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Dynamics of a nonlinear operators generated from $\xi^{(as)}$-Quadratic Stochastic Operators

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Abstract

A quadratic stochastic operator (QSO) exhibits the time development of various species in biology. Several QSOs have been examined by Lotka and Volterra. The main problem in a non linear operators is to explore their behavior. The behavior of a non linear operators have not been studied in comprehensively even QSOs which are the simplest a nonlinear operators. To address this problem, many classes of QSO were introduced. This paper aims to examine the behavior of six an operators selected from different classes of $\xi^{(as)}$-QSO.

1. INTRODUCTION

Many systems are presented by a nonlinear operators. One of the most simplest nonlinear case is quadratic one. A quadratic dynamical systems have been demonstrated to be exporter for various fields to study dynamical properties and modeling, such as biology [1, 24, 9], physics [14, 20, 28], economics and mathematics [9, 10, 11, 21]

The theory of Markov processes is a quickly evolving field with various applications to numerous branches of physics and mathematics. Nevertheless, several physical and biological systems can't be studyed by Markov processes. One of these systems is given by quadratic stochastic operators (QSO). The system of a QSO which relates to genetics population has appeared in [1]. A QSO typically is utilized to exhibit the time evolution of various species in biology, which emerges as follows. Let a population consists of m species $1, 2, \cdots, m$. Let $x^{(0)} = (x_1^{(0)}, \cdots, x_m^{(0)})$ taken as a probability distribution of species at an initial state and let $p_{ij,k}$ be probability which the species i^{th} and j^{th} will be interbreed to produce k^{th} species. Then, a probability distribution $x^{(1)} = (x_1^{(1)}, \cdots, x_m^{(1)})$ of the species in the first generation can be described as a total probability as below,

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1, m}.$$

Consequently, it shows that the association $x^{(0)} \to x^{(1)}$ represents a mapping V, which is known as evolution operator. Starting from initial state $x^{(0)}$, the population develops to the first generation $x^{(1)} = V(x^{(0)})$ and then to the second generation $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})) = V^{(2)}\left(x^{(0)}\right)$ and so on. Hence, the discrete dynamical system debates the population system evolution states by follows:

$$x^{(0)}, \quad x^{(1)} = V\left(x^{(0)}\right), \quad x^{(2)} = V^{(2)}\left(x^{(0)}\right), \quad \cdots$$

In another sense, if the present generation distribution is given then, a QSO describes a distribution of the next generation. The a wonderful implementations of QSO to population genetics were provided in [11]. The new accomplishments and open problems in theory of the QSO have been described and explained in [26]. The difficulty of the problem relies on the given cubic matrix coefficient $(P_{ijk})_{i,j,k=1}^{m}$. Numerous researchers dedicated their study to present a special class of QSO and investigated its behavior, such as, F-QSO [17], Volterra-QSO [6, 25], permutated Volterra-QSO [7, 8], ℓ-Volterra-QSO [15, 16], Quasi-Volterra-QSO [4], non-Volterra-QSO [5, 19], strictly non-Volterra-QSO [18] and non Volterra operators which produced by measurements [2, 3]. Nevertheless, all these classes jointly will not cover a system of QSOs. Therefore, there exist numerous classes of QSO need to be studied.

Recently, a new class of QSO was introduced depending on a partition of the coupled index set, $\mathbf{P}_m = \{(i,j) : i < j\} \subset I \times I$ and $\mathbf{\Delta}_m\{(i,i) : i \in I\} \subset I \times I$, where $I = \{1, \cdots, m\}$. In [22], the $\xi^{(s)}$-QSO related to $|\xi_1| = 2$ of $\mathbf{P_m}$ with a point partition of $\mathbf{\Delta_m}$ was investigated. In [12], the $\xi^{(a)}$-QSO related to $|\xi_1| = 2$ of $\mathbf{P_m}$ with a trivial partition of $\mathbf{\Delta_m}$ was studied. The $\xi^{(s)}$-QSO related to $|\xi_1| = 3$ of $\mathbf{P_m}$ with a point partition of $\mathbf{\Delta_m}$ was examined in [13]. Furthermore, the $\xi^{(s)}$-QSO and $\xi^{(a)}$-QS related to $|\xi_1| = 1$ of $\mathbf{P_m}$ with a point and a trivial partitions of $\mathbf{\Delta_m}$ respectively were also discussed in[23].

In [27], it has been classified the $\xi^{(as)}$-QSO into 18 non-conjugate classes which generated by the partitions of $\mathbf{P}_3$ with $|\xi_1| = 2$ related to the partitions of $\mathbf{\Delta_3}$ with $|\xi_2| = 2$. Therefore, we proceed with study the dynamics of some classes which are not studied. The paper is arranged as follows. Section 2, provides several preliminary definitions. Sections

3 presents the behavior of V_{28} and V_{30} obtained from class G_{16} and G_{18} respectively. Section 4 presents the study of the dynamics of classes G_5 and G_{11} by investigating the behavior of V_5 and V_{17}. In the last section, the behavior of V_{13} and V_{19} obtained from G_7 and G_8 respectively are studied.

2. PRELIMINARIES

In this section, some essential definitions are reviewed.

Definition 1 *QSO is a mapping of the simplex*

$$S^{m-1} = \{(x_1, \cdots, x_m) \in \mathbb{R}^m : (x_1 + x_2 + \cdots + x_m) = 1, x_i \geq 0, \quad i = 1, \cdots, m\} \tag{1}$$

into itself of the form

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1,m}, \tag{2}$$

where $V(x) = x' = (x'_1, \cdots, x'_m)$ *and* $P_{ij,k}$ *is a coefficient of heredity, which verifies the succeeding conditions*

$$\sum_{k=1}^{m} P_{ij,k} = 1 P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad . \tag{3}$$

The above definition suggests that each QSO QSO $V : S^{m-1} \to S^{m-1}$ is definedrather distinctively by a cubic matrix $\mathcal{P} = \left(P_{ijk}\right)_{i,j,k=1}^{m}$ with conditions (3).

For $V : S^{m-1} \to S^{m-1}$, the set of fixed points, limiting point and k-periodic are denoted by $Fix(V)$, $\omega_V(x^{(0)})$ and $Per_k(V)$ respectively.

