

Numerical Solution of Singularly Perturbed BVPs using an Optimal Fitted One-Step Integration Scheme via Initial Value Method

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Abstract

In this paper, a numerical solution of singularly perturbed BVPs (SPBVPs) using an optimal fitted one-step integration scheme via initial value method is present. The original second order SPBVP is replaced by an asymptotically approximate first order initial value singularly perturbed problem (IVSPP) and solved using an optimal fitted one-step integration scheme. The error analysis is present. Some SPBVPs are solved and the numerical results confirm that the suggested method is accurate and efficient in solving the considered problems.

Keywords: Singularly perturbed BVPs, Initial value methods, Exponentially fitted finite difference schemes.

I. INTRODUCTION

Singularly perturbed boundary value problems (SPBVPs) are common in applied sciences and engineering. They often occur in, for example, fluid dynamics, quantum mechanics, chemical reactions, electrical networks, etc. A well-known fact is that the solution of such problems has a multiscale character, i.e. there are thin transition layers where the solution varies very rapidly, while away from the layers the solution behaves regularly and varies slowly. For a detailed discussion on the analytical and numerical treatment of SPBVPs we may refer the reader to the books of O'Malley [1], Doolan *et al.* [2], Roos *et al.* [3], Miller *et al.* [4] and references therein [5-25]. In this paper, a numerical solution of singularly perturbed BVPs (SPBVPs) using an optimal fitted one-step integration scheme via initial value method is present. The original second order SPBVP is replaced by an asymptotically approximate first order initial value singularly perturbed problem (IVSPP) and solved by using an optimal fitted one-step integration scheme. The error analysis is present. Some SPBVPs are solved and the numerical results confirm that the suggested method is accurate and efficient in solving the considered SPBVPs.

II. DESCRIPTION OF THE METHOD

Consider the two point second order SPBVP

$$\varepsilon y'' + p(x)y' + q(x, y) = h(x), \quad x \in [0, 1], \quad (1)$$

with boundary conditions

$$y(0) = \alpha \text{ and } y(1) = \beta, \quad (2)$$

Where $0 < \varepsilon \ll 1$, α and β are given constants, $p(x)$, $q(x, y)$ and $h(x)$ are assumed to be sufficiently continuously differentiable functions, and $p(x) \geq M > 0$ for $x \in [0, 1]$ where M is some positive constant. Under these assumptions, the problem (1-2) has a solution which, in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$.

Following the same procedure suggested by El-Zahar and EL-Kabeir [9], equation (1) can be written as

$$\varepsilon y'' + (p(x)y)' = F(x, y), \quad (3)$$

where $F(x, y) = h(x) + p'(x)y - q(x, y)$.

Now, let $u(x)$ be the solution of the reduced problem

$$p(x)u' + q(x, u) = h(x), \quad (4)$$

with initial condition $u(1) = \beta$.

Also, Eq. (4) can be written as

$$(p(x)u)' = F(x, u), \quad (5)$$

where $F(x, u) = h(x) + p'(x)u - q(x, u)$.

Subtracting Eq.(5) from Eq.(3) and integrating the resulting equation, we have

$$\int_1^x \left(\varepsilon \frac{d^2 y(s)}{ds^2} + \frac{d}{dx} (p(s)y(s)) \right) ds = \int_1^x \left(\frac{d}{dx} (p(s)u(s)) \right) ds + E, \quad (6)$$

where

$$E = \int_1^x (F(s, y(s)) - F(s, u(s))) ds. \quad (7)$$

From (6) and (7), we get

$$\varepsilon y' + p(x)y = p(x)u + K + E, \quad y(0) = \alpha, \quad (8)$$

where

$$K = \left(\varepsilon y'(s) + p(s)(y(s) - u(s)) \right)_{s=1} = \varepsilon y'(1)$$

Lemma 1. Let $y(x)$ and $u(x)$ be respectively the solutions of the SPBVP (1-2) and the reduced problem (4), then

$$|y(x) - u(x)| \leq C \left(\varepsilon + e^{-Mx/\varepsilon} \right), x \in [0,1].$$

Proof. See (J. Lorenz [13], theorem 3).

