stagnation point and the second order effect in the generalized constitutive equation of the inelastic fluid.

We have included twelve figures to analyze various features. The graphs in the Figs 1 and 2 correspond to velocity profiles in the boundary layer, while those in Figs 3–8 relate to the effect of inclusion of various order terms in the perturbation expansion. In these graphs (Figs 1–8), we have fixed $c = 0.5$ and $\epsilon = 0.3$. In Figs 9 and 10, we have included counterparts of the Figs 1 and 2, respectively, allowing the parameter c to vary for fixed values of the remaining two parameters, while the final sets of graphs in Figs 11 and 12 show the influence of ϵ on the percentage increases in $f(\eta)$ and $f'(\eta)$ corresponding to the third order effects in the perturbation expansion. It is worth stating that we have included up to third order effects in order to determine if the retention of terms after second order in the perturbation series, commonly used in the literature, is indeed desirable for such dilatant fluid flows.

In Figs 1 and 2 we have included plots of $f(\eta)$, which is directly related to the magnitude of the transverse component of velocity and of $f'(\eta)$, which is related to the longitudinal component of velocity, respectively, to show how the key non-Newtonian parameter α affects the two-dimensional boundary layer velocity profiles. We have included two small values of the non-Newtonian parameter α ($= 0.1$ and 0.9) to determine the extent to which non-Newtonian variations influence the flow. It is quite apparent from both Fig 1 as well as Fig 2 that for small values of α (< 1), the effect of this parameter on the velocity profiles is moderate. It may be noted that as the rheological effects enhance, α assumes higher values. It is seen that the magnitude of

the transverse velocity decreases with increase in α while the opposite trend is observed for the longitudinal velocity. It is worth remarking here that the overshooting feature seen in the case of boundary layer flow of viscoelastic fluids, for such type of flows [26, 28], is apparently not visible for the dilatant fluids being considered here.

We shall now discuss some interesting but important features emanating from the sets of graphs, namely, Fig 3 through Fig 8. Here, our aim is to find out, through the qualitative as well as quantitative analysis of the graphs of both velocity components, the effect of inclusion of various order terms in the perturbation expansion. The main focus here is to explore whether the higher order terms are desirable in perturbation expansions involving small parameters. For this purpose, we have computed percentage increases in $f(\eta)$ and $f'(\eta)$ with respect to the corresponding Newtonian flow, and exhibited them through three curves in each graph in the Figs 3 through 8. For example, in line with one of our recent studies [23], in the set of graphs for $f(\eta)$ (see Figs 3–5) corresponding to the transverse velocity component v , the three curves relate to percentage increases for $f(\eta)$ using the following expressions:

$$\text{Curve 1 (first order effects): } f_{v1} = \frac{\alpha f_1}{f_0} \times 100$$

$$\text{Curve 2 (second order effects): } f_{v2} = \frac{\alpha f_1 + \alpha^2 f_2}{f_0} \times 100$$

$$\text{Curve 3 (third order effects): } f_{v3} = \frac{\alpha f_1 + \alpha^2 f_2 + \alpha^3 f_3}{f_0} \times 100$$

We have similarly calculated percentages increases for f' corresponding to the longitudinal velocity component u , and shown them in Figs 6–8 using the above formulae by replacing f by its primed quantity.

Some interesting conclusions can be drawn from the analyses of Figs 3 – 8. First and foremost, one may note that the inclusion of higher order terms in the perturbation expansion is indeed necessary when investigating the boundary layer flow of dilatant fluids of the type considered in this study. This fact becomes increasingly important for higher values of the governing non-Newtonian parameter α . The reason for this is quite obvious and clearly borne out from the graphs in the Figs 4 and 5 as well as Figs 7 and 8, where one may note that plots of curve 3, related to the inclusion of terms up to and including third order in the perturbation expansion, all lie between curves 1 and curves 2. This striking feature clearly demands that one needs to go beyond second order terms in the perturbation expansions in order to predict more accurately the flow features for the type of flow considered in this work. Such observations have been noted in a number of previous works, for instance, in the boundary layer flow of viscoelastic fluids.

