

# Single Finite Fourier Sine Integral Transform Method for the Flexural Analysis of Rectangular Kirchhoff Plate with Opposite Edges Simply Supported, Other Edges Clamped for the Case of Triangular Load Distribution

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## Abstract

This paper presents the single finite Fourier sine integral transform method for the flexural analysis of rectangular Kirchhoff plate with opposite edges simply supported, and the other edges clamped for the case of triangular load distribution on the plate domain. The finite sine transform method is adopted because the sinusoidal integral kernel function satisfies all the Dirichlet boundary conditions along the simply supported edges. The governing domain equation which is a nonhomogeneous biharmonic equation is converted by the transformation to an integral equation over the solution domain. The integral equation is reduced by the linearity properties of the transformation, Leibnitz rule and boundary conditions along the simply supported edges to ordinary differential equations (ODEs). Classical methods of solving ODEs in closed-form were used to obtain the solutions that satisfy the boundary conditions along the clamped edges. Thus exact solutions to the boundary value problem that satisfy the governing equation at all points in the solution domain as well as at all points along the four edges are obtained. The resulting expression for the deflection is a single infinite series with demonstrated rapidly convergent properties. The bending moments  $M_{xx}$ ,  $M_{yy}$  expressions were obtained using the bending moment-deflection relations as infinite series with demonstrated rapid convergence. The converged values of the maximum deflection and bending moments  $M_{xx}$ ,  $M_{yy}$  evaluated at the plate centre are in excellent agreement with the previous exact results to the problem obtained by researchers who used the Kantorovich-Vlasov method; Levy's method; and the superposition method.

**Keyword:** Single finite Fourier sine integral transform, method, Kirchhoff plate theory, Inhomogeneous biharmonic differential equation, Single infinite series, Deflection, Bending moment.

## 1. INTRODUCTION

Plate problems of bending, stability and vibration are important themes in structural analysis due to the extensive applications of plates in the fields of structural, mechanical, naval, marine, aeronautical, and aerospace engineering. The problems have attracted intensive research interest leading to the development of several theories of plate behaviour and

several methods for the analytical and numerical treatment of plate problems.

Broadly, plates have been classified as thin, moderately thick and thick plates depending on the ratios of their transverse dimensions to the least inplane dimension. They have also been classified based on their material properties as anisotropic, isotropic, orthotropic or transversely isotropic or homogeneous, nonhomogeneous, linearly elastic or nonlinearly elastic [1 – 10].

Formulations of governing theories for each of those classifications of plates have been made in the literature. Plate governing theories have been developed for thin plates, moderately thick plates and thick plates. They have also been developed for anisotropic plates, isotropic plates, homogeneous and nonhomogeneous plates that are either thin, moderately thick or thick.

Thin plate idealizations rely on the relative smallness of the transverse dimension of the plate compared to the lateral or inplane dimensions thus reducing the three-dimensional (3D) nature of plates to two-dimensional (2D) simplifications. This simplification has its attendant consequence of introduction of errors which become really insignificant when the ratio of the plate thickness to the least inplane dimension is less than 0.05, which is the threshold for the definition of thin plates.

The following theories have been derived and formulated for plate problems: Kirchhoff plate theory, Mindlin plate theory, Reissner plate theory, Reddy plate theory, Levinson plate theory, shear deformation plate theories, and higher order shear deformation plate theories.

Formulations, development and solutions of moderately thick plate problems have been presented by amongst others, by Ike, [11], Ike [12], Ike et al [13], Nwoji et al [14], Ike [15], Nwoji et al [16] and Ike et al [17].

Plate problems have been solved using numerical and analytical methods. Some of the numerical techniques used in presenting and solving plate problems include: Ordinary Finite Difference Method by Ezeh et al [18], Finite Element Methods and Energy methods by Shames and Dym [2].

Some of the analytical methods that have been used for plate problems are (i) superposition methods; (ii) Navier double

Fourier series method; (iii) Levy's single trigonometric series method; (iv) Galerkin-Vlasov method; (v) Ritz variational method; (vi) Kantorovich method and variants of the Kantorovich method and (vii) Fourier series methods:

Superposition methods have been used by Timoshenko and Woinowsky-Krieger [1]. Navier's double Fourier series (trigonometric series) method have been used for plate problems by Timoshenko and Woinowsky-Krieger, [1], and Szilard [7].

Levy's single trigonometric series method have been used for plate problems by Timoshenko and Woinowsky-Krieger [1], Shames and Dym [2].

Galerkin-Vlasov method have been used for plate problems by Osadebe et al [19], Nwoji et al [20], Ike [21], Mama and Ike [22], Onyia et al [23], Mama et al [24].

Ritz variational method has been used for plate problems by Ike [25], Ike [26] and Nwoji et al [27].

Kantorovich variational method has been used for plate problems by Ike [28], Ike and Nwoji [29], Nwoji et al [30], Onah et al [31], Ike and Mama [32], Ike et al [33].

Finite Fourier sine transform methods have been used to solve plate problems by Ike [34], Nwoji et al [35], Mama et al [36], Mama et al [37], Mama et al [38], Onah et al [39], Ike [40], Oguaghamba and Ike [41], and Onyia et al [42].

Other significant contributions to the formulation and solutions of plate problems include the following publications: Ghannadiasi and Noorzad [43], Bigdoli et al [44], Zargaripoor et al [45], Goodarzi et al [46], Javidi et al [47], Makvandi et al [48], Sayyad and Ghugal [49], Mirzapour et al [50], Shahabian et al [51], Abdollahzadeh and Ghobadi [52], Cuba et al [53] and Jahanpour and Roozbabani [54].

In this paper, the single finite Fourier sine integral transform method is used for the flexural analysis of rectangular thin plates with two opposite clamped edges and the other two edges simply supported for the case where the plate is subject to triangular load distribution over the entire domain. Though the methodology adopted has been previously applied by the authors to plate stability and flexural problems, this is a first attempt at applying the method to thin CSCS plate flexure problem under triangular load distribution.

### Justification for the use of the single finite Fourier sine integral transform method

The single finite Fourier sine integral transform method is adopted for the study because the considered plate is simply supported along the two edges  $x = 0$  and  $x = a$  and the sinusoidal kernel function in the transformation a priori satisfies the Dirichlet boundary conditions along the edges,  $x = 0$  and  $x = a$ . Thus, the method renders unnecessary prior selection of the shape function for solving the boundary value problem. Another justification, which is evident in past problems solved by the method is the great simplification offered by the use of the method in reducing the boundary value problem to ordinary differential equations and ultimately to algebraic problems [42].

## 2. THEORY

Kirchhoff small deflection thin plate theory is used for the plate flexure problem. The governing domain equation is given by the inhomogeneous biharmonic equation:

$$D\nabla^4 u_z(x, y) = p(x, y) \quad (1)$$

$$\text{where } D = \frac{Eh^3}{12(1 - \mu^2)} \quad (2)$$

$D$  is the flexural rigidity of the plate,  $u_z(x, y)$  is the transverse deflection of the plate,  $p(x, y)$  is the distributed transverse load over the plate domain.  $h$  is the plate thickness,  $\mu$  is the Poisson's ratio of the plate material.  $x$  and  $y$  are the inplane Cartesian coordinates.  $E$  is the Young's modulus of elasticity of the plate material.