Lemma 2. Let $y(x)$ be the solution of the SPBVP (1-2), then

$$|y^{(m)}(x)| \leq C \left(1 + \varepsilon^{-m} e^{-Mx/\varepsilon} \right), x \in [0,1], m = 0,1,2, \dots$$

Proof. See (R. Vulanovic [14], theorem1).

With the help of these lemmas 1 and 2 we can prove the following theorem.

Theorem 1. Let $y(x), u(x)$ and $w(x)$ be respectively the solutions of the SPBVP (1-2), the reduced problem (4) and the following initial-value problem

$$\varepsilon w' + p(x)w = p(x)u, \quad (9)$$

with initial condition $w(0) = \alpha$.

Then,

$$|y(x) - w(x)| \leq C \varepsilon, \quad \forall x \in [0,1].$$

Proof. From Eq.(7) and by using lemma 1 we get the following bounded error

$$|E| = \left| \int_1^x (F(s, y(s)) - F(s, u(s))) ds \right|, \\ \leq \int_1^x \left(|F_y(s, \xi)| |y(s) - u(s)| \right) ds \leq C \varepsilon, \quad (10)$$

where, ξ lies between $u(x)$ and $y(x)$.

Let $m=1$ in lemma 2, then we get the following bounded error

$$|K| = |\varepsilon y'(1)| \leq C \varepsilon. \quad (11)$$

Therefore, Eq.(8) becomes

$$\varepsilon y' + p(x)y = p(x)u + O(\varepsilon), \quad y(0) = \alpha, \quad (12)$$

To estimate the error involved in the solution $w(x)$ of Eq. (9) we proceed as follows

Let $z(x) = y(x) - w(x)$. Then $z(x)$ satisfies the following IVP

$$\varepsilon z'(x) + p(x)z(x) = O(\varepsilon), \quad z(0) = 0, \quad (13)$$

where, ζ lies between $w(x)$ and $y(x)$.

By integrating Eq. (13), it can be shown that

$$|z(x)| = |y(x) - w(x)| \leq C \varepsilon.$$

The proof of theorem 1 is completed.

Thus, by Theorem 1 the solution of the two-point boundary

value problem (1-2) is approximated by that of the first order initial value problem (9).

We solve the initial value problem (9) by using an optimal fitted finite difference one-step integration scheme as given in next section.

III. EXPONENTIALLY FITTED ONE-STEP SCHEME

Discretizing SPIVP (9) by the Exponentially Fitted One-Step Scheme (EFFD) derived by Salama and Bakr [15] results in the following optimal fitted one-step integration scheme

$$\frac{\varepsilon}{h} \left(\sigma(-\rho \tilde{p}_j) w_{j+1} - \sigma(\rho \tilde{p}_j) w_j \right) = \tilde{p}_j u(x_{j+1}), \quad (17)$$

where $w_0 = \alpha$, $h = 1/N$, $0 \leq j < N-1$, $\rho = h/\varepsilon$,

$$\tilde{p}_j = \left[p(x_j) + p(x_{j+1}) \right] / 2, \quad \sigma(\rho \tilde{p}_j) = \rho \tilde{p}_j / [\exp(\rho \tilde{p}_j) - 1]$$

$$\sigma(-\rho \tilde{p}_j) = \rho \tilde{p}_j / [1 - \exp(-\rho \tilde{p}_j)].$$

Theorem 2. Let $w(x)$ be the solution of SPIVP (9) and w_j be the numerical solution obtained by the two-term recurrence relation (17). Then, at each mesh point x_j , we have the following error estimate:

$$|w(x_j) - w_j| \leq C \min(h^2, \varepsilon), \quad (18)$$

where C is independent of j , ε and h .