Remember that a Volterra-QSO is defined by (2), (3) and the extra assumption

$$P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\}. \tag{4}$$

One can see that a Volterra-QSO adopts the following form:

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k \in I, \tag{5}$$

where

$$a_{ki} = 2P_{ik,k} - 1 \ \text{ for } i \neq k \text{ and } a_{ii} = 0, \ i \in I. \tag{6}$$

Moreover,

$$a_{ki} = -a_{ik} \ \text{ and } \ |a_{ki}| \leq 1.$$

The notation of ℓ-Volterra-QSO, was inserted in [13] which popularize a concept of Volterra-QSO. The definition is as follows.

Let $\ell \in I$ be fixed, and we take it that the heredity coefficient $\{P_{ij,k}\}$ verify the following conditions

$$P_{ij,k} = 0 \ \text{ if } \ k \notin \{i, j\} \ \text{ for any } \ k \in \{1, \ldots, \ell\}, \ \ i, j \in I, \tag{7}$$

$$P_{i_0 j_0,k} > 0 \ \text{ for some } (i_0, j_0), \ i_0 \neq k, \ j_0 \neq k, \ k \in \{\ell + 1, \ldots, m\}. \tag{8}$$

Remark 1 *The following properties are established:*

(i) Easy to note that ℓ-Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.

(ii) A periodic trajectory does not exist for Volterra-QSO [6]. However, periodic trajectories can be found for ℓ-Volterra-QSO [15].

In accordance with [22], each element $x \in S^{m-1}$ is considered as a probability distribution of the set $I = \{1, ..., m\}$. Let $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ be two vectors selected from S^{m-1}. We call that x *and* y *are equivalent* if $x_k = 0 \Leftrightarrow y_k = 0$ and this relation is symbolized by $x \sim y$. Let $supp(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We call that x *and* y *are singular* if $supp(x) \cap supp(y) = \emptyset$, and this relation is symbolized by $x \perp y$.
It is denoted sets of coupled indexes by

$$\mathbf{P}_m = \{(i, j) : \ i < j\} \subset I \times I, \quad \Delta_m = \{(i, i) : i \in I\} \subset I \times I.$$

For a given pair $(i, j) \in \mathbf{P}_m \cup \Delta_m$, we set a vector $\mathbb{P}_{ij} = (P_{ij,1}, \cdots, P_{ij,m})$. Clearly, because of condition (3), $\mathbb{P}_{ij} \in S^{m-1}$.

Let $\xi_1 = \{A_i\}_{i=1}^{N}$ and $\xi_2 = \{B_i\}_{i=1}^{M}$ be some fixed partitions of $\mathbf{P}_m$ and Δ_m, respectively, i.e. $A_i \bigcap A_j = \emptyset$, $B_i \bigcap B_j = \emptyset$, and $\bigcup_{i=1}^{N} A_i = \mathbf{P}_m$, $\bigcup_{i=1}^{M} B_i = \Delta_m$, where $N, M \leq m$.

Definition 2 *[22] QSO $V : S^{m-1} \to S^{m-1}$ given by* (2), (3), *is said a $\xi^{(as)}$-QSO w.r.t. the partitions ξ_1, ξ_2, if the following conditions are satisfied:*

(i) For each $n \in \{1, \ldots, N\}$ and any (u_1, v_1), $(u_2, v_2) \in A_n$, one has $\mathbb{P}_{u_1 v_1} \sim \mathbb{P}_{u_2 v_2}$;

(ii) For any $n \neq s$, $n, s \in \{1, \ldots, N\}$ and any $(u_1, v_1) \in A_n$ and $(u_2, v_2) \in A_s$ one has $\mathbb{P}_{u_1 v_1} \perp \mathbb{P}_{u_2 v_2}$;

(iii) For each $b \in \{1, \ldots, M\}$ and any (u_1, u_1), $(v_1, v_1) \in B_b$, one has $\mathbb{P}_{u_1 u_1} \sim \mathbb{P}_{v_1 v_1}$;

(iv) For any $d \neq q$, $d, q \in \{1, \ldots, M\}$ and any $(u_1, u_1) \in B_d$ and $(v_1, v_1) \in B_q$, one has $\mathbb{P}_{u_1 u_1} \perp \mathbb{P}_{v_1 v_1}$.

3. DYNAMICS OF CLASSES G_{16} AND G_{18}

This section is dedicated to examine the dynamics of classes G_{16} and G_{18}, by studying the dynamics of V_{28} and V_{30} selected from G_{16} and G_{18} respectively. To begin, V_{28} is rewritten as follows:

$$V_{28} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = \alpha(x^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \\ z' = (1-\alpha)\,(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \quad (9)$$

The operator V_{28} can be redrafted as a convex combination $V_{28} = \alpha W_1 + (1-\alpha)\, W_2$,
where

$$W_1 := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = 2x^{(0)} - (x^{(0)})^2 \\ z' = 2y^{(0)}z^{(0)} \end{cases} \quad (10)$$

and

$$W_2 := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = 2x^{(0)}\left(1 - x^{(0)}\right) \\ z' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \quad (11)$$

Theorem 1 *Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (10)and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_1) \cup Per_2(W_1)$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has

$$Fix(W_1) = \left\{(1 - \frac{1}{2}\sqrt{2}, \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}\sqrt{2})\right\}, \quad (12)$$

(ii) One has

$$Per_2(W_1) = \left\{e_1, e_2, (\frac{3}{2} - \frac{1}{2}\sqrt{5}, \frac{1}{2}\sqrt{5} - \frac{1}{2}, 0)\right\}, \quad (13)$$

(iii) One has

$$\omega_{W_1}(x_1^{(0)}) = \{e_1, e_2\}. \quad (14)$$

Proof. Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (10), $x_1^{(0)} \notin Fix(W_1) \cup Per_2(W_1)$ be any initial point in simplex S^2, and $\{W_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of W_1 starting from point $x_1^{(0)}$.

(i) The set of fixed points of W_1 are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = y^2 + z^2 \\ y = 2x - x^2 \\ z = 2yz \end{cases} \quad (15)$$

By adding the first equation to the third equation in system (15), we derive $z = x^2 - 3x + 1$. Subsequently, the last equation and second equation are substituted to the first equation in system (15), we obtain $(1 - \frac{1}{2}\sqrt{2}, \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}\sqrt{2})$ is solution for system (15). Hence, the set of fixed point only consists of $(1 - \frac{1}{2}\sqrt{2}, \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}\sqrt{2})$.