$\nabla^4$  is the biharmonic operator given explicitly by:

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad (3)$$

The bending moments  $M_{xx}$  and  $M_{yy}$  are given in terms of transverse deflections  $u_z(x, y)$  as:

$$M_{xx} = -D \left( \frac{\partial^2 u_z}{\partial x^2} + \mu \frac{\partial^2 u_z}{\partial y^2} \right) \quad (4)$$

$$M_{yy} = -D \left( \frac{\partial^2 u_z}{\partial y^2} + \mu \frac{\partial^2 u_z}{\partial x^2} \right) \quad (5)$$

The CSCS thin plate considered is shown in Figure 1, which also shows the definition of coordinates and the applied triangular load over the plate.

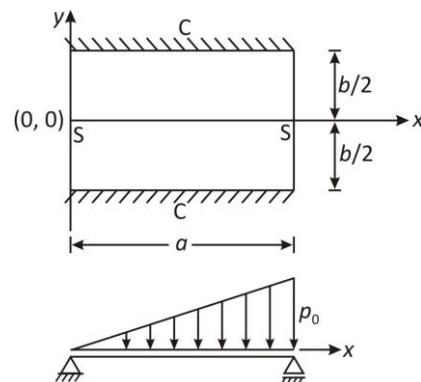


Figure 1: Kirchhoff CSCS plate subjected to triangular load distribution  $p(x, y) = p_0 x/a$

The coordinates system are chosen to take advantage of the symmetrical nature of the problem about the  $x$  coordinate axis. The CSCS plate is simply supported at the edges  $x = 0$  and  $x = a$ , and clamped at the edges  $y = 0$  and  $y = b$ . The triangular load distribution is given by

$$p(x, y) = \frac{p_0 x}{a} \quad (6)$$

where  $p_0$  is the intensity of the triangular load distribution at  $x = a$ , as shown in Figure 1.

The boundary conditions for the CSCS thin plate shown in Figure 1 are:

$$\text{Along the simply supported edges } x = 0, x = a, \quad u_z(x = 0, y) = u_z(x = a, y) = 0 \quad (7)$$

$$\frac{\partial^2 u_z}{\partial x^2}(x=0, y) = \frac{\partial^2 u_z}{\partial x^2}(x=a, y) = 0 \quad (8)$$

Along the clamped edges  $y = \pm b/2$ ,

$$u_z(x, y = \pm b/2) = 0 \quad (9)$$

$$\frac{\partial u_z}{\partial y}(x, y = \pm b/2) = 0 \quad (10)$$

### 3. METHODOLOGY

#### 3.1 Definitions

The single finite Fourier sine integral transform of a function  $v_s(x)$  of the independent variable  $x$  denoted by  $S_n(v(x))$  is defined as: [55, 56, 57, 58, 59]

$$S_n(v(x)) = \frac{2}{l} \int_0^l v(x) \sin \frac{n\pi x}{l} dx = V_s(n) \quad (11)$$

where  $0 \leq x \leq l$ , and  $n = 1, 2, 3, \dots$

$n$  is the single finite Fourier sine integral transform parameter. Other scholarly publications on integral transformations are found in references listed as [60 – 63].

The transforms of the derivatives of  $v(x)$  are found using integration by parts and Leibnitz rule as follows: [55, 56, 57, 58, 59]

$$S_n(v'(x)) = S_n\left(\frac{dv(x)}{dx}\right) = v(x=l) \sin n\pi - v(x=0) \sin 0 - \frac{n\pi}{l} \frac{2}{l} \int_0^l \cos \frac{n\pi x}{l} dx \quad (12)$$

$$S_n(v'(x)) = (v(x=l) \sin n\pi - v(x=0) \sin 0) - \frac{n\pi}{l} C_n v(x) = -\frac{n\pi}{l} C_n v(x) = -\frac{n\pi}{l} V_c(n) \quad (13)$$

$C_n(v(x)) = V_c(n)$  is the single finite Fourier cosine transform of  $v(x)$

$$S_n(v''(x)) = S_n\left(\frac{d^2 v(x)}{dx^2}\right) = -\left(\frac{n\pi}{l}\right) ((-1)^n v(x=l) - v(x=0)) - \left(\frac{n\pi}{l}\right)^2 \frac{2}{l} \int_0^l v(x) \sin \frac{n\pi x}{l} dx \quad (14)$$

$$S_n(v''(x)) = -\left(\frac{n\pi}{l}\right) ((-1)^n v(x=l) - v(x=0)) - \left(\frac{n\pi}{l}\right)^2 V_s(n) \quad \dots(15)$$

$$S_n(v'''(x)) = -\left(\frac{n\pi}{l}\right) \left( (-1)^n \frac{dv}{dx}(x=l) - \frac{dv}{dx}(x=0) \right) + \left(\frac{n\pi}{l}\right)^3 \frac{2}{l} \int_0^l v(x) \cos \frac{n\pi x}{l} dx \quad (16)$$

$$S_n(v'''(x)) = -\left(\frac{n\pi}{l}\right) ((-1)^n v'(x=l) - v'(x=0)) + \left(\frac{n\pi}{l}\right)^3 V_c(n) \quad \dots(17)$$

$$S_n(v^{iv}(x)) = S_n\left(\frac{d^4 v(x)}{dx^4}\right) = \left(\frac{n\pi}{l}\right)^4 \frac{2}{l} \int_0^l v(x) \sin \frac{n\pi x}{l} dx +$$

$$\left(\frac{n\pi}{l}\right)^3 (v(x=l)(-1)^n - v(x=0)) - \left(\frac{n\pi}{l}\right) ((-1)^n v''(x=l) - v''(x=0)) \quad (18)$$

When the function  $v$  is a function of two independent variables  $v(x, y)$ , we have:

$$S_n(v(x, y)) = \frac{2}{l} \int_0^l v(x, y) \sin \frac{n\pi x}{l} dx = V_s(n, y) \quad (19)$$

$$S_n\left(\frac{\partial^2 v(x, y)}{\partial x^2}\right) = -\left(\frac{n\pi}{l}\right) ((-1)^n v(x=l, y) - v(x=0, y)) - \left(\frac{n\pi}{l}\right)^2 V_s(n, y) \quad (20)$$

$$S_n\left(\frac{\partial^2 v(x, y)}{\partial x^4}\right) = \left(\frac{n\pi}{l}\right)^4 V_s(n, y) + \left(\frac{n\pi}{l}\right)^3 ((-1)^n (v(x=l, y) - v(x=0, y)) - \left(\frac{n\pi}{l}\right) ((-1)^n v''(x=l, y) - v''(x=0, y)) \quad (21)$$

For problems with Dirichlet boundary conditions,

$$v(x=l, y) = v(x=0, y) = 0 \quad (22)$$

$$v''(x=l, y) = v''(x=0, y) = 0 \quad (23)$$

The single finite Fourier sine integral transforms of the second and fourth derivatives of  $v(x, y)$  with respect to  $x$  simplify greatly to:

$$S_n\left(\frac{\partial^2 v(x, y)}{\partial x^2}\right) = -\left(\frac{n\pi}{l}\right)^2 V_s(n, y) \quad (24)$$

$$S_n\left(\frac{\partial^4 v(x, y)}{\partial x^4}\right) = \left(\frac{n\pi}{l}\right)^4 V_s(n, y) \quad (25)$$