Proof. See Salama and Bakr [15], and Doolan et al. [2]

Theorem 3. Let $y(x)$ be the solution of SPBVP (1) and w_j be the numerical solution obtained by the two-term recurrence relation (17). Then, at each mesh point x_j , we have the following error estimate:

$$|y(x_j) - w_j| \leq C \left(\varepsilon + \min(h^2, \varepsilon) \right), \quad (19)$$

where C is independent of j , ε and h .

Proof. We have

$$|y(x_j) - w_j| \leq |y(x_j) - w(x_j)| + |w(x_j) - w_j|$$

By applying Theorem 1 and Theorem 2 to the right hand side of the above inequality, we get

$$|y(x_j) - w_j| \leq C \left(\varepsilon + \min(h^2, \varepsilon) \right).$$

The proof of Theorem 3 is completed

IV. NUMERICAL RESULTS

In this section, five SPBVPs are solved to illustrate the accuracy of the method. These SPBVPs have been discussed in the literature and their approximate solutions are available for comparison. To get more information about the behavior

of the solution in the boundary layer region, the solution is computed for $h = 2\varepsilon$ over a narrow region $x_j \in [0, 20\varepsilon]$ and $h = 0.1$ over the outer region $x_j \in (20\varepsilon, 1]$ at different values of ε

Example 1. Consider the following homogeneous SPBVP [10,21]

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \quad x \in [0,1], \quad (20)$$

with boundary conditions $y(0) = 1$ and $y(1) = 1$. The exact solution is given by

$$y(x) = \frac{(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}}{e^{m_2} - e^{m_1}},$$

where

$$m_1 = (-1 + \sqrt{1 + 4\varepsilon}) / (2\varepsilon) \text{ and } m_2 = (-1 - \sqrt{1 + 4\varepsilon}) / (2\varepsilon).$$

The solution error $|y(x_i) - w_i|_\infty$ of Example 1 is shown in Table 1 at different values of the perturbation parameter ε .

Example 2. Consider the following non-homogenous SPBVP from fluid dynamics for fluid of small viscosity [22]

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad x \in [0,1], \quad (21)$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{1 - e^{-1/\varepsilon}}.$$

The solution error of Example 2 is shown in Table 2 at different values of the perturbation parameter ε .

Table 1. Solution error of Example 1 at different values ε

Nodes	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
2ε	2.4663e-4	2.4658e-5	2.4657e-6	2.4659e-7
4ε	4.2781e-4	4.2787e-5	4.2788e-6	4.2790e-7
6ε	4.7247e-4	4.7244e-5	4.7244e-6	4.7246e-7
8ε	4.8140e-4	4.8120e-5	4.8118e-6	4.8120e-7
10ε	4.8317e-4	4.8277e-5	4.8273e-6	4.8275e-7
12ε	4.8365e-4	4.8306e-5	4.8300e-6	4.8301e-7
14ε	4.8391e-4	4.8312e-5	4.8304e-6	4.8305e-7
16ε	4.8414e-4	4.8315e-5	4.8305e-6	4.8306e-7
18ε	4.8436e-4	4.8318e-5	4.8305e-6	4.8306e-7
20ε	4.8458e-4	4.8320e-5	4.8305e-6	4.8306e-7
0.1	3.6535e-4	3.6586e-5	3.6591e-6	3.6593e-7
0.2	3.5889e-4	3.5941e-5	3.5946e-6	3.5948e-7
0.3	3.4704e-4	3.4755e-5	3.4760e-6	3.4763e-7
0.4	3.2873e-4	3.2923e-5	3.2928e-6	3.2931e-7
0.5	3.0274e-4	3.0321e-5	3.0326e-6	3.0328e-7
0.6	2.6765e-4	2.6808e-5	2.6812e-6	2.6814e-7
0.7	2.2184e-4	2.2220e-5	2.2224e-6	2.2226e-7
0.8	1.6344e-4	1.6372e-5	1.6374e-6	1.6376e-7
0.9	9.0308e-5	9.0466e-6	9.0482e-7	9.0489e-8
1.0	0	0	0	0