The effect of the parameter c is shown in the plots of velocity components in Figs 9–10, assuming other two parameters fixed. One may note that the longitudinal velocity component is more sensitive to changes in c , particularly for values of c beyond unity. In the next set of figures, Figs 11–12, we have analyzed the effect of the higher order rheological parameter ϵ on f_{v3} and f_{u3} assuming $\alpha = c = 0.5$. Here also, one can

observe the non-monotonic behavior of the higher order effects in the constitutive equation of the shear thickening fluid considered. This feature, not commonly analyzed in the flow of inelastic fluids, clearly emphasizes the need of considering higher order terms in the constitutive equation for dilatant fluids — similar to our observations for perturbation expansions of the physical variables.

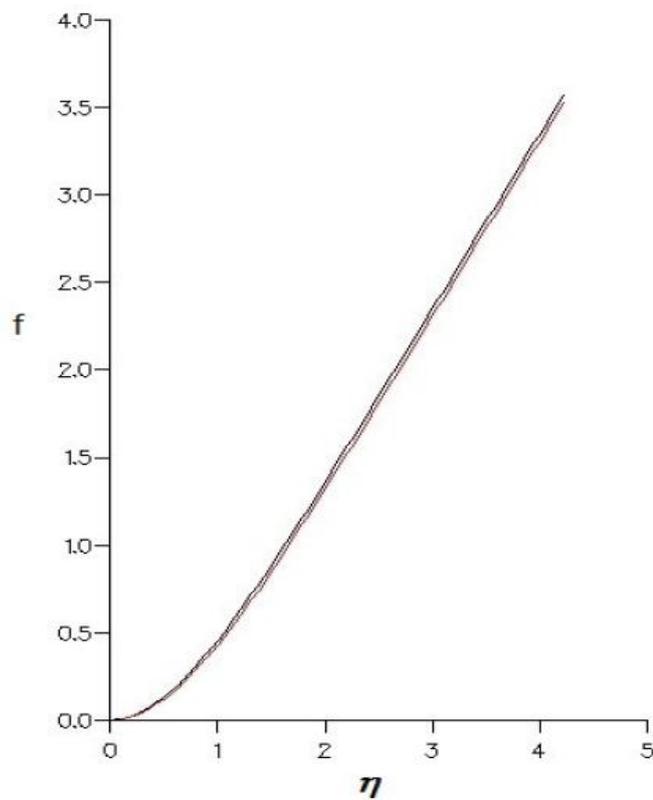


Fig 1. Variation of f : $\alpha = 0.1$ (uppercurve), 0.9 (lowercurve)

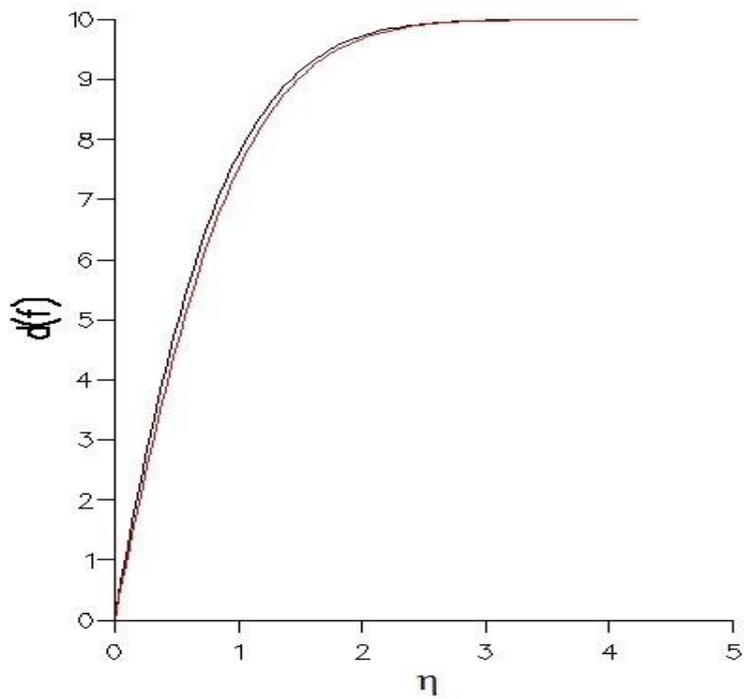


Fig 2. Variation of f' [$\equiv d(f)$]. $\alpha = 0.1$ (lower curve), 0.9 (upper curve)