(ii) To find $2-$periodic points of W_1, we should prove that W_1 has no any order periodic points in set $W_1 \setminus L_3$, where $L_3 = \left\{x^{(0)} \in S^2 : z^{(0)} = 0\right\}$. The third coordinate of W_1 increases if $y^{(n)} \geq \frac{1}{2}$ and decreases if $y^{(n)} \leq \frac{1}{2}$. In both cases, W_1 has no any order $2-$periodic points in set $W_1 \setminus L_3$ because the third coordinate of W_1 increases or decreases along the iteration of $W_1 \setminus L_3$. Therefore, finding $2-$periodic points of W_1 over L_3 is sufficient. To find $2-$periodic points of W_1, the following system of equations should be solved:

$$\begin{cases} x = (2x - x^2)^2 \\ y = 2y^2 - y^4 \\ z = 0 \end{cases} \quad (16)$$

The solution for the first equation in system (16) is easily to find. Therefore, the $2-$periodic points of W_1 are e_1, e_2 and $(\frac{3}{2} - \frac{1}{2}\sqrt{5}, \frac{1}{2}\sqrt{5} - \frac{1}{2}, 0)$.

(iii) Let $x_1^{(0)} \notin Fix(W_1) \cup Per_2(W_1)$ and $x_1^{(0)} \in L_3$. Obviously, the line L_3 is invariant under W_1. Therefore, the first coordinate of W_1 can be redrafted as $\eta^{(1)}(x^{(0)}) = (1 - x^{(0)})^2$ and $\eta^{(2)}(x^{(0)}) = (2x^{(0)} - (x^{(0)})^2)^2$. One shows that $\eta^{(1)}$ and $\eta^{(2)}$ are decreasing and increasing on $[0, 1]$ respectively. As discussed in proof part (ii) of this theorem, we derive $Fix(\eta^{(1)}) \cap [0, 1] = \{\frac{3}{2} - \frac{\sqrt{5}}{2}\}$, $Fix(\eta^{(2)}) \cap [0, 1] = \{0, \frac{3}{2} - \frac{\sqrt{5}}{2}, 1\}$, which indicate that intervals $[0, \frac{3}{2} - \frac{\sqrt{5}}{2}]$ and $[\frac{3}{2} - \frac{\sqrt{5}}{2}, 1]$ are invariant under the function $\eta^{(2)}$. Evidently, $\eta^{(2)}(x^{(0)}) \leq x^{(0)}$, for all $x^{(0)} \in [0, \frac{3}{2} - \frac{\sqrt{5}}{2}]$ and $\eta^{(2)}(x^{(0)}) \geq x^{(0)}$, for all $x^{(0)} \in [\frac{3}{2} - \frac{\sqrt{5}}{2}, 1]$. If $x^{(0)} \in [0, \frac{3}{2} - \frac{\sqrt{5}}{2}]$, then $\omega_{\eta^{(2)}}(x^{(0)}) = \{0\}$; if $x^{(0)} \in [\frac{3}{2} - \frac{\sqrt{5}}{2}, 1]$, then $\omega_{\eta^{(2)}}(x^{(0)}) = \{1\}$.
In another way,

$$W_1^{(n)}(x_1^{(0)}) = \begin{cases} \left(\eta^{(2n)}(x^{(0)}), 1 - \eta^{(2n)}(x^{(0)}), 0\right) & , if\ n\ is\ even \\ \left(\eta^{(2n)}(\eta(x^{(0)})), 1 - \eta^{(2n)}(\eta(x^{(0)})), 0\right) & , if\ n\ is\ odd \end{cases} \quad (17)$$

Therefore, the set of limiting point $\omega_{W_1}(x_1^{(0)})$ consists of e_1, e_2 whenever $x_1^{(0)} \in L_3$. To explore the behavior of W_1 when $x_1^{(0)} \notin L_3$, the following result is required.

Claim 1 *For any $x_1^{(0)} \notin Fix(W_1) \cup Per_2(W_1)$ there exist n_o, such that $\left\{z^{(n)}\right\}_{n=n_o}^{\infty} \subseteq L_3$.*

Suppose that $\ell_1 = \{x_1^{(0)} \in S^2 : z^{(0)} > 0\}$ is invariant region, which means a new region $\ell_2 = \{x_1^{(0)} \in S^2 : y^{(0)} \geq \frac{1}{2}\}$ is invariant region. Evidently, the second coordinate of W_1 is a bounded increasing sequence that converges to a fixed point, *where this a fixed point is greater than or equal to $\frac{1}{2}$. Thereby making $z^{(n+1)} \geq z^{(n)}$ i.e., it is a bounded increasing sequence that converges to the fixed point one. But one is not fixed point, which contradicts our assumption, which results in $z^{(n)}$ is a bounded decreasing sequence that converges to zero and $y^{(n)}$ is impossible always to be greater than $\frac{1}{2}$. Hence, there exist n_o, such that $\left\{z^{(n)}\right\}_{n=n_o}^{\infty} \subseteq L_3$.*

Hence, exploring the behavior of W_1 over line L_3 is sufficient, the proof is completed.

□

Theorem 2 *Let $W_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (11) and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_2) \cup Per_2(W_2)$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has

$$Fix(W_2) = \left\{(x^{\ltimes}, y^{\ltimes}, z^{\ltimes})\right\}, \tag{18}$$

where

$$x^{\ltimes} = \frac{-1}{24(t_1^{\frac{1}{6}} t^{\frac{1}{4}})} \left(\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t} + 99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}} - (\sqrt[4]{t^3} - 15\sqrt[6]{t_1}\sqrt[4]{t})\right),$$

$$y^{\ltimes} = \frac{-1}{144(t\sqrt{t_1})}((\sqrt[4]{t^3}\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t} + 99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[6]{t} - 3\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t}$$

$$+99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[3]{t_1}\sqrt[4]{t} + 360\sqrt[3]{t_1^2} - 36\sqrt{t_1}\sqrt{t} - 108\sqrt{3}\sqrt{397} + 630\sqrt[3]{t_1} - 18),$$

$$z^{\ltimes} = \frac{1}{144(t\sqrt{t_1})}((\sqrt[4]{t^3}\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t} + 99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[6]{t} - 3\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t}$$

$$+99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[3]{t_1}\sqrt[4]{t} + 18 + 108\sqrt{3}\sqrt{397} + 234\sqrt[3]{t_1^2} + 18\sqrt{t}\sqrt{t_1} - 630\sqrt[3]{t_1},$$

$$t = (-6\sqrt[3]{(1 + 6\sqrt{3}\sqrt{397})^2} + 21\sqrt[3]{1 + 6\sqrt{3}\sqrt{397} + 210}) \quad \textit{and} \quad t_1 = 1 + 6\sqrt{3}\sqrt{397}.$$

(ii) One has

$$\omega_{W_2}(x_1^{(0)}) = \left\{(x^{\ltimes}, y^{\ltimes}, z^{\ltimes})\right\}. \tag{19}$$

Proof. Let $W_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (11), $x_1^{(0)} \notin Fix(W_2) \cup Per_2(W_2)$ be any initial point in simplex S^2, such that $y^{(0)} \neq 1$ and $\{W_2^{(n)}\}_{n=1}^{\infty}$ be a trajectory of W_2 starting from point $x_1^{(0)}$.