Table 2. Solution error of Example 2 at different values ε

Nodes	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
2 ε	1.9965e-3	1.9997e-4	2.0000e-5	2.0000e-6
4 ε	2.2644e-3	2.2700e-4	2.2706e-5	2.2707e-6
6 ε	2.2982e-3	2.3064e-4	2.3072e-5	2.3073e-6
8 ε	2.3004e-3	2.3111e-4	2.3121e-5	2.3122e-6
10 ε	2.2984e-3	2.3115e-4	2.3128e-5	2.3129e-6
12 ε	2.2957e-3	2.3113e-4	2.3128e-5	2.3130e-6
14 ε	2.2930e-3	2.3110e-4	2.3128e-5	2.3130e-6
16 ε	2.2902e-3	2.3108e-4	2.3128e-5	2.3130e-6
18 ε	2.2875e-3	2.3105e-4	2.3128e-5	2.3130e-6
20 ε	2.2847e-3	2.3102e-4	2.3128e-5	2.3130e-6
0.1	1.8000e-3	1.8000e-4	1.8000e-5	1.8000e-6
0.2	1.6000e-3	1.6000e-4	1.6000e-5	1.6000e-6
0.3	1.4000e-3	1.4000e-4	1.4000e-5	1.4000e-6
0.4	1.2000e-3	1.2000e-4	1.2000e-5	1.2000e-6
0.5	1.0000e-3	1.0000e-4	1.0000e-5	1.0000e-6
0.6	8.0000e-4	8.0000e-5	8.0000e-6	8.0000e-7
0.7	6.0000e-4	6.0000e-5	6.0000e-6	6.0000e-7
0.8	4.0000e-4	4.0000e-5	4.0000e-6	4.0000e-7
0.9	2.0000e-4	2.0000e-5	2.0000e-6	2.0000e-7
1.0	0	0	0	1.1102e-16

Table 3. Solution error of Example 3 at different values ε

Nodes	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
2 ε	6.7803e-5	6.7681e-6	6.7669e-7	6.7668e-8
4 ε	7.7279e-5	7.6871e-6	7.6830e-7	7.6826e-8
6 ε	7.8859e-5	7.8144e-6	7.8073e-7	7.8066e-8
8 ε	7.9370e-5	7.8345e-6	7.8244e-7	7.8234e-8
10 ε	7.9737e-5	7.8402e-6	7.8270e-7	7.8257e-8
12 ε	8.0085e-5	7.8439e-6	7.8276e-7	7.8260e-8
14 ε	8.0433e-5	7.8473e-6	7.8280e-7	7.8261e-8
16 ε	8.0782e-5	7.8507e-6	7.8284e-7	7.8261e-8
18 ε	8.1133e-5	7.8541e-6	7.8287e-7	7.8262e-8
20 ε	8.1486e-5	7.8575e-6	7.8290e-7	7.8262e-8
0.1	0	0	0	0
0.2	0	0	0	0
0.3	0	0	0	0
0.4	0	0	0	0
0.5	0	0	0	0
0.6	0	0	0	0
0.7	0	0	0	0
0.8	0	0	0	0
0.9	0	0	0	0
1.0	0	0	0	0

Example 3. Consider the variable coefficient SPBVP [23,24]

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0; \quad x \in [0,1], \quad (22)$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. The exact solution is approximated in [24] as

$$y(x) = \frac{1}{2-x} - \frac{1}{2} e^{-(x-x^2/4)/\varepsilon}$$

By considering the given problem solution as our exact solution, Table 3 presents the maximum absolute error $|y(x_i) - w_i|_{\infty}$ for the numerical solution of Example 3 at different values of ε .