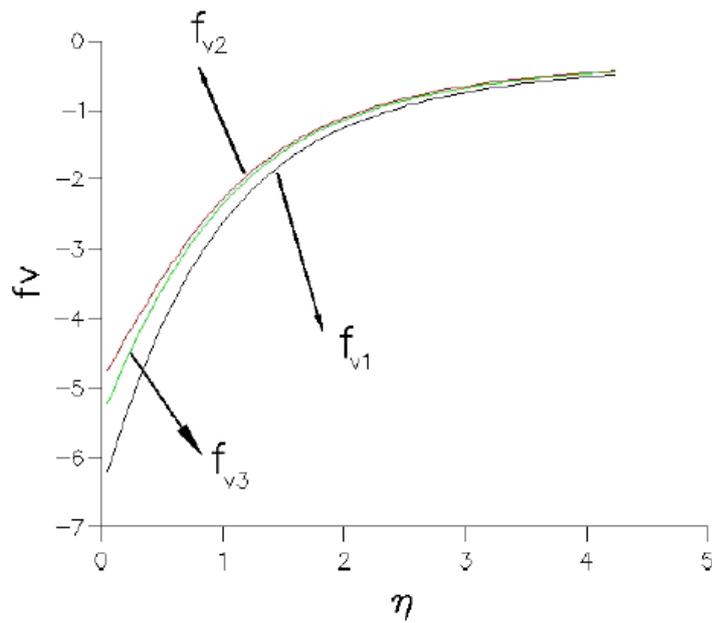


Fig 3. Percentage increase of f with different orders of α . ($\alpha = 0.3$)

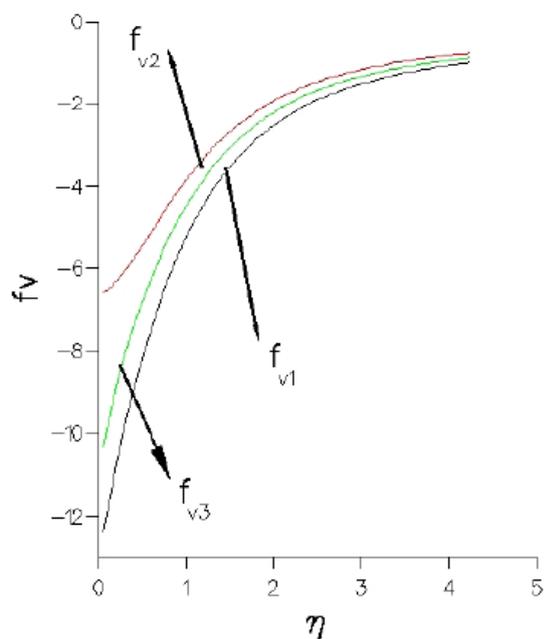


Fig 4. Percentage increase of f' with different orders of α . ($\alpha = 0.6$)