#### 3.2 Single finite Fourier sine integral transformation of the governing domain equation

The partial differential equation (PDE) for the domain for the problem considered is given explicitly by the fourth order inhomogeneous equation:

$$D\left(\frac{\partial^4 u_z}{\partial x^4} + 2\frac{\partial^4 u_z}{\partial x^2 \partial y^2} + \frac{\partial^4 u_z}{\partial y^4}\right) = p(x, y) = \frac{p_0 x}{a} \quad (26)$$

$$\text{or, } \frac{\partial^4 u_z}{\partial x^4} + 2\frac{\partial^4 u_z}{\partial x^2 \partial y^2} + \frac{\partial^4 u_z}{\partial y^4} - \frac{p_0 x}{Da} = 0 \quad (27)$$

Application of the transformation gives the integral equation:

$$\frac{2}{a} \int_0^a \left( \frac{\partial^4 u_z}{\partial x^4} + \frac{2\partial^4 u_z}{\partial x^2 \partial y^2} + \frac{\partial^4 u_z}{\partial y^4} - \frac{p_0 x}{Da} \right) \sin \frac{n\pi x}{a} dx = 0 \quad (28)$$

From the linearity properties of the transformation and Leibnitz rule, the integral equation becomes the following inhomogeneous ordinary differential equation (ODE):

$$\left(\frac{n\pi}{a}\right)^4 \bar{U}_z(n, y) - 2\left(\frac{n\pi}{a}\right)^2 \frac{d^2}{dy^2} \bar{U}_z(n, y) + \frac{d^4}{dy^4} \bar{U}_z(n, y) - \frac{2p_0}{Da^2} \int_0^a x \sin \frac{n\pi x}{a} dx = 0 \quad (29)$$

where  $\bar{U}_z(n, y)$  is the single finite Fourier sine integral transform of  $u_z(x, y)$ , and is given by:

$$\bar{U}_z(n, y) = \frac{2}{a} \int_0^a u_z(x, y) \sin \frac{n\pi x}{a} dx \quad (30)$$

The ordinary differential equation (ODE) becomes upon evaluating the integral term:

$$\frac{d^4 \bar{U}_z(n, y)}{dy^4} - 2 \left( \frac{n\pi}{a} \right)^2 \frac{d^2 \bar{U}_z(n, y)}{dy^2} + \left( \frac{n\pi}{a} \right)^4 \bar{U}_z(n, y) = \frac{2p_0(-1)^{n+1}}{n\pi D} \quad (31)$$

The solution to Equation (31) is the sum of the homogeneous solution and the particular integral.

#### 4. RESULTS

##### Homogeneous solution

The homogeneous solution to Equation (31) is sought in exponential form. Thus, let

$$\bar{U}_{zh}(n, y) = \Gamma \exp \lambda y \quad (32)$$

where  $\Gamma$  and  $\lambda$  are parameters of the assumed (trial) homogeneous solution.

For  $\bar{U}_{zh}(n, y)$  to be solution, the homogeneous differential equation becomes

$$\left( \lambda^4 - 2 \left( \frac{n\pi}{a} \right)^2 + \left( \frac{n\pi}{a} \right)^4 \right) \Gamma e^{\lambda y} = 0 \quad (33)$$

For nontrivial solutions  $\Gamma e^{\lambda y} \neq 0$

Hence,

$$\lambda^4 - 2 \left( \frac{n\pi}{a} \right)^2 + \left( \frac{n\pi}{a} \right)^4 = \left( \lambda^2 - \left( \frac{n\pi}{a} \right)^2 \right)^2 = 0 \quad (34)$$

The solutions for  $\lambda$  are:

$$\lambda_{1,2} = \frac{n\pi}{a} \text{ twice} \quad (35)$$

$$\lambda_{3,4} = -\left( \frac{n\pi}{a} \right) \text{ twice} \quad (36)$$

The homogeneous solution becomes:

$$\bar{U}_{zh}(n, y) = c_{1n} \cosh \frac{n\pi y}{a} + c_{2n} \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} + c_{3n} \frac{n\pi y}{a} \cosh \frac{n\pi y}{a} + c_{4n} \sinh \frac{n\pi y}{a} \quad (37)$$

where  $c_{1n}, c_{2n}, c_{3n}$  and  $c_{4n}$  are integration constants.

The problem considered is symmetrical about the  $x$  axis. Hence,  $\bar{U}_{zh}(n, y)$  is expected to be a symmetrical function in  $y$ .

The symmetry of  $\bar{U}_{zh}(n, y)$ , demands that:

$$c_{3n} = c_{4n} = 0 \quad (38)$$

Hence the homogeneous solution becomes:

$$\bar{U}_{zh}(n, y) = c_{1n} \cosh \frac{n\pi y}{a} + c_{2n} \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} \quad (39)$$

##### Particular solution $\bar{U}_{zp}(n, y)$

The applied load  $p(x, y) = p_0 x/a$  does not depend upon  $y$ . Hence we expect that the derivatives of the particular solution  $\bar{U}_{zp}(n, y)$  with respect to  $y$  would vanish.

According:

$$\bar{U}_{zp}''(n, y) = U_{zp}^{iv}(n, y) = 0 \quad (40)$$

Then, Equation (23) becomes:

$$\left( \frac{n\pi}{a} \right)^4 \bar{U}_{zp}(n, y) = \frac{2p_0(-1)^{n+1}}{n\pi D} \quad (41)$$

Solving,

$$\bar{U}_{zp}(n, y) = \frac{2p_0 a^4 (-1)^{n+1}}{(n\pi)^5 D} \quad (42)$$

##### General solutions $\bar{U}_z(n, y)$

The general solution in the finite integral transform space is found as the superposition of the homogeneous solution and the particular solution.

Then,

$$\bar{U}_z(n, y) = \bar{U}_{zh}(n, y) + \bar{U}_{zp}(n, y) \quad (43)$$

$$\bar{U}_z(n, y) = c_{1n} \cosh \frac{n\pi y}{a} + c_{2n} \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} + \frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \dots (44)$$

##### General solution for transverse deflection in the physical domain

The general solution for transverse deflection in the problem domain is found by the inverse finite Fourier sine integral transform of  $\bar{U}_z(n, y)$ . By inversion,

$$u_z(n, y) = \sum_{n=1}^{\infty} \bar{U}_z(n, y) \sin \frac{n\pi x}{a} \quad (45)$$

Then, substituting the expression for  $\bar{U}_z(n, y)$  from Equation (44), we obtain the single infinite series:

$$u_z(x, y) = \sum_{n=1}^{\infty} \left( c_{1n} \cosh \frac{n\pi y}{a} + c_{2n} \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} + \frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \right) \sin \frac{n\pi x}{a} \quad (46)$$

The general solution for  $u_z(x, y)$  is expressed in terms of constants of integration  $c_{1n}$  and  $c_{2n}$  which are found using the boundary conditions along the clamped edges ( $y = \pm b/2$ ) of the plate.