Example 4. Consider the non-linear SPBVP [21]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0, \quad x \in [0,1] \quad (23)$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. The problem (23) has a uniformly valid approximation [21] for comparison,

$$y(x) = \log_e(2/(1+x)) - (\log_e 2) e^{-2x/\varepsilon}.$$

The solution error of Example 4 is shown in Table 4 at different values of the perturbation parameter ε .

Table 4. Solution error of Example 4 at different values ε

Nodes	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
2ε	3.6595e-5	3.6628e-6	3.6631e-7	3.6631e-8
4ε	3.7192e-5	3.7291e-6	3.7301e-7	3.7302e-8
6ε	3.7130e-5	3.7296e-6	3.7313e-7	3.7314e-8
8ε	3.7057e-5	3.7289e-6	3.7312e-7	3.7314e-8
10ε	3.6983e-5	3.7281e-6	3.7311e-7	3.7314e-8
12ε	3.6910e-5	3.7274e-6	3.7311e-7	3.7314e-8
14ε	3.6837e-5	3.7266e-6	3.7310e-7	3.7314e-8
16ε	3.6765e-5	3.7259e-6	3.7309e-7	3.7314e-8
18ε	3.6692e-5	3.7252e-6	3.7308e-7	3.7314e-8
20ε	3.6620e-5	3.7244e-6	3.7308e-7	3.7314e-8
0.1	0	0	0	0
0.2	0	0	0	0
0.3	0	0	0	0
0.4	0	0	0	0
0.5	0	0	0	0
0.6	0	0	0	0
0.7	0	0	0	0
0.8	0	0	0	0
0.9	0	0	0	0
1.0	0	0	0	0

Table 5. Solution error of Example 5 at different values ε

Nodes	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
2ε	2.6959e-4	2.7056e-5	2.7066e-6	2.7067e-7
4ε	1.5996e-4	1.6069e-5	1.6077e-6	1.6078e-7
6ε	2.7986e-4	2.8225e-5	2.8249e-6	2.8251e-7
8ε	3.0348e-4	3.0715e-5	3.0752e-6	3.0756e-7
10ε	3.0676e-4	3.1157e-5	3.1206e-6	3.1211e-7
12ε	3.0630e-4	3.1222e-5	3.1282e-6	3.1288e-7
14ε	3.0522e-4	3.1222e-5	3.1293e-6	3.1300e-7
16ε	3.0404e-4	3.1211e-5	3.1294e-6	3.1302e-7
18ε	3.0284e-4	3.1199e-5	3.1293e-6	3.1302e-7
20ε	3.0166e-4	3.1187e-5	3.1292e-6	3.1302e-7
0.1	0	0	0	0
0.2	0	0	0	0
0.3	0	0	0	0
0.4	0	0	0	0
0.5	0	0	0	0
0.6	0	0	0	0
0.7	0	0	0	0
0.8	0	0	0	0
0.9	0	0	0	0
1.0	0	0	0	0

Example 5. Consider the non-linear SPBVP [25] given by

$$\varepsilon y''(x) + y'(x) + (y(x))^2 = 0 \quad x \in [0,1] \quad (24)$$

with boundary conditions $y(0) = 0$ and $y(1) = 0.5$. The problem (24) has a uniformly valid approximation [25] for comparison,

$$y(x) = \frac{1}{1+x} - \frac{e^{-x/\varepsilon}}{(1+x)^2}.$$

The solution error of Example 5 is shown in Table 5 at different values of the perturbation parameter ε . The numerical results show that the proposed method approximates the exact solution very well. Moreover, the numerical solution improves in accuracy as the perturbation parameter ε tends to zero.

V. CONCLUSIONS

In this paper, a numerical solution of SPBVPs having left end boundary layer is presented using an optimal fitted one-step integration scheme via initial value method. The original second order SPBVP is replaced by an asymptotically

approximate first order IVSPP and solved by an optimal fitted one-step integration scheme. The error analysis is present. Several SPBVPs are solved and the numerical results show that the proposed method approximates the exact solution very well. Moreover, the numerical solution improves in accuracy as the perturbation parameter ε tends to zero.

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