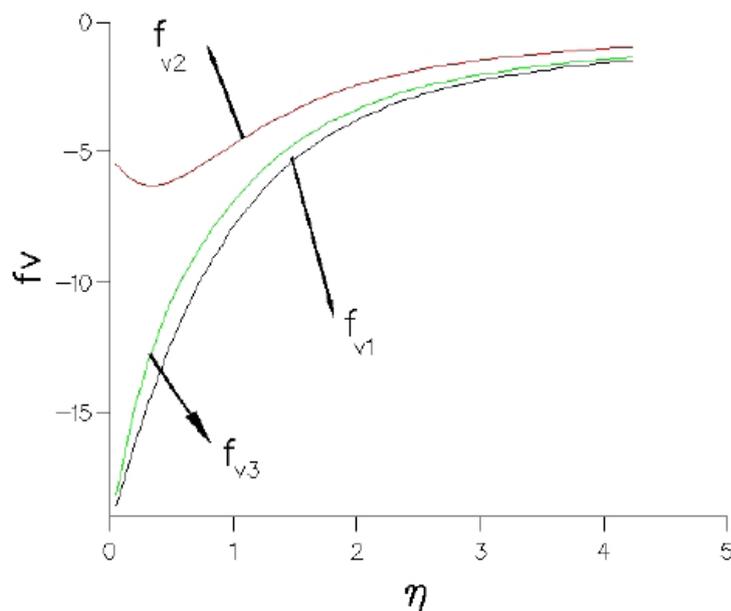


Fig 5. Percentage increase of f with different orders of α . ($\alpha = 0.9$)

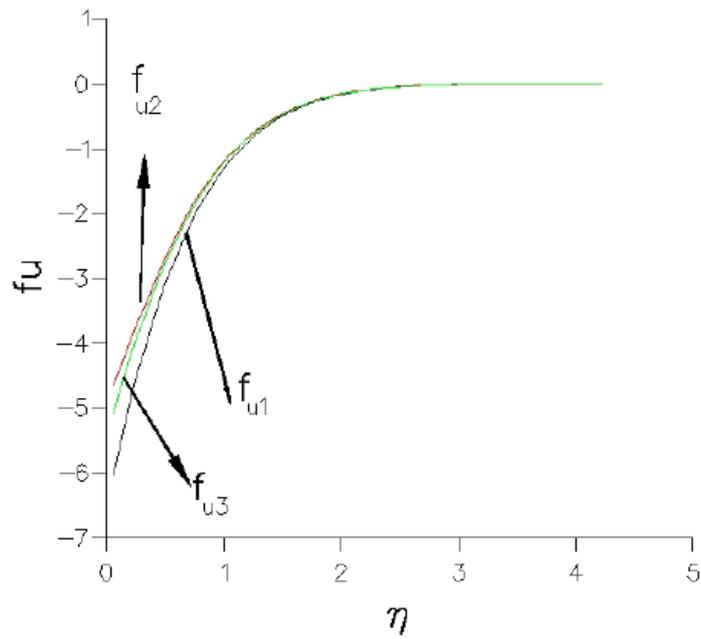


Fig 6. Percentage increase of f' with different orders of α . ($\alpha = 0.3$)

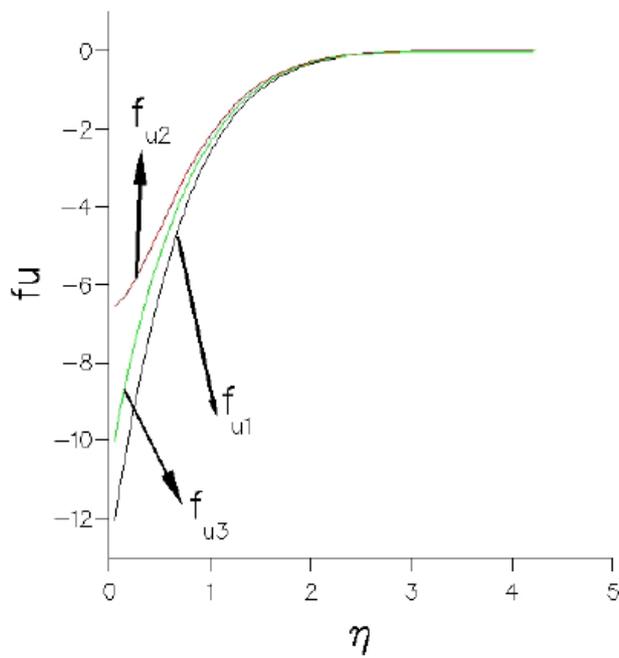


Fig 7. Percentage increase of f' with different orders of α . ($\alpha = 0.6$)