##### Application of boundary conditions

Substituting Equation (46) in Equation (9) we have the following boundary condition equation:

$$u_z \left( x, y = \pm \frac{b}{2} \right) = \sum_{n=1}^{\infty} \left( c_{1n} \cosh \frac{n\pi b}{2a} + c_{2n} \frac{n\pi b}{2a} \sinh \frac{n\pi b}{2a} + \right)$$

$$\left. \frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \right) \sin \frac{n\pi x}{a} = 0 \quad (47)$$

Substitution of Equation (46) in Equation (10) gives another boundary condition equation regarding slopes at the clamped edges as follows:

$$\frac{\partial u_z}{\partial z} \left( x, y = \pm \frac{b}{2} \right) = \sum_{n=1}^{\infty} \left( c_{1n} \frac{n\pi}{a} \sinh \frac{n\pi b}{2a} + c_{2n} \frac{n\pi}{a} \left( \frac{n\pi b}{2a} \cosh \frac{n\pi b}{2a} + \sinh \frac{n\pi b}{2a} \right) \right) \sin \frac{n\pi x}{a} = 0 \quad (48)$$

Equation (48) is simplified to:

$$c_{1n} \frac{n\pi}{a} \sinh \frac{n\pi b}{2a} = -c_{2n} \frac{n\pi}{a} \left( \frac{n\pi b}{2a} \cosh \frac{n\pi b}{2a} + \sinh \frac{n\pi b}{2a} \right) \quad (49)$$

Hence,

$$c_{1n} = \frac{-c_{2n} \frac{n\pi}{a} \left( \frac{n\pi b}{2a} \cosh \frac{n\pi b}{2a} + \sinh \frac{n\pi b}{2a} \right)}{\frac{n\pi}{a} \sinh \frac{n\pi b}{2a}} \quad (50)$$

Simplifying further, Equation (50) becomes:

$$c_{1n} = -c_{2n} \left( \frac{\frac{n\pi b}{2a} \cosh \frac{n\pi b}{2a}}{\sinh \frac{n\pi b}{2a}} + 1 \right) \quad (51)$$

Equation (47) simplifies to:

$$c_{1n} \cosh \frac{n\pi b}{2a} + c_{2n} \frac{n\pi b}{2a} \sinh \frac{n\pi b}{2a} = -\frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \quad (52)$$

Substituting  $c_{1n}$  in Equation (51) into Equation (52) gives:

$$\begin{aligned} -c_{2n} \left( \frac{\frac{n\pi b}{2a} \cosh \frac{n\pi b}{2a}}{\sinh \frac{n\pi b}{2a}} + 1 \right) \cosh \frac{n\pi b}{2a} + c_{2n} \frac{n\pi b}{2a} \sinh \frac{n\pi b}{2a} \\ = \frac{-2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \end{aligned} \quad (53)$$

Simplifying,

$$\begin{aligned} -c_{2n} \frac{n\pi b}{2a} \frac{\cosh^2 \frac{n\pi b}{2a}}{\sinh \frac{n\pi b}{2a}} - c_{2n} \cosh \frac{n\pi b}{2a} + c_{2n} \frac{n\pi b}{2a} \sinh \frac{n\pi b}{2a} \\ = \frac{-2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \end{aligned} \quad (54)$$

Further simplification gives:

$$\begin{aligned} c_{2n} \frac{n\pi b}{2a} \left( \sinh \frac{n\pi b}{2a} - \frac{\cosh^2 \frac{n\pi b}{2a}}{\sinh \frac{n\pi b}{2a}} \right) - c_{2n} \cosh \frac{n\pi b}{2a} \\ = \frac{-2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \end{aligned} \quad (55)$$

Further simplification of Equation (55) yields:

$$c_{2n} \frac{n\pi b}{2a} \left( \frac{\sinh^2 \frac{n\pi b}{2a} - \cosh^2 \frac{n\pi b}{2a}}{\sinh \frac{n\pi b}{2a}} \right) - c_{2n} \cosh \frac{n\pi b}{2a}$$

$$= -\frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \quad (56)$$

We recall the hyperbolic identity:

$$\sinh^2 \theta - \cosh^2 \theta = -1 \quad (57)$$

Using the hyperbolic identity in Equation (57), Equation (56) becomes:

$$-c_{2n} \frac{n\pi b}{2a} \frac{1}{\sinh \frac{n\pi b}{2a}} - c_{2n} \cosh \frac{n\pi b}{2a} = \frac{-2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \quad (58)$$

Thus,

$$c_{2n} \left( \frac{n\pi b}{2a} \frac{1}{\sinh \frac{n\pi b}{2a}} + \cosh \frac{n\pi b}{2a} \right) = c_{2n} F_1 = \frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \quad \dots(59)$$

$$\text{where, } F_1 = \frac{n\pi b}{2a} \frac{1}{\sinh \frac{n\pi b}{2a}} + \cosh \frac{n\pi b}{2a} \quad (60)$$

$$\text{Then } c_{2n} = \frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \frac{1}{F_1} \quad (61)$$

Then,

$$c_{1n} = \frac{-2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5 F_1} \left( 1 + \frac{n\pi b}{2a} \coth \frac{n\pi b}{2a} \right) \quad (62)$$

In compact form, Equation (62) is expressed as:

$$c_{1n} = \frac{-2p_0 a^4 (-1)^{n+1} F_2}{D (n\pi)^5 F_1} \quad (63)$$

where,

$$F_2 = 1 + \frac{n\pi b}{2a} \coth \frac{n\pi b}{2a} \quad (64)$$

Thus, the transverse deflection is determined as the single infinite series:

$$\begin{aligned} u_z(x, y) = \sum_{n=1}^{\infty} \left\{ \frac{-2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5 F_1} F_2 \cosh \frac{n\pi y}{a} + \right. \\ \left. \frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} + \frac{2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5} \right\} \sin \frac{n\pi x}{a} \quad \dots(65) \end{aligned}$$

From Equation (65) it is confirmed that the transverse deflection expression satisfies all the Dirichlet boundary conditions along the simply supported edges given as:

$$\begin{aligned} \frac{\partial^2 u_z}{\partial x^2} (x=0, y) = \frac{\partial^2 u_z}{\partial x^2} (x=a, y) = u_z(x=0, y) \\ = u_z(x=a, y) = 0 \end{aligned}$$

$$\text{Since, } \sin \frac{n\pi x}{a} \Big|_{x=0} = 0 = \sin \frac{n\pi x}{a} \Big|_{x=a}$$

### Deflection at the centre of the plate

At the centre of the plate,  $x = a/2$ ,  $y = 0$ , and the deflection at the plate centre is obtained by substitution of the centre coordinates into Equation (65) to obtain:

$$u_z \left( x = \frac{a}{2}, y = 0 \right) = \sum_{n=1}^{\infty} \left\{ \frac{-2p_0 a^4 (-1)^{n+1}}{D (n\pi)^5 F_1} F_2 \cosh 0 + \right.$$

$$\frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5 F_1} 0 + \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \left\} \sin \frac{n\pi}{2} \quad (66)$$

Simplification of the series expression in Equation (66) gives:

$$u_z \left( x = \frac{a}{2}, y = 0 \right) = \sum_{n=1}^{\infty} \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \left( 1 - \frac{F_2}{F_1} \right) \quad (67)$$

Further simplification by factoring out the constants in Equation (67) gives:

$$u_z \left( x = \frac{a}{2}, y = 0 \right) = \frac{p_0a^4}{D} \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(n\pi)^5} \left( 1 - \frac{F_2}{F_1} \right) \quad (68)$$

The expression for the deflection of the centre of the plate can be given in general as:

$$u_z \left( x = \frac{a}{2}, y = 0 \right) = \alpha_1 \frac{p_0a^4}{D} \quad (69)$$

$$\text{where } \alpha_1 = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(n\pi)^5} \left( 1 - \frac{F_2}{F_1} \right) \quad (70)$$

$\alpha_1$  is the deflection coefficient for the centre of rectangular CSCS Kirchhoff plate subjected to triangular load distribution given by  $p(x, y) = p_0x/a$ .