(i) The set of fixed points of W_2 are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = y^2 + z^2 \\ y = 2x(1-x) \\ z = x^2 + 2yz \end{cases} \tag{20}$$

By adding the first equation to the third equation in system (20), we derive $z = 2x^2 - 3x + 1$. The last equation and second equation in the first equation are substituted in system (20), we obtain only fixed point $(x^{\ltimes}, y^{\ltimes}, z^{\ltimes})$.

(ii) To investigate the behavior of W_2, the following regions are introduced:

$$B_1 : \;=\; \{x_1^{(0)} \in S^2 : 0 \leq x^{(0)}, y^{(0)}, z^{(0)} \leq \frac{1}{2}\},$$

$$B_2 : \;=\; \{x_1^{(0)} \in S^2 : \frac{1}{2} < x^{(0)} < 1\},$$

$$B_3 : \;=\; \{x_1^{(0)} \in S^2 : \frac{1}{2} < y^{(0)} < 1\},$$

$$B_4 : \;=\; \{x_1^{(0)} \in S^2 : \frac{1}{2} < z^{(0)} < 1\},$$

$$B_5 : \;=\; \{x_1^{(0)} \in S^2 : 0 < x^{(0)} \leq z^{(0)} \leq y^{(0)} < \frac{1}{2}\},$$

$$B_6 : \;=\; \{x_1^{(0)} \in S^2 : 0 \leq y^{(0)} \leq z^{(0)} \leq x^{(0)} \leq \frac{1}{2}\},$$

$$B_7 : \;=\; \{x_1^{(0)} \in S^2 : \frac{1}{3} \leq z^{(0)} \leq y^{(0)} \leq \frac{1}{2}, \quad 0 < x^{(0)} \leq \frac{1}{3}\}.$$

Subsequently, the behavior of W_2 across all aforementioned regions is explored. Then, the behavior of W_2 will be described. To achieve this objective, the following results should be shown:

(1) Let $x_1^{(0)} \in B_1$. Then, $0 \leq x^{(0)}, y^{(0)}, z^{(0)} \leq \frac{1}{2}$. It can be easily to see that $-1 \leq 3x^{(0)} - 1 \leq \frac{1}{2}$, by squaring and adding $-3(y^{(0)} - z^{(0)})^2$, we obtain $9(x^{(0)})^2 - 6x^{(0)} + 1 - 3(y^{(0)} - z^{(0)})^2 \leq 1$. Dividing the previous inequality by three after adding two to both part for inequality, will derive

$$3(x^{(0)})^2 - 2x^{(0)} + 1 - (y^{(0)} - z^{(0)})^2 \leq 1.$$

Therefore,

$$2(x^{(0)})^2 + (y^{(0)} + z^{(0)})^2 - (y^{(0)} - z^{(0)})^2 \leq 1.$$

Then, $2(x^{(0)})^2 + 4y^{(0)}z^{(0)} \leq 1$, which implies that $z' \leq \frac{1}{2}$. Subsequently, one shows that $y' \leq \frac{1}{2}\ \forall x_1^{(0)}$. Evidently, $0 \leq (y^{(0)})^2, (z^{(0)})^2 \leq \frac{1}{4}$, thereby indicating that $x' \leq \frac{1}{2}$. Hence, B_1 is an invariant region.

(2) The second coordinate of W_2 is less than or equal $\frac{1}{2}$ at any $n \in \mathbb{N}$. We can easily see that $x' \leq y^{(0)} + z^{(0)}$, which indicates $x^{(n)} \leq \frac{1}{2}$. Therefore, if $x_1^{(0)} \in B_2 \cup B_3$, then $n_{k_1}, n_{k_2} \in \mathbb{N}$, such that sequences $y^{(n_{k_1})}$ and $x^{(n_{k_2})}$ return to the invariant region B_1.

(3) Thereafter, we intend to show that if $x_1^{(0)} \in B_4$, then $n_{k_3} \in \mathbb{N}$, such that sequences $z^{(n_{k_3})}$ returns to the invariant region B_1. To achieve this objective, suppose that B_4 is invariant region. However,

$$z' \leq (x^{(0)})^2 + (y^{(0)})^2 + (z^{(0)})^2 \leq z^{(0)}(x^{(0)} + y^{(0)} + z^{(0)}) = z^{(0)}.$$

Then, $\frac{z'}{z^{(0)}} < 1$, this means $z^{(n)}$ is a bounded decreasing sequence that converges to the fixed point zero, which contradicts our assumption. Hence, the region B_4 is not invariant region.

(4) Suppose that $z' \geq y'$ and $x' \geq z'$. By depending on the first coordinate, we derive $x' = (y^{(0)})^2 + (z^{(0)})^2 \leq 2(z^{(0)})^2 \leq 2z^{(0)}x^{(0)}$. This thereby indicating x' is greater than or equal $\dfrac{1}{2}$, which is a contradiction. Then, we consider a new sequence $x' + z' = 2(x^{(0)})^2 - 2x^{(0)} + 1$. The new sequence has a minimum value $\dfrac{1}{2}$, which indicates that all coordinates are greater than zero and less than $\frac{1}{2}$. Hence, if $x_1^{(0)} \in B_6 \cup B_1$, then $n_{k_4}, n_{k_5}, n_{k_6} \in \mathbb{N}$, such that the sequences $z^{(n_{k_4})}$, $y^{(n_{k_5})}$ and $z^{(n_{k_5})}$ return to the invariant region B_5.

(5) Let $z^{(0)} \geq \frac{1}{3}$. Whether the minimum value of $z' = (1 - z^{(0)} - y^{(0)})^2 + 2y^{(0)}z^{(0)}$ occurs when $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Thus, $z^{(n)} \geq \frac{1}{3}$ and $y^{(n)} \geq \frac{1}{3}$, since all coordinates are equal one, we obtain $x^{(n)} \leq \frac{1}{3}$. This indicates to that if $x_1^{(0)} \in B_5$, then $n_{k_6} \in \mathbb{N}$, such that sequences $W_2^{(n_{k_6})}$ returns to the invariant region B_7.