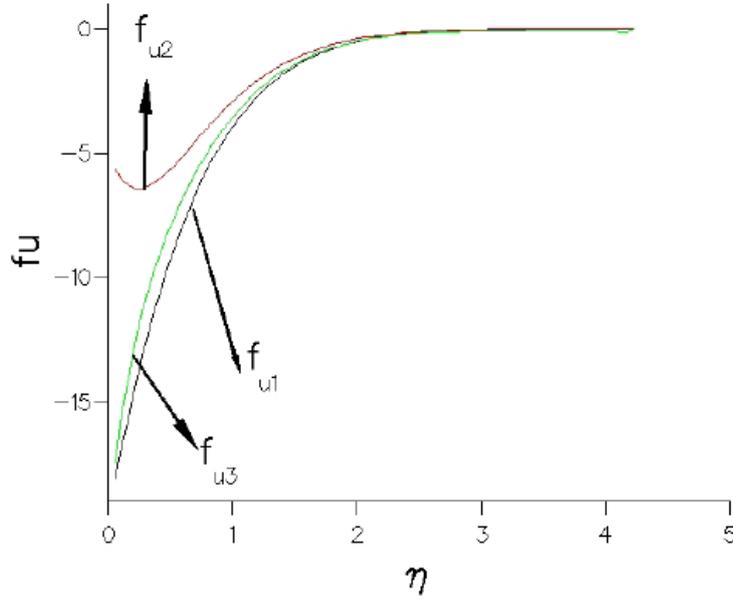


Fig 8. Percentage increase of f' with different orders of α . ($\alpha = 0.9$)

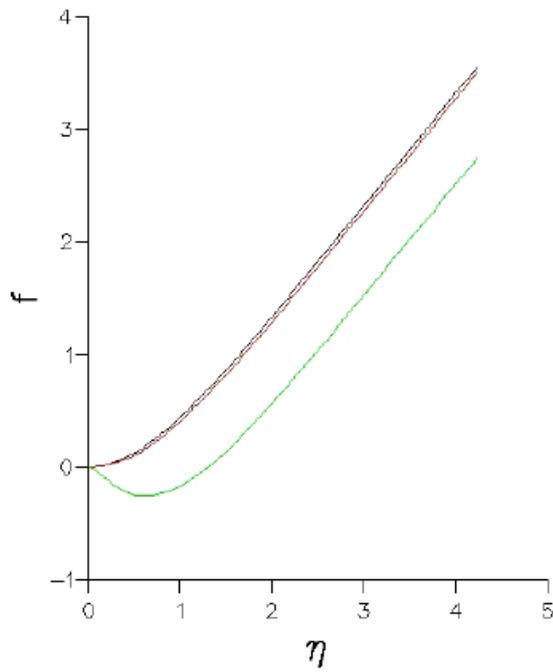


Fig 9. Effect of c on f : $c = 0.5$ (upper), 1.0 (middle), 2.0 (lower)

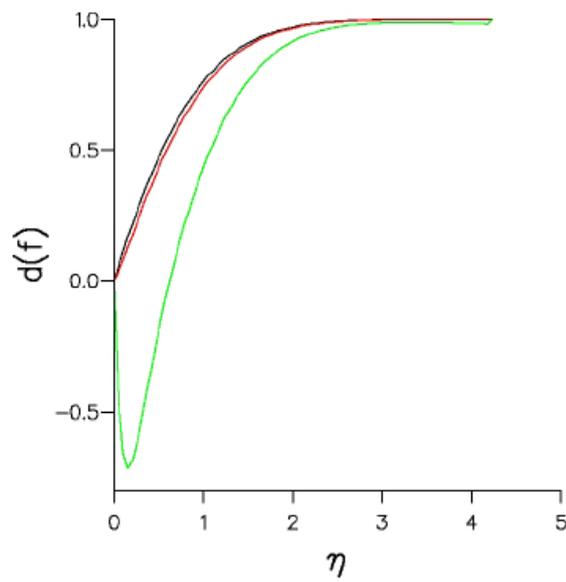


Fig 10. Effect of c on f' : $c = 0.5$ (upper), 1.0 (middle), 2.0 (lower)