### Bending moment distributions

By partial differentiations of  $u_z(x, y)$ , we have:

$$\frac{\partial^2 u_z}{\partial x^2} = - \sum_{n=1}^{\infty} \left( c_{1n} \cosh \frac{n\pi y}{a} + c_{2n} \frac{n\pi y}{a} \sinh \frac{n\pi y}{a} + \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \right) \left( \frac{n\pi}{a} \right)^2 \sin \frac{n\pi x}{a} \quad (71)$$

$$\frac{\partial^2 u_z}{\partial y^2} = \sum_{n=1}^{\infty} \left\{ c_{1n} \left( \frac{n\pi}{a} \right)^2 \cosh \frac{n\pi y}{a} + c_{2n} \left( \left( \frac{n\pi}{a} \right)^3 y \sinh \frac{n\pi y}{a} + 2 \left( \frac{n\pi}{a} \right)^2 \cosh \frac{n\pi y}{a} \right) \right\} \sin \frac{n\pi x}{a} \quad (72)$$

$$\frac{\partial^2 u_x}{\partial x^2} \left( x = \frac{a}{2}, y = 0 \right) = - \sum_{n=1}^{\infty} \left( c_{1n} \cosh 0 + \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \right) \left( \frac{n\pi}{a} \right)^2 \sin \frac{n\pi}{2} \quad (73)$$

$$\frac{\partial^2 u_z}{\partial x^2} \left( x = \frac{a}{2}, y = 0 \right) = - \sum_{n=1}^{\infty} \left( c_{1n} + \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \right) \left( \frac{n\pi}{a} \right)^2 \sin \frac{n\pi}{2} \quad (74)$$

$$\frac{\partial^2 u_z}{\partial y^2} \left( x = \frac{a}{2}, y = 0 \right) = \sum_{n=1}^{\infty} \left\{ c_{1n} \left( \frac{n\pi}{a} \right)^2 + c_{2n} \left( 2 \left( \frac{n\pi}{a} \right)^2 \right) \right\} \sin \frac{n\pi}{2} = \sum_{n=1}^{\infty} (c_{1n} + 2c_{2n}) \left( \frac{n\pi}{a} \right)^2 \sin \frac{n\pi}{2} \quad (75)$$

Hence at the plate centre, the bending moments  $M_{xx}$  and  $M_{yy}$  are obtained by substituting Equations (74) and (75) into Equations (4) and (5) and simplifying:

$$M_{xx} = -D \sum_{n=1}^{\infty} \left( \frac{n\pi}{a} \right)^2 \left\{ \mu(c_{1n} + 2c_{2n}) - c_{1n} - \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \right\} \sin \frac{n\pi}{2} \quad \dots(76)$$

$M_{xx}$  expressed by Equation (76) is presented using bending moment coefficients  $\alpha_2$  and  $\alpha_3$  as:

$$M_{xx} = \alpha_2 p_0 a^2 = \alpha_3 p_0 b^2 \quad (77)$$

$\alpha_2$  is the bending moment coefficient for  $M_{xx}$  at the plate centre, when  $M_{xx}$  is expressed using  $a$ , and  $\alpha_3$  is the bending moment coefficient for  $M_{xx}$  at the plate centre when  $M_{xx}$  is expressed using  $b$ .

$$M_{yy} = -D \sum_{n=1}^{\infty} \left( \frac{n\pi}{a} \right)^2 \left\{ (c_{1n} + 2c_{2n}) - \mu \left( c_{1n} + \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \right) \right\} \sin \frac{n\pi}{2} \quad \dots(78)$$

$M_{yy}$  expressed by Equation (78) is presented using bending moment coefficients  $\alpha_4$  and  $\alpha_5$  as:

$$M_{yy} = \alpha_4 p_0 a^2 = \alpha_5 p_0 b^2 \quad (79)$$

$\alpha_4$  is the bending moment coefficient for  $M_{yy}$  at the plate centre where  $M_{yy}$  is expressed using  $a$ ,  $\alpha_5$  is the bending moment coefficient for  $M_{yy}$  at the plate centre where  $M_{yy}$  is expressed using  $b$ .

### Solutions for square CSCS plates

For square CSCS plates  $b/a = 1$

$$F_1 \left( \frac{b}{a} = 1 \right) = \frac{n\pi}{2} \frac{1}{\sinh \frac{n\pi}{2}} + \cosh \frac{n\pi}{2} \quad (80)$$

$$F_2 \left( \frac{b}{a} = 1 \right) = 1 + \frac{n\pi}{2} \coth \frac{n\pi}{2} \quad (81)$$

$$c_{1n} \left( \frac{b}{a} = 1 \right) = \frac{-2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \frac{\left( \frac{n\pi}{2} \frac{1}{\sinh \frac{n\pi}{2}} + \cosh \frac{n\pi}{2} \right)}{\left( 1 + \frac{n\pi}{2} \coth \frac{n\pi}{2} \right)} \quad (82)$$

$$c_{2n} \left( \frac{b}{a} = 1 \right) = \frac{2p_0a^4(-1)^{n+1}}{D(n\pi)^5} \left( \frac{1}{\frac{n\pi}{2} \frac{1}{\sinh \frac{n\pi}{2}} + \cosh \frac{n\pi}{2}} \right) \quad (83)$$

Thus,  $u_z(x, y)$ ,  $M_{xx}$  and  $M_{yy}$  can be calculated for square CSCS plates from the obtained values of  $F_1$ ,  $F_2$ ,  $c_{1n}$  and  $c_{2n}$ . Similarly, for any plate aspect ratio,  $F_1$ ,  $F_2$ ,  $c_{1n}$ ,  $c_{2n}$  are calculated and  $u_z(x, y)$ ,  $M_{xx}$ ,  $M_{yy}$  determined. An illustration of the solution obtained for a one term series approximation is presented for square CSCS plates. For the one term series approximation,  $n = 1$

$$F_1 \left( n = 1, \frac{b}{a} = 1 \right) = \frac{\pi}{2} \frac{1}{\sinh \frac{\pi}{2}} + \cosh \frac{\pi}{2} = 3.19175 \quad (84)$$

$$F_2 \left( n = 1, \frac{b}{a} = 1 \right) = 1 + \frac{\pi}{2} \coth \frac{\pi}{2} = 2.71269 \quad (85)$$