We have proven that if $x_1^{(0)} \in B_i$, $i \in \{1, \ldots, 6\}$, then the trajectory of W_2 goes to invariant region B_7. Thus, exploring the dynamics of W_2 over this region is sufficient. Evidently, $y^{(n)}$ is a bounded increasing sequence. Since $y^{(n)} + x^{(n)}$ is bounded increasing sequence and $x^{(n)} = x^{(n)} + y^{(n)} - y^{(n)}$, we conclude sequence $x^{(n)}$ converges to $x^{\ltimes}$. Thus, we have $y^{(n)}$ converging to $y^{\ltimes}$. Hence, $\omega_{W_2}(x_1^{(0)}) = \{(x^{\ltimes}, y^{\ltimes}, z^{\ltimes})\}$, which is the desired conclusion.

Remark: If $y^{(0)} = 1$, then $V^{(1)}((0,1,0)) = (1,0,0)$ and $V^{(2)}((0,1,0)) = (0,0,1)$. When the same preceding process is performed for next iteration, we will obtain $V^{(2n)}((0,1,0)) = (0,0,1)$ and $V^{(2n+1)}((0,1,0)) = (1,0,0)$. One can find e_1 and e_3 are $2-$periodic points for W_2. □

Now, we are going to explore the behavior of V_{30}.

$$V_{30} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = (1-\alpha)(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = \alpha(x^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \end{cases} \tag{21}$$

We redraft V_{30} as a convex combination $V_{30} = \alpha W_1 + (1-\alpha) W_2$, where

$$W_1 := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ z' = 2x^{(0)}\left(1 - x^{(0)}\right) \end{cases} \tag{22}$$

and

$$W_2 := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 \\ y' = 2y^{(0)}z^{(0)} \\ z' = 2x^{(0)} - (x^{(0)})^2 \end{cases} \tag{23}$$

Corollary 1 *Let $W_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (22)and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_1) \cup Per_2(W_1)$ be any initial point in simplex S^2. Then, the following statements are true:*

(1) One has

$$Fix(W_1) = (x^{\ltimes}, y^{\ltimes}, z^{\ltimes}), \tag{24}$$

Where

$$x^{\ltimes} = \frac{-1}{24(t_1^{\frac{1}{6}} t^{\frac{1}{4}})} \left(\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t} + 99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}} - (\sqrt[4]{t^3} - 15\sqrt[6]{t_1}\sqrt[4]{t})\right);$$

$$z^{\ltimes} = \frac{-1}{144(t\sqrt{t_1})}((\sqrt[4]{t^3}\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t} + 99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[6]{t} - 3\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t}$$

$$+99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[3]{t_1}\sqrt[4]{t} + 360\sqrt[3]{t_1^2} - 36\sqrt{t_1}\sqrt{t} - 108\sqrt{3}\sqrt{397} + 630\sqrt[3]{t_1} - 18);$$

$$y^{\ltimes} = \frac{1}{144(t\sqrt{t_1})}((\sqrt[4]{t^3}\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t} + 99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[6]{t} - 3\sqrt{6}(7\sqrt[3]{t_1}\sqrt{t} + \sqrt[3]{t_1^2}\sqrt{t}$$

$$+99\sqrt{t_1} - 35\sqrt{t})^{\frac{1}{2}}\sqrt[3]{t_1}\sqrt[4]{t} + 18 + 108\sqrt{3}\sqrt{397} + 234\sqrt[3]{t_1^2} + 18\sqrt{t}\sqrt{t_1} - 630\sqrt[3]{t_1};$$

$$t = (-6\sqrt[3]{(1+6\sqrt{3}\sqrt{397})^2} + 21\sqrt[3]{1+6\sqrt{3}\sqrt{397}+210}) \quad and \quad t_1 = 1+6\sqrt{3}\sqrt{397}$$

(ii) One has

$$\omega_{W_1}(x_1^{(0)}) = \left\{(x^{\ltimes}, y^{\ltimes}, z^{\ltimes})\right\}. \tag{25}$$

For W_2, let $W_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (23)*and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(W_2) \cup Per_2(W_2)$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has that

$$Fix(W_2) = \left\{\left(1 - \frac{1}{2}\sqrt{2}, -\frac{1}{2} + \frac{1}{2}\sqrt{2}, \frac{1}{2}\right)\right\}, \tag{26}$$

(ii) One has

$$Per_2(W_2) = \{e_1, e_3\}, \tag{27}$$

(iii) One has

$$\omega_{W_1}(x^{(0)}) = \{e_1, e_3\}. \tag{28}$$

4. DYNAMICS OF CLASSES G_5 AND G_{11}

This section explores the dynamics of $V_{5,17} : S^2 \to S^2$ selected from G_5 and G_{11} respectively. To begin, V_5 can be rewritten as follows:

$$V_5 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = (1-\alpha)(x^{(0)})^2 \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \end{cases} \tag{29}$$

Theorem 3 *Let $V_5 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by* (29) *and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_5)$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has

$$Fix(V_5) = \{e_3\}, \tag{30}$$

(ii) One has

$$\omega_{V_5}(x_1^{(0)}) = \{e_3\}. \tag{31}$$

Proof. Let $V_5 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (29), $x_1^{(0)} \notin Fix(W_2)$ be any initial point in S^2 and $\{x_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of V_5 starting from point $x_1^{(0)}$.

(i) The set of fixed points of V_5 are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = \alpha x^2 + 2yz \\ y = (1-\alpha)x^2 \\ z = y^2 + z^2 + 2x(1-x) \end{cases} \tag{32}$$

The second and equation $z = 1 - y - x$ are substituted to the first equation in system (32). Then, the first equation in system (32) becomes as follows: $(2-\alpha)x^2 - 2(1-\alpha)^2x^4 + 2(1-\alpha)x^3 = x$. We easily check that the only solution for the last equation, such that $x \in [0, 1]$ is zero. Hence, the only fixed point is e_3.