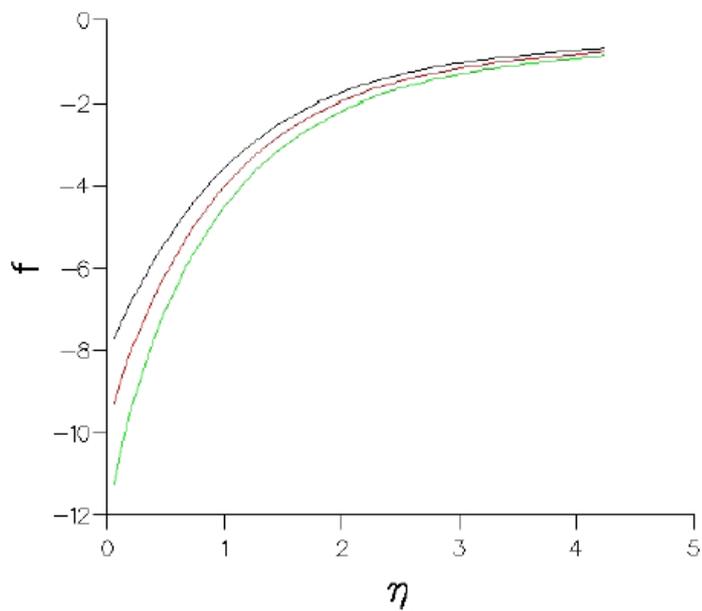


Fig 11. Effect of ϵ on $fv3$: $\epsilon = 0.1$ (lower), 0.5 (upper), 0.9 (middle)

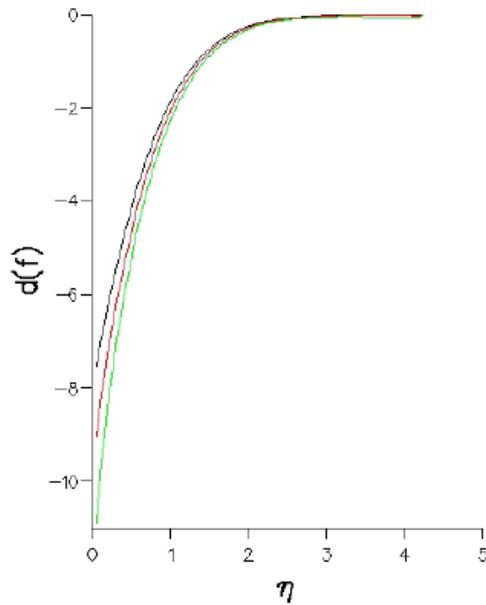


Fig 12. Effect of ϵ on $fu3$: $\epsilon = 0.1$ (lower), 0.5 (upper), 0.9 (middle)

Table 1: Skin friction τ ($c = 0.5$, $\epsilon = 0.5$)

α	τ_1	τ_2	τ_3
0.0	0.8716	0.8716	0.8716
0.3	0.8781	0.8953	0.8869
0.6	0.8464	0.9476	0.8691
0.9	0.7979	1.0567	0.7623

In Table 1, the computed values of the coefficient of skin friction τ at the bounding wall have been given for different values of α , including $\alpha = 0$, for a direct comparison with the corresponding Newtonian incompressible fluid. The values of the coefficient of skin friction in the three columns, namely, τ_1 , τ_2 , τ_3 , respectively, refer to the first order, second order and third order perturbation expansions in our analysis. Here, we have fixed $c = 0.5$ and $\epsilon = 0.5$ while computing these coefficients. It is quite apparent from the comparison of values of the skin friction coefficients in the three columns (compare particularly second and third columns corresponding to τ_2 and τ_3) how vitally important it is to include the third order terms in the perturbation expansions, a fact already highlighted earlier.

In conclusion, this work clearly indicates that neglecting the higher order terms in a perturbation method may not always yield the correct results in the boundary layer flow of inelastic shear thickening fluids.

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