Then,

$$u_z \left( n = 1, \frac{b}{a} = 1 \right) = \frac{2p_0a^4(-1)^2}{D\pi^5} \left( 1 - \frac{2.71269}{3.19175} \right)$$

$$= 9.8093 \times 10^{-4} \frac{p_0 a^4}{D} \quad (86)$$

$$c_{1n} \left( n = 1, \frac{b}{a} = 1 \right) = \frac{-2p_0 a^4 (-1)^2 F_2}{D \pi^5 F_1} = -5.5546 \times 10^{-3} \frac{p_0 a^4}{D} \quad \dots(87)$$

$$c_{2n} \left( n = 1, \frac{b}{a} = 1 \right) = \frac{2p_0 a^4 (-1)^2}{D \pi^5 F_1} = 2.0476 \times 10^{-3} \frac{p_0 a^4}{D} \quad (88)$$

$$M_{xx} \left( \frac{b}{a} = 1, n = 1 \right) = -D \left( \frac{\pi}{a} \right)^2 \left( 0.3(c_1 + 2c_2) - c_1 - \frac{2p_0 a^4}{D \pi^5} \right) = 0.01402 p_0 a^2 \quad (89)$$

$$M_{yy} \left( n = 1, \frac{b}{a} = 1 \right) = -D \left( \frac{\pi}{a} \right)^2 \left( c_1 + 2c_2 - 0.3 \left( c_1 + \frac{2}{\pi^5} \right) \right) = 0.017307 p_0 a^2 \quad (90)$$

**Table 1:** Converged results for the expression for deflections evaluated at the centre of rectangular thin CSCS plate under triangular load distribution  $p(x, y) = p_0 x/a$  over the plate domain; for various values of the plate aspect ratio and comparison with the centre deflections evaluated and obtained previously by Onah et al [31] and Shames and Dym [2]

| $b/a$ | $w_c = \alpha_1 \frac{p_0 a^4}{D}$ |                              |                                 |
|-------|------------------------------------|------------------------------|---------------------------------|
|       | $(\alpha_1)$ Present study         | $(\alpha_1)$ Onah et al [31] | $(\alpha_1)$ Shames and Dym [2] |
| 0.5   | $8.1576 \times 10^{-5}$            | $8.1576 \times 10^{-5}$      | $8.1576 \times 10^{-5}$         |
| 1.0   | $9.5855 \times 10^{-4}$            | $9.5855 \times 10^{-4}$      | $9.5855 \times 10^{-4}$         |
| 2.0   | $4.2224 \times 10^{-3}$            | $4.2224 \times 10^{-3}$      | $4.2224 \times 10^{-3}$         |

**Table 2:** Converged results for bending moments  $M_{xx}$  at the centre of rectangular thin CSCS plate subjected to triangular load distribution  $p(x, y) = p_0 x/a$  over the domain of the plate ( $\mu = 0.30$ ) for various values of the plate aspect ratio and comparison with previously obtained converged results by Onah et [31] and Timoshenko and Woinowsky-Krieger [1] for  $M_{xx}$  evaluated at the plate centre

| $b/a$ aspect ratio | $M_{xx}$               |                          |   |
|--------------------|------------------------|--------------------------|---|
|                    | Present study $M_{xx}$ | Onah et al [31] $M_{xx}$ | Timoshenko and Woinowsky-Krieger [1] $M_{xx}$ |
| 0.5                | $0.007 p_0 b^2$        | $0.007 p_0 b^2$          | $0.007 p_0 b^2$                               |
| 0.75               | $0.011 p_0 b^2$        | $0.011 p_0 b^2$          | $0.011 p_0 b^2$                               |
| 1.0                | $0.0121760 p_0 a^2$    | $0.012 p_0 a^2$          | $0.012 p_0 a^2$                               |
| 1.25               | $0.021 p_0 a^2$        | $0.021 p_0 a^2$          | $0.021 p_0 a^2$                               |
| 1.50               | $0.030 p_0 a^2$        | $0.030 p_0 a^2$          | $0.030 p_0 a^2$                               |
| 2                  | $0.043 p_0 a^2$        | $0.043 p_0 a^2$          | $0.043 p_0 a^2$                               |
| $\infty$           | $0.063 p_0 a^2$        | $0.063 p_0 a^2$          | $0.063 p_0 a^2$                               |

**Table 3:** Converged results for bending moments  $M_{yy}$  at the centre of rectangular thin CSCS plate subjected to triangular load distribution  $p(x, y) = p_0 x/a$  over the domain of the plate (for  $\mu = 0.30$ ) for various values of the plate aspect ratio; and

comparative results for  $M_{yy}$  at the plate centre for different values of plate aspect ratio as previously obtained by Onah et al [31] and Timoshenko and Woinowsky-Krieger [1]

| aspect ratio $b/a$ | $M_{yy}$               |                          |   |
|--------------------|------------------------|--------------------------|---|
|                    | Present study $M_{yy}$ | Onah et al [31] $M_{yy}$ | Timoshenko and Woinowsky-Krieger [1] $M_{yy}$ |
| 0.5                | $0.021 p_0 b^2$        | $0.021 p_0 b^2$          | $0.021 p_0 b^2$                               |
| 0.75               | $0.020 p_0 b^2$        | $0.020 p_0 b^2$          | $0.020 p_0 b^2$                               |
| 1.0                | $0.016617 p_0 a^2$     | $0.017 p_0 a^2$          | $0.017 p_0 a^2$                               |
| 1.25               | $0.021 p_0 a^2$        | $0.021 p_0 a^2$          | $0.021 p_0 a^2$                               |
| 1.50               | $0.023 p_0 a^2$        | $0.023 p_0 a^2$          | $0.023 p_0 a^2$                               |
| 2                  | $0.024 p_0 a^2$        | $0.024 p_0 a^2$          | $0.024 p_0 a^2$                               |
| $\infty$           | $0.019 p_0 a^2$        | $0.019 p_0 a^2$          | $0.019 p_0 a^2$                               |

**Table 4:** Convergence studies of the single series expression for the deflection at the centre of square CSCS Kirchhoff plate subjected to triangular load distribution  $p(x, y) = p_0 x/a$ , over the entire plate domain. Comparative results for the convergence investigation of the expressions for deflection at the centre of CSCS thin plate obtained using Kantorovich-Vlasov method by Onah et al [31] and Levy's method by Shames and Dym [2].  $u_z = \alpha_1 p_0 a^4 / D$

| Number of terms in series | $u_z = (\alpha_1 p_0 a^4 / D)$<br>$\alpha_1$ | Onah et al [31], Shames and Dym [2] ( $\alpha_1$ ) |
|---------------------------|--|--|
| 1                         | $9.8093 \times 10^{-4}$                      | $9.8093 \times 10^{-4}$                            |
| 2                         | $9.8093 \times 10^{-4}$                      | $9.8093 \times 10^{-4}$                            |
| 3                         | $9.5679 \times 10^{-4}$                      | $9.5679 \times 10^{-4}$                            |
| 4                         | $9.5679 \times 10^{-4}$                      | $9.5679 \times 10^{-4}$                            |
| 5                         | $9.5887 \times 10^{-4}$                      | $9.5887 \times 10^{-4}$                            |
| 6                         | $9.5887 \times 10^{-4}$                      | $9.5887 \times 10^{-4}$                            |
| 7                         | $9.5848 \times 10^{-4}$                      | $9.5848 \times 10^{-4}$                            |
| 8                         | $9.5848 \times 10^{-4}$                      | $9.5848 \times 10^{-4}$                            |
| 9                         | $9.5859 \times 10^{-4}$                      | $9.5859 \times 10^{-4}$                            |
| 10                        | $9.5859 \times 10^{-4}$                      | $9.5859 \times 10^{-4}$                            |
| 11                        | $9.5855 \times 10^{-4}$                      | $9.5855 \times 10^{-4}$                            |
| 12                        | $9.5855 \times 10^{-4}$                      | $9.5855 \times 10^{-4}$                            |