(ii) It can be easily to see that $y' \leq x'$. Thus, $y^{(n)} \leq x^{(n)}$ for any $n \in \mathbb{N}$. Consider a new sequence $x' + y' = (x^{(0)})^2 + 2y^{(0)}z^{(0)}$. Then,

$$x' + y' \leq (x^{(0)} + y^{(0)})^2 + (x^{(0)} + y^{(0)})z^{(0)} = x^{(0)} + y^{(0)}.$$

Now, we are going to prove that $\frac{x'+y'}{x^{(0)}+y^{(0)}} < 1$. To achieve this objective, suppose that $x' \geq \frac{1}{2}$. However,

$$\begin{aligned} x' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} &\leq (x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ &\leq x^{(0)}(x^{(0)} + y^{(0)} + z^{(0)}) \\ &\leq x^{(0)}, \end{aligned}$$

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then $\frac{x'}{x^{(0)}} \leq 1$. Therefore, $x^{(n)}$ is a decreasing bounded sequence that converges to a fixed point, but the set of fixed point has no this point, which is a contradiction. Hence, $x^{(n)}$ is less than $\frac{1}{2}$, which results in $\frac{x'+y'}{x^{(0)}+y^{(0)}} < 1$. Therefore, the new sequence is a bounded decreasing that converges to zero. It is known that sequences $x^{(n)}$, $y^{(n)}$ and $z^{(n)}$ are always greater than or equal to zero. Hence, $x^{(n)}$ converges to zero and $y^{(n)}$ converges to zero. Since the set of limiting points $\omega_{V_5}(x_1^{(0)})$ is not empty set, we derive $\omega_{V_5}(x_1^{(0)}) = \{e_3\}$, the proof is completed. □

Now, we explore the behavior of V_{17}.

$$V_{17} := \begin{cases} x' = (1-\alpha)(x^{(0)})^2 + 2y^{(0)}z^{(0)} \\ y' = \alpha(x^{(0)})^2 \\ z' = (z^{(n)})^2 + (y^{(n)})^2 + 2x^{(0)}\left(1 - x^{(0)}\right) \end{cases} \tag{33}$$

Corollary 2 *Let $V_{17} : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by* (33) *and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_5)$ be any initial point in simplex S^2. Then, the following statements are true:*

(1) One has
$$Fix(V_{17}) = \{e_3\}, \tag{34}$$

(2) One has
$$\omega_{V_{17}}(x_1^{(0)}) = \{e_3\}. \tag{35}$$

5. DYNAMICS OF CLASSES G_7 AND G_8

In this section, we explore the dynamics of classes G_7 and G_8 by studying the dynamics of V_{13} and V_{19}. This section needs useful facts about properties of the function $g_\alpha : [0,1] \to [0,1]$ given by
$$g_\alpha(x) = (1-\alpha)x^2 + 2x(1-x) \tag{36}$$
Thereafter, the dynamics of the behavior V_{13} and V_{19} will be ready to explore.

Proposition 1 *Let $g_\alpha : [0,1] \to [0,1]$ be a function given by* (36). *Then, the following statements are true:*

(i) One has $Fix(g_\alpha) = \left\{0, \frac{1}{1+\alpha}\right\}$,

(ii) One has $\omega_{g_\alpha}(x_0) = \left\{\frac{1}{1+\alpha}\right\}$, where $x_0 \notin Fix(g_\alpha(x))$.

Proof. *(i) The set of fixed points of g_α are obtained by solving the following equation:*

$$(1-\alpha)x^2 + 2x(1-x) = x \tag{37}$$

By solving the equation (37) *with regard to variable x, we find $x = 0$ or $x = \frac{1}{1+\alpha}$. Therefore, $Fix(g_\alpha) = \left\{0, \frac{1}{1+\alpha}\right\}$.*

(ii) Firstly, the interval $[0,1]$ is divided into tow intervals $I_1 = [0, \frac{1}{1+\alpha}]$ and $I_2 = [\frac{1}{1+\alpha}, 1]$. Let $x_0 \in I_1$, it can be easily to see that $g_\alpha(x_0) \geq x_0$ for any $x_0 \in I_1$, then $g_\alpha^{(n+1)}(x_0) \geq g_\alpha^{(n)}(x_0)$ for any $n \in \mathbb{N}$, which indicates that g_α is a bounded increasing sequence on I_1. Accordingly, it converges to x^* and x^* should be a fixed point, that is $x^* = \frac{1}{1+\alpha}$. Hence the limiting point is $\omega_{g_\alpha}(x_0) = \{\frac{1}{1+\alpha}\}$. Similarly, one can show that if $x_0 \in I_2$, then the limiting point is $\omega_{g_\alpha}(x_0) = \{\frac{1}{1+\alpha}\}$, this process completes the proof. □

The operator V_{13} is rewritten as follows:

$$V_{13} := \begin{cases} x' = (1-\alpha)(x^{(0)})^2 + 2x^{(0)}(1-x^{(0)}) \\ y' = \alpha(x^{(0)})^2 \\ z' = (z^{(0)})^2 + (y^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \tag{38}$$

Theorem 4 *Let $V_{13} : S^2 \to S^2$ given by* (38) *be a $\xi^{(as)}$-QSO and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_{13})$ be any initial point in simplex S^2. Then, the following statement are true:*

(i) One has
$$Fix(V_{13}) = \{e_3, (\tilde{x}, \tilde{y}, \tilde{z})\},$$
where $\tilde{x} = \frac{1}{1+\alpha}$, $\tilde{y} = \frac{\alpha}{(1+\alpha)^2}$ and $\tilde{z} = \frac{\alpha^2}{(1+\alpha)^2}$.

(ii) One has
$$\omega_{V_{13}}(x_1^{(0)}) = \begin{cases} e_3, & if\ \ x^{(0)} \in L_1; \\ (\tilde{x}, \tilde{y}, \tilde{z}), & if\ \ x^{(0)} \notin L_1. \end{cases}$$

Proof. Let $V_{13} : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (38), $x_1^{(0)} \notin Fix(W_2)$ be any initial point in simplex S^2 and $\{x_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of V_{13} starting from point $x_1^{(0)}$.

(i) The set of fixed points of V_{13} are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = (1-\alpha)x^2 + 2x(1-x) \\ y = \alpha x^2 \\ z = y^2 + z^2 + 2yz \end{cases} \tag{39}$$

By finding the solution for the first equation in system (39), we obtain $x = 0$ or $x = \frac{1}{1+\alpha}$. If $x = 0$, then $y = 0$ and $z = 1$; if $x = \frac{1}{1+\alpha}$, then $y = \tilde{y} = \frac{\alpha}{(1+\alpha)^2}$ and $z = \tilde{z} = \frac{\alpha^2}{(1+\alpha)^2}$. Therefore, $Fix(V_{13}) = \{e_3, (\tilde{x}, \tilde{y}, \tilde{z})\}$.