**Table 5:** Convergence studies of the single series expressions for bending moments  $M_{xx}$  and  $M_{yy}$  at the centre of square CSCS Kirchhoff plate subjected to triangular load distribution  $p(x, y) = p_0 x/a$  over the entire domain of the plate. Comparative results for the investigation of the convergence of expressions of bending moments,  $M_{xx}$  and  $M_{yy}$  obtained by using Kantorovich-Vlasov variational method by Onah et al [31] and Levy's method by Shames and Dym [2]

| Number of terms in series | $M_{xx} = \alpha_2 p_0 a^2$<br>$\alpha_2$ | Onah et al [31]; Shames and Dym [2] $\alpha_2$ | $M_{yy} = \alpha_4 p_0 a^2$<br>$\alpha_4$ | Onah et al [31]; Shames and Dym [2] $\alpha_4$ |
|---------------------------|---|--|---|--|
| 1                         | $1.4002 \times 10^{-2}$                   | $1.4002 \times 10^{-2}$                        | $1.7307 \times 10^{-2}$                   | $1.7307 \times 10^{-2}$                        |
| 2                         | $1.4002 \times 10^{-2}$                   | $1.4002 \times 10^{-2}$                        | $1.7307 \times 10^{-2}$                   | $1.7307 \times 10^{-2}$                        |

|    |                         |                         |                         |                         |
|----|-------------------------|-------------------------|-------------------------|-------------------------|
|    | $10^{-2}$               | $10^{-2}$               | $10^{-2}$               | $10^{-2}$               |
| 3  | $1.1810 \times 10^{-2}$ | $1.1810 \times 10^{-2}$ | $1.6505 \times 10^{-2}$ | $1.6505 \times 10^{-2}$ |
| 4  | $1.1810 \times 10^{-2}$ | $1.1810 \times 10^{-2}$ | $1.6505 \times 10^{-2}$ | $1.6505 \times 10^{-2}$ |
| 5  | $1.2324 \times 10^{-2}$ | $1.2324 \times 10^{-2}$ | $1.6661 \times 10^{-2}$ | $1.6661 \times 10^{-2}$ |
| 6  | $1.2324 \times 10^{-2}$ | $1.2324 \times 10^{-2}$ | $1.6661 \times 10^{-2}$ | $1.6661 \times 10^{-2}$ |
| 7  | $1.2136 \times 10^{-2}$ | $1.2136 \times 10^{-2}$ | $1.6605 \times 10^{-2}$ | $1.6605 \times 10^{-2}$ |
| 8  | $1.2136 \times 10^{-2}$ | $1.2136 \times 10^{-2}$ | $1.6605 \times 10^{-2}$ | $1.6605 \times 10^{-2}$ |
| 9  | $1.2224 \times 10^{-2}$ | $1.2224 \times 10^{-2}$ | $1.6632 \times 10^{-2}$ | $1.6632 \times 10^{-2}$ |
| 10 | $1.2224 \times 10^{-2}$ | $1.2224 \times 10^{-2}$ | $1.6632 \times 10^{-2}$ | $1.6632 \times 10^{-2}$ |
| 11 | $1.2176 \times 10^{-2}$ | $1.2176 \times 10^{-2}$ | $1.6617 \times 10^{-2}$ | $1.6617 \times 10^{-2}$ |
| 12 | $1.2176 \times 10^{-2}$ | $1.2176 \times 10^{-2}$ | $1.6617 \times 10^{-2}$ | $1.6617 \times 10^{-2}$ |

## 5. DISCUSSION

The flexural problem of rectangular thin plate with simply supported edges  $x = 0$ ,  $x = a$  and clamped edges  $y = \pm b/2$  where the plate is subjected to triangular load distribution  $p(x, y) = p_0x/a$  over the domain has been solved in this work using the single finite Fourier sine integral transform method. The plate considered was modelled using the small deformation Kirchhoff thin plate theory. The governing domain equation of equilibrium is the inhomogeneous biharmonic partial differential equation expressed as Equation (1). The solution was sought to satisfy the boundary conditions of CSCS plates given by Equations (7 – 10).

The application of the finite sine transformation to the explicit form of the governing PDE resulted in the integral equation formulation of the boundary value problem as Equation (28). The linearity properties of the integral transformation used and Leibnitz rule was used together with the boundary conditions to express the integral equation as the ordinary differential equation (ODE) – Equation (31). The general solution to the ODE – Equation (31) – was found using the trial function method, the expected symmetrical nature of the solution and the nondependence of the triangular loading function on  $y$  as Equation (44) which contains two integration constants  $c_{1n}$  and  $c_{2n}$ .

By inversion of  $\bar{U}_z(n, y)$  the general solution in the problem domain was obtained as Equation (46). Enforcement of the boundary conditions equation at the clamped edges  $y = \pm b/2$  led to the determination of  $c_{2n}$  as Equation (61) and  $c_{1n}$  as Equation (63). The deflection was thus found as the single infinite series expression given by Equation (65). The deflection expression was confirmed to satisfy the Dirichlet boundary conditions at the simply supported edges  $x = 0$ , and  $x = a$ . The deflection was evaluated in general at the centre of

the plate and the deflection expression at the plate centre was found as Equation (68). Equation (68) is presented in terms of deflection coefficients  $\alpha_1$  as Equation (69) where  $\alpha_1$  is expressed as the single infinite series expressed by Equation (70).

The bending moment expressions were found by using the single infinite series expression for  $u_z(x, y)$  in the bending moment-deflection equations. The expression for  $M_{xx}$  was found as the single infinite series given generally by Equation (76), which can be expressed using bending moment coefficients for  $M_{xx}$  ( $\alpha_2$  and  $\alpha_3$ ) as Equation (77). Similarly, the expression for  $M_{yy}$  was found as the single infinite series given generally by Equation (78), which can be expressed using bending moment coefficients for  $M_{yy}$  ( $\alpha_4$  and  $\alpha_5$ ) as Equation (79).

The expressions obtained were used to compute the solution for square CSCS plate under triangular loading distribution. A one term approximation was found for the centre deflection as Equation (86), and the bending moments at the centre as Equation (89) for  $M_{xx}$  and Equation (90) for  $M_{yy}$ .

The single infinite series for the deflection was used to obtain the converged solutions for deflection of the centre of CSCS plate for various values of the aspect ratio ( $b/a$ ) and the results presented in Table 1, which shows that the results are in excellent agreement with previously presented results by Shames and Dym [2] and Onah et al [31]. The expression for  $M_{xx}$  was used to determine the bending moments  $M_{xx}$  at the centre of the plate for  $\mu = 0.30$  and various values of the aspect ratio ( $b/a$ ) and the results were presented in Table 2. Table 2 illustrates the excellent agreement of the present results for  $M_{xx}$  with the previous results obtained by Timoshenko and Woinowsky-Krieger [1] and Onah et al [31].