(ii) Let $x_1^{(0)} \notin Fix(V_{13})$. The dynamics of V_{13} will be discussed when $x_1^{(0)} \in L_1$ and $x_1^{(0)} \notin L_1$, where $L_1 = \{x_1^{(0)} \in S^2 : x^{(0)} = 0\}$. Thus, two cases should be discussed separately.

(a) Let $x_1^{(0)} \in L_1$. We can easily show that L_1 is invariant line under V_{13}. Therefore, $\left\{x^{(n)}\right\}_{n=1}^{\infty}$ converges to zero.

Accordingly, sequence $\{y^{(n+1)}\}_{n=1}^{\infty}$ converges to zero. By using $x^{(n)} + y^{(n)} + z^{(n)} = 1$, we obtain that $\{z^{(n)}\}_{n=1}^{\infty}$ converges to one. Hence, the limiting point in this case is $\omega_{V_{13}}(x_1^{(0)}) = \{e_3\}$.

(b) Let $x_1^{(0)} \notin L_1$. Due to proposition(1), we have $\{x^{(n)}\}_{n=1}^{\infty}$ converging to $\tilde{x} = \frac{1}{1+\alpha}$. The second coordinate depends on the first coordinate of V_{13}, then $\{y^{(n)}\}_{n=1}^{\infty}$ converges to $\tilde{y} = \frac{\alpha}{(1+\alpha)^2}$. Accordingly, $\{z^{(n)}\}_{n=1}^{\infty}$ converges to $\tilde{z} = \frac{\alpha^2}{(1+\alpha)^2}$. Hence, the limiting point whenever $x^{(0)} \notin L_1$ is $\omega_{V_{13}}(x_1^{(0)}) = \{(\tilde{x}, \tilde{y}, \tilde{z})\}$, the proof is completed. □

Subsequently, we are going to explore the behavior of $\xi^{(as)}$-QSO $V_{19} : S^2 \to S^2$. The operator V_{19} is rewritten as follows:

$$V_{19} := \begin{cases} x' = (1-\alpha)(x^{(0)})^2 + 2x^{(0)}(1-x^{(0)}) \\ y' = (y^{(0)})^2 + (z^{(0)})^2 \\ z' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)} \end{cases} \quad (40)$$

Theorem 5 *Let $V_{19} : S^2 \to S^2$ given by (40) be a $\xi^{(as)}$-QSO and $x_1^{(0)} = (x^{(0)}, y(0), z(0)) \notin Fix(V_{19})$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has

$$Fix(V_{19}) = \left\{e_2, (0, \frac{1}{2}, \frac{1}{2}), (x^+, y^+, z^+)\right\},$$

where $x^+ = \frac{1}{1+\alpha}$, $y^+ = \frac{3\alpha+1-\sqrt{\alpha^2+6\alpha+1}}{4(\alpha+1)}$ *and* $z^+ = \frac{\alpha-1+\sqrt{\alpha^2+6\alpha+1}}{4(\alpha+1)}$.

(ii) One has

$$\omega_{V_{19}}\left(x_1^{(0)}\right) = \begin{cases} (0, \frac{1}{2}, \frac{1}{2}), & if \ x^{(0)} \in L_1 \\ (x^+, y^+, z^+), & if \ x^{(0)} \notin L_1 \end{cases}$$

Proof. Let $V_{19} : S^2 \to S^2$ be $\xi^{(as)}$-QSO given by (40), $x_1^{(0)} \notin Fix(V_{19})$ and $\{x_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of V_{19} starting from point $x_1^{(0)}$.

(i) The set of fixed points of V_{19} are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = (1-\alpha)x^2 + 2x(1-x) \\ y = y^2 + z^2 \\ z = \alpha x^2 + 2yz \end{cases} \quad (41)$$

By finding the solution for the first equation in system (41), we obtain that $x = 0$ or $x = \frac{1}{1+\alpha}$. If $x = 0$, then $y = 1$ and $z = 0$ or $y = z = \frac{1}{2}$; if $x = \frac{1}{1+\alpha}$, it follows by finding the solution for the second equation in system (41) that is $y = y^+$ and $z = z^+$. Therefore, $Fix(V_{19}) = \{e_2, (0, \frac{1}{2}, \frac{1}{2}), (x^+, y^+, z^+)\}$.

(ii) Let $x_1^{(0)} \notin Fix(V_{13})$. The dynamics of V_{19} will be explored when $x_1^{(0)} \in L_1$ and $x_1^{(0)} \notin L_1$. Thus, two cases should be discussed separately:

(a) Let $x_1^{(0)} \in L_1$. Since the first coordinate of V_{19} is equal to the first coordinate of V_{13}, which indicates L_1 is invariant line under V_{19}. Therefore, $\{x^{(n)}\}_{n=1}^{\infty}$ converges to zero. To study the behavior of the third coordinate, we consider the third coordinate of V_{19}, namely, $z' = h(z^{(0)}) = 2z^{(0)}(1 - z^{(0)})$. Firstly, the interval $[0, 1]$ is divided into two intervals as the follows: $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. One can see that $h(I_2) \subseteq I_1$, thereby indicating that I_1 is invariant interval under h. Thus, exploring the dynamics of h over I_1 is sufficient. Let $z^{(0)} \in I_1$. Evidently, $h^{(n+1)}(z^{(0)}) \geq h^{(n)}(z^{(0)})$ i.e., $h^{(n)}$ is a bounded increasing sequence. Accordingly, $\{h^{(n)}(z^{(0)})\}_{n=1}^{\infty}$ converges to a fixed point of h that is $\frac{1}{2}$. Therefore, sequence $\{z^{(n)}\}_{n=1}^{\infty}$ converges to $\frac{1}{2}$. By using $x^{(n)} + y^{(n)} + z^{(n)} = 1$, we obtain that $\{y^{(n)}\}_{n=1}^{\infty}$ converges to $\frac{1}{2}$. Hence, the limiting point in this case is $\omega_{V_{19}}(x_1^{(0)}) = \{(0, \frac{1}{2}, \frac{1}{2})\}$.

(b) Let $x_1^{(0)} \notin L_1$. Due to proposition (1), we have $\{x^{(n)}\}_{n=1}^{\infty}$ converging to $\frac{1}{1+\alpha}$. It can be easily to check that $\frac{1}{1+\alpha} \geq \frac{1}{2}$ at any $\alpha \in [0, 1]$. Thus, exploring the behavior of second or third coordinates on a region less than $\frac{1}{2}$ is sufficient. To explore the behavior of second coordinate, we consider the second coordinate of V_{19}, namely,

$$y' = M(y^{(0)}) = 2(y^{(0)})^2 - \frac{2\alpha}{\alpha+1}y^{(0)} + (\frac{\alpha}{\alpha+1})^2, \quad (42)$$

where $y^{(0)} \in [0, \frac{1}{2})$ and $\alpha \in [0, 1]$. The interval $[0, \frac{1}{2}]$ is divided into the following intervals:

$I_1 = [0, s(\alpha))]$, $I_2 = [s(\alpha), y^-(\alpha)]$, $I_3 = [y^-(\alpha), \frac{\alpha}{2(\alpha+1)}]$, $I_4 = [\frac{\alpha}{2(\alpha+1)}, \frac{\alpha}{\alpha+1}]$, and

$I_5 = [\frac{\alpha}{\alpha+1}, y^+(\alpha)]$, where $s(\alpha) = \frac{\alpha^2}{2(\alpha+1)^2}$, $y^-(\alpha) = \frac{3\alpha+1-\sqrt{\alpha^2+6\alpha+1}}{4(\alpha+1)}$, $y^+(\alpha) = \frac{3\alpha+1+\sqrt{\alpha^2+6\alpha+1}}{4(\alpha+1)}$,

$k(\alpha) = \frac{\alpha^2(\alpha^2+2\alpha+2)}{2(\alpha+1)^4}$ and $y^+(\alpha) = \frac{3\alpha+1+\sqrt{\alpha^2+6\alpha+1}}{4(\alpha+1)} \geq \frac{1}{2}$. It can be easily to check

$$0 \leq s(\alpha) \leq y^-(\alpha) \leq k(\alpha) \leq \frac{\alpha^2}{(\alpha+1)^2} \leq \frac{\alpha}{2(\alpha+1)} \leq \frac{\alpha}{\alpha+1} \leq y^+(\alpha).$$

and

$$M(I_1) \subseteq [k(\alpha), \frac{\alpha^2}{(\alpha+1)^2}] \subseteq [y^-(\alpha), \frac{\alpha}{2(\alpha+1)}] = I_3$$
$$M(I_4) \subseteq [s(\alpha), \frac{\alpha^2}{(\alpha+1)^2}] \subseteq I_2 \cup I_3$$

Thus, exploring the dynamics of $M(y^{(0)})$ on the intervals I_2, I_3 and I_5 is sufficient. Firstly, we are going to explore the dynamics of $M(y^{(0)})$ on I_5. To achieve this objective, the following claim is required.

Claim 2 *Let* $y^{(0)} \in I_5$. *Then,* $n_k \in \mathbb{N}$*, such that* $M^{(n_k)}(y^{(0)}) \in I_2 \cup I_3$.

Proof. Suppose that I_5 is an invariant interval and let $y^{(0)} \in I_5$, which results in that $y^{(n)} \in I_5$ for any $n \in \mathbb{N}$. Obviously, $M^{(n+1)}(y^{(0)}) \leq M^{(n)}(y^{(0)})$, which indicates that $\{M^{(n)}(y^{(0)})\}_{n=1}^{\infty}$ is a bounded decreasing sequence that converges to a fixed point of M. However, $Fix(M) \cap [0, \frac{1}{2}] = \emptyset$, which is a contradiction. Hence, $n_k \in \mathbb{N}$, such that $M^{(n_k)}(y^{(0)}) \in I_2 \cup I_3$. □

Accordingly, it is enough to explore the dynamics of $M(y^{(0)})$ on $I_2 \cup I_3$. Evidently, $M(y)$ is a decreasing on I_2 and I_3. $M^{(2)}(y^{(0)}) \geq y^{(0)}$ if $y^{(0)} \in I_2$ and $M^{(2)}(y^{(0)}) \leq y^{(0)}$, if $y^{(0)} \in I_3$ are easily shown. Thus, to explore the behavior of V_{19} on the interval I_2 and I_3, we should examine two separately cases.

(1) Let $y^{(0)} \in I_2$. Then, $M(y^{(0)}) \in I_3$ and $M^{(2)}(y^{(0)}) \in I_2$. Define the function $G(y^{(0)}) = M^{(2)}(y^{(0)}) - y^{(0)}$. Evidently, $G(y^{(0)}) \geq 0$ on I_2. This means that $M^{(2n)}(y^{(0)}) \geq M^{(2(n-1))}(y^{(0)})$, which indicates that $\{M^{(2n)}(y^{(0)})\}_{n=1}^{\infty}$ is an increasing bounded sequence. Moreover, sequence $\{M^{(2n)}(y^{(0)})\}_{n=1}^{\infty}$ converges to a fixed point of $M^{(2)}$. It can be easily to see that $y^{-}(\alpha)$ is fixed point of $M^{(2)}$ that is the only possible point the convergence trajectory. Hence, $y^{(2n)}$ converges to $y^{-}(\alpha)$.

(2) Let $y^{(0)} \in I_3$. Then, $M(y^{(0)}) \in I_2$ and $M^{(2)}(y^{(0)}) \in I_3$. In the same manner, define the function $D(y^{(0)}) = y^{(0)} - M^{(2)}(y^{(0)})$, it is easy to see that $D(y^{(0)}) \geq 0$ on I_3. Thus, $M^{(2n)}(y^{(0)}) \leq M^{(2(n-1))}(y^{(0)})$, which indicates that $\{M^{(2n)}(y^{(0)})\}_{n=1}^{\infty}$ is a bounded decreasing sequence. Furthermore, sequence$\{M^{(2n)}(y^{(0)})\}_{n=1}^{\infty}$ converges to a fixed point of $M^{(2)}$. One can find that $y^{-}(\alpha)$ is fixed point for $M^{(2)}$ that is the only possible point of the convergence trajectory. Thus, $y^{(2n)}$ converges to $y^{-}(\alpha)$. Hence, $M^{(n)}$ converges to $y^{-}(\alpha) = y^{+}$.

In accordance with what we have proven, we have $\{x^{(n)}\}_{n=1}^{\infty}$ converging to x^{+} and $\{y^{(n)}\}_{n=1}^{\infty}$ converging to y^{+}. One concludes that the limiting point when $x_1^{(0)} \notin L_1$ is $\omega_{V_{19}} = \{(x^{+}, y^{+}, z^{+})\}$, the proof is completed. □

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