The expression for  $M_{yy}$  was used to determine the bending moment  $M_{yy}$  at the centre of the plate for  $\mu = 0.30$ , and various values of the plate aspect ratio ( $b/a$ ) and the results were presented in Table 3. Table 3 illustrates the excellent agreement of the present results for  $M_{yy}$  with the previous results obtained by Timoshenko and Woinowsky-Krieger [1] and Onah et al [31].

The results of the investigation of the convergence properties of the infinite series for the deflection, which are presented in Table 4 for square CSCS plate show the series for the deflection is a rapidly convergent series. Table 4 also demonstrates that the results of the convergence studies of the expression for deflection are identical with previous results presented by Shames and Dym [2] and Onah et al [31], who used Levy's method and Kantorovich-Vlasov's method respectively. The investigation of the convergence properties of the expressions for bending moments  $M_{xx}$  and  $M_{yy}$  which are presented in Table 5 for square CSCS plates show that both series are rapidly convergent. Table 5 shows that the results of the convergence studies for the bending moment expressions at the centre of square CSCS plate are identical with previous results of convergence investigations presented by Shames and Dym [2] and Onah et al [31].



## 6. CONCLUSION

- (1) The single finite Fourier sine integral transform method is a very effective mathematical tool for solving the flexural analysis problem of rectangular Kirchhoff CSCS plates subject to triangular load distribution over the plate region.
- (2) The governing domain equation which is an inhomogeneous biharmonic partial differential equation is converted by the transformation to an integral equation over the solution domain.
- (3) The integral equation is reduced by the linearity properties of the transformation, the Leibnitz rule and the boundary conditions along the simply supported edges to ordinary differential equations (ODEs).
- (4) The general solution to the ODEs obtained using the method of trial functions but which could also be obtained using other methods of solving ODEs such as Differential Operator (D-operator) methods, variational of parameters methods, etc gave the expression for  $u_z(x, y)$  that satisfies the domain PDE at all points in the solution domain in terms of the integration constants  $c_{1n}$  and  $c_{2n}$ .
- (5) The enforcement of boundary conditions along the clamped edges  $y = \pm b/2$  was used to obtain the expressions for the integration constants  $c_{1n}$  and  $c_{2n}$ , thus leading to the full determination of the solution for the deflection  $u_z(x, y)$  for all points in the solution domain and on the clamped boundaries ( $y = \pm b/2$ ).
- (6) The obtained solution for transverse deflection  $u_z(x, y)$  is a single infinite series with rapidly convergent properties.
- (7) The expression for transverse deflection  $u_z(x, y)$  is used to determine the expressions for the bending moments  $M_{xx}$  and  $M_{yy}$  at any point in the plate. The expressions for  $M_{xx}$  and  $M_{yy}$  are single infinite series that are also rapidly convergent as demonstrated in the calculations done and presented for square CSCS plate considered.
- (8) Maximum deflection was obtained at the plate centre and this result agrees with the symmetrical features of the presented problem.
- (9) The converged values of deflection and bending moments calculated at the plate centre are in excellent agreement with previous results published by Timoshenko and Woinowsky-Krieger [1], Onah et al [31] and Shames and Dym [2] and that employed superposition method, Kantorovich-Vlasov's and Levy's methods respectively..
- (10) The single finite integral transform method has yielded exact solutions for the flexural problem of rectangular CSCS plates under triangular loading distribution since the governing domain equation is satisfied at every point in the solution domain, and at every point on the boundaries of the CSCS plate considered.

### Nomenclature/Notations

|             |   |
|-------------|---|
| ODE(s)      | Ordinary Differential Equations(s)  |
| 3D          | three-dimensional   |
| 2D          | two-dimensional   |
| CSCS plate: | plate with two opposite sides simply supported and the other two sides clamped. |
| $x, y, z$   | Cartesian coordinates   |

|                                  |   |
|----------------------------------|---|
| $\mu$                            | Poisson's ratio   |
| $h$                              | plate thickness   |
| $p(x, y)$                        | transverse load distribution over the plate domain  |
| $D$                              | flexural rigidity of plate material   |
| $u_z(x, y)$                      | transverse deflection of the plate middle surface.  |
| $M_{xx} M_{yy}$                  | Bending moment distributions  |
| $E$                              | Young's modulus of elasticity of the plate material   |
| $a$                              | inplane dimension of the plate in the $x$ direction   |
| $b$                              | inplane dimension of the plate in the $y$ direction.  |
| $p_0$                            | intensity of triangular load distribution at $x = a$ .  |
| $S_n$                            | single finite Fourier sine integral transform operator  |
| $0 \leq x \leq l$                | interval of definition of $x$ for $v(x)$  |
| $v(x)$                           | a function defined in terms of $x$  |
| $v(x, y)$                        | a function defined in terms of $x$ and $y$  |
| $V_s(n)$                         | single finite Fourier sine transform of $v(x)$  |
| $C_n(v(x))$ or $V_c(n)$          | single finite Fourier cosine integral transform of $v(x)$   |
| $n$                              | single finite Fourier sine integral transform parameter.  |
| $\frac{d^k}{dx^k}$               | $k$ th derivative with respect to $x$   |
| $\int$                           | integral  |
| $\int_0^a ( ) dx$                | integration with respect to $x$ between the intervals $x = 0$ and $x = a$ .   |
| $\bar{U}_z(n, y)$                | single finite Fourier integral transform of the deflection $u_z(x, y)$ of the plate.  |
| $\Gamma, \lambda$                | parameters of the assumed homogeneous solution for the ordinary differential equation in the transform space resulting from the application of the transformation to the governing domain equation. |
| $\bar{U}_{zh}(n, y)$             | homogeneous component of the general solution for $\bar{U}_z(n, y)$   |
| $\lambda_{1,2}, \lambda_{3,4}$   | four roots of the resulting algebraic eigenequation involving $\lambda$   |
| $c_{1n}, c_{2n}, c_{3n}, c_{4n}$ | integration constants   |
| $\bar{U}_{zp}(n, y)$             | particular integral (solution) for the general solution of the ordinary differential equation in transform space  |
| $\Sigma$                         | summation   |
| $\infty$                         | infinity  |
| $F_1$                            | parameter defined in terms of $n, a$ and $b$  |
| $F_2$                            | another parameter defined in terms of $n, a$ and $b$  |
| $\alpha_1$                       | deflection coefficient for the centre of rectangular CSCS Kirchhoff plate subjected to triangular load distribution over the entire plate domain  |
| $\alpha_2$                       | bending moment coefficient for $M_{xx}$ at the plate centre when $M_{xx}$ is expressed using $a$  |
| $\alpha_3$                       | bending moment coefficient for $M_{xx}$ at the plate centre when $M_{xx}$ is expressed using $b$  |

|                                   |   |
|-----------------------------------|---|
| $\alpha_4$                        | bending moment coefficient for $M_{yy}$ at the plate centre where $M_{yy}$ is expressed using $a$ |
| $\alpha_5$                        | bending moment coefficient for $M_{yy}$ at the plate centre where $M_{yy}$ is expressed using $b$ |
| $\frac{\partial^k}{\partial x^k}$ | $k$ th partial derivative with respect to $x$   |
| cos                               | cosine function   |
| sin                               | sine function   |
| cosh                              | hyperbolic cosine function  |
| sinh                              | hyperbolic sine function  |
| coth                              | hyperbolic cotangent function.  |

### Conflict of interest

The authors declare no conflict of interest in the publication of this paper